

$$\begin{aligned} \text{Im } \mathbf{x}_1 &= -\frac{1}{\sqrt{4\eta - \xi^2}} \left(\frac{1}{2} \xi \mathbf{v}_p + \mathbf{v}_{p+1} \right) \\ &= [-1.983328 \quad 0.872197 \quad 3.752865 \quad -1.463834]^T \times 10^3 \end{aligned} \quad (q)$$

Following the same procedure with A^T as with A , we obtain the same values for ξ and η as the values given by Eqs. (n), as well as the iterates \mathbf{w}_p and \mathbf{w}_{p+1} . Hence, the eigenvalues λ_1 and $\bar{\lambda}_1$ remain the same as those given by Eq. (p), as expected. On the other hand, using the analogy with Eqs. (q) in conjunction with the iterates \mathbf{w}_p and \mathbf{w}_{p+1} , we compute $\text{Re } \mathbf{y}_1$ and $\text{Im } \mathbf{y}_1$. At this point, we are in a position to normalize the right and left eigenvectors belonging to λ_1 by writing $\mathbf{y}_1^T \mathbf{x}_1 = 1$, according to Eq. (6.185) with $r = s = 1$. The normalized right and left eigenvectors and the complex conjugates are

$$\begin{aligned} \mathbf{x}_1 &= \begin{bmatrix} 0.138171 \mp 0.156517i \\ -0.054440 \pm 0.069777i \\ 0.372785 \pm 0.391081i \\ -0.167784 \mp 0.155893i \end{bmatrix}, & \mathbf{y}_1 &= \begin{bmatrix} 1.259528 \pm 1.241648i \\ -1.074222 \mp 1.053273i \\ 0.526929 \mp 0.435837i \\ -0.423504 \pm 0.395909i \end{bmatrix} \end{aligned} \quad (r)$$

To compute the eigensolutions $\lambda_3, \mathbf{x}_3, \bar{\lambda}_3, \bar{\mathbf{x}}_3$, we must produce the deflated matrix A_3 . Introducing Eqs. (c), (p) and (r) into Eq. (6.202), we obtain

$$\begin{aligned} A_3 &= A - \lambda_1 \mathbf{x}_1 \mathbf{y}_1^T - \bar{\lambda}_1 \bar{\mathbf{x}}_1 \bar{\mathbf{y}}_1^T = A - 2 \text{Re} (\lambda_1 \mathbf{x}_1 \mathbf{y}_1^T) \\ &= \begin{bmatrix} 0.020244 & -0.010725 & -0.087772 & 0.926615 \\ 0.064174 & -0.057336 & 0.463307 & 0.606787 \\ 2.292088 & -2.204316 & 0.148014 & -0.120941 \\ -1.102962 & 0.639655 & -0.060470 & -0.025354 \end{bmatrix} \end{aligned} \quad (s)$$

Using the same matrix iteration process with A_3 as with A_1 we compute the subdominant eigenvalues

$$\begin{aligned} \lambda_3 &= -0.027404 \pm 0.545795i \\ \bar{\lambda}_3 & \end{aligned} \quad (t)$$

and the normalized right and left eigenvectors

$$\begin{aligned} \mathbf{x}_3 &= \begin{bmatrix} 0.433803 \mp 0.452506i \\ 0.516529 \mp 0.525161i \\ 0.235088 \pm 0.249168i \\ 0.272475 \pm 0.296311i \end{bmatrix}, & \mathbf{y}_3 &= \begin{bmatrix} 0.186794 \pm 0.111816i \\ 0.339785 \pm 0.366585i \\ 0.275539 \mp 0.274295i \\ 0.655774 \mp 0.636294i \end{bmatrix} \end{aligned} \quad (u)$$

6.14 REDUCTION OF NONSYMMETRIC MATRICES TO HESSENBERG FORM

As discussed in Sec. 6.8, Givens' method for the computation of the eigenvalues of real symmetric matrices requires that the matrix be in tridiagonal form. Then, in Sec. 6.9, we presented the QR method, designed for the same purpose. Convergence of the QR method can be very slow, unless the matrix is in tridiagonal form. However, in general matrices defining the eigenvalue problem for vibrating conservative systems, albeit symmetric, tend to be fully populated. Hence, if the interest lies in solving the eigenvalue problem by one of these two algorithms, it is necessary to transform the original matrix to tridiagonal form. But, because the tridiagonal matrix must have the same eigenvalues as the original matrix, the two

matrices must be related by a similarity transformation. In the case of symmetric matrices, the transformation is actually orthogonal, i.e., a special case of the similarity transformation. Three methods capable of transforming a real symmetric matrix to tridiagonal form are Givens' method (Sec. 6.5), Householder's method (Sec. 6.6) and Lanczos' method (Sec. 6.7), among others. Denoting the original matrix by A and the tridiagonal matrix by T , the orthogonal transformation has the form

$$T = P^T A P \quad (6.210)$$

where the transformation matrix P is orthonormal, i.e., it satisfies $P^T P = I$. In Givens' method, P is obtained through a sequence of $(n-1)(n-2)/2$ rotations, where n is the order of the matrix A , and in Householder's method through a series of $n-2$ reflections. In contrast, Lanczos' method is a direct method whereby the matrix P is obtained by means of a recursive process.

In the case of a nonsymmetric matrix A , significant simplification of the eigenvalue problem accrues by reducing the matrix to *Hessenberg form*, denoted here by H . There are two such forms, an *upper Hessenberg* and a *lower Hessenberg* form. Of the two, the upper Hessenberg form is more commonly encountered, because various algorithms for the computation of eigenvalues are based on such forms. The upper Hessenberg matrix, defined by $h_{ij} = 0, i \geq j + 2$, has the form

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \dots & h_{1,n-1} & h_{1n} \\ h_{21} & h_{22} & h_{23} & \dots & h_{2,n-1} & h_{2n} \\ 0 & h_{32} & h_{33} & \dots & h_{3,n-1} & h_{3n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & h_{n-1,n-1} & h_{n-1,n} \\ 0 & 0 & 0 & \dots & h_{n,n-1} & h_{nn} \end{bmatrix} \quad (6.211)$$

A real nonsymmetric matrix can be reduced to Hessenberg form by means of either Givens' method or Householder's method. In this regard, a slight inconvenience arises, because Givens' method and Householder's method are commonly formulated so as to reduce symmetric matrices to tridiagonal form through annihilation of the upper off-diagonal elements. Of course, as a by-product due to symmetry, the corresponding lower off-diagonal elements are also annihilated. In the case of nonsymmetric matrices, Givens' algorithm and Householder's algorithm, as presented in Secs. 6.5 and 6.6, respectively, result in lower Hessenberg matrices. Modification of the algorithms to produce upper Hessenberg matrices is relatively trivial and amounts to an interchange of the subscripts involved.

In this section, we consider a direct method for reducing a real nonsymmetric matrix to upper Hessenberg form. The approach reminds of the Gaussian elimination for the reduction of a general matrix to upper triangular form through elementary operations presented in Sec. 6.1. Using the same idea, and recognizing that the current objective is to solve an eigenvalue problem, instead of solving nonhomogeneous algebraic equations, we propose to reduce the nonsymmetric $n \times n$ matrix A to an upper Hessenberg form H by means of a transformation that preserves the eigenvalues. Hence, whereas in both cases use is made of elementary transformations, our interest lies in a transformation to Hessenberg form by means of a similarity

transformation. In view of this, we consider the transformation

$$P^{-1}AP = H \tag{6.212}$$

where P is a unit lower triangular matrix of the type encountered in Sec. 6.1. But, because our objective is to produce only an upper Hessenberg matrix, instead of an upper triangular matrix, the transformation matrix P has the form

$$P = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & p_{32} & 1 & \dots & 0 & 0 \\ 0 & p_{42} & p_{43} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & p_{n-1,2} & p_{n-1,3} & \dots & 1 & 0 \\ 0 & p_{n2} & p_{n3} & \dots & p_{n,n-1} & 1 \end{bmatrix} \tag{6.213}$$

The problem consists of determining not only the matrix H but also the matrix P . To solve this problem, we premultiply Eq. (6.212) by P and obtain

$$AP = PH \tag{6.214}$$

which represents a set of n^2 equations in n^2 unknowns, $(n^2 - 3n + 2)/2$ nonzero p_{ij} and $(n^2 + 3n - 2)/2$ nonzero h_{ij} .

Because of the structure of P and H , we can determine the nonzero elements p_{ij} and h_{ij} by equating every column of AP to the corresponding column of PH . In particular, by equating the r th columns, we determine column r of H ($r = 1, 2, \dots, n$) and column $r + 1$ of P ($r = 1, 2, \dots, n - 2$). We note that it is only necessary to determine $n - 2$ columns of P , because the first and last columns are known. The solution can be carried out in a recursive manner, beginning by equating the first column on both sides of Eq. (6.214), or

$$Ae_1 = Ph_1 \tag{6.215}$$

where $e_1 = [1 \ 0 \ 0 \ \dots \ 0]^T$ is recognized as the first standard unit vector and $h_1 = [h_{11} \ h_{21} \ 0 \ \dots \ 0]^T$. Considering Eq. (6.213), Eq. (6.215) yields

$$h_{11} = a_{11}, \quad h_{21} = a_{21} \tag{6.216a}$$

which determines the first column of H and

$$p_{i2} = \frac{a_{i1}}{h_{21}}, \quad i = 3, 4, \dots, n \tag{6.216b}$$

which defines the second column of P . Note that, in choosing the first standard unit vector e_1 as the first column of P , we obtain the simplest solution. Indeed, we could have used any other vector instead of e_1 and still obtained a solution. Equating the second column on both sides of Eq. (6.214), we have

$$Ap_2 = Ph_2 \tag{6.217}$$

where $\mathbf{p}_2 = [0 \ 1 \ p_{32} \ p_{42} \ \dots \ p_{n2}]^T$, $\mathbf{h}_2 = [h_{12} \ h_{22} \ h_{32} \ 0 \ \dots \ 0]^T$. Following the same pattern, the first three rows of Eq. (6.217) determine the second column of H , or by components

$$h_{i2} = a_{i2} + \sum_{k=3}^n a_{ik} p_{k2} - \sum_{k=2}^{i-1} p_{ik} h_{k2}, \quad i = 1, 2, 3 \quad (6.218a)$$

and the remaining $n - 3$ rows can be used to determine the third column of P in the form

$$p_{i3} = \frac{1}{h_{32}} \left(a_{i2} + \sum_{k=3}^n a_{ik} p_{k2} \right) - p_{i2} h_{22}, \quad i = 4, 5, \dots, n \quad (6.218b)$$

The procedure can be generalized by writing

$$h_{ir} = a_{ir} + \sum_{k=r+1}^n a_{ik} p_{kr} - \sum_{k=2}^{i-1} p_{ik} h_{kr},$$

$$r = 1, 2, \dots, n - 1; \quad i = 1, 2, \dots, r + 1 \quad (6.219a)$$

$$p_{i,r+1} = \frac{1}{h_{r+1,r}} \left(a_{ir} + \sum_{k=r+1}^n a_{ik} p_{kr} - \sum_{k=2}^r p_{ik} h_{kr} \right),$$

$$r = 1, 2, \dots, n - 2; \quad i = r + 2, r + 3, \dots, n \quad (6.219b)$$

$$h_{in} = a_{in} - \sum_{k=2}^{i-1} p_{ik} h_{kn}, \quad i = 1, 2, \dots, n \quad (6.219c)$$

where it must be remembered that $p_{k1} = 0$ ($k = 2, 3, \dots, n$). The computations alternate between Eqs. (6.219a) and (6.219b), until $r = n - 1$, at which point the computations skip from Eq. (6.219a) to Eq. (6.219c). This can be explained by the fact that the matrix P has only $n - 2$ unknown columns.

Equation (6.219b) contains a potential source of difficulties in that it requires division by $h_{r+1,r}$. Indeed, if $h_{r+1,r}$ is zero, the algorithm breaks down. Even when $h_{r+1,r}$ is only very small, and not necessarily zero, numerical instability can occur. In such cases, it becomes necessary to carry out suitable row and column interchanges, which can be done by means of permutation matrices, Eq. (6.36). However, unlike in Gaussian elimination, the transformation using the permutation matrix must be a similarity transformation so as to preserve the eigenvalues. But, because the inverse of a permutation matrix is equal to the permutation matrix itself, the similarity transformation implies premultiplication and postmultiplication by the same permutation matrix, which amounts to an interchange of both rows and the corresponding columns. As an example, if rows s and t are to be interchanged, so are columns s and t . In the case in which the matrix A represents the coefficient matrix in the state equations, such row and column interchanges are an absolute necessity. In fact, entire blocks must be interchanged, leading to the matrix

$$A' = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \begin{bmatrix} -M^{-1}C & -M^{-1}K \\ I & 0 \end{bmatrix} \quad (6.220)$$

Example 6.16

Derive the upper Hessenberg matrix corresponding to the coefficient matrix of Example 6.15.

Inserting Eq. (c) of Example 6.15 into Eq. (6.220), we obtain the coefficient matrix

$$A' = \begin{bmatrix} -M^{-1}C & -M^{-1}K \\ I & 0 \end{bmatrix} = \begin{bmatrix} -0.4 & 0.2 & -5 & 4 \\ 0.1 & -0.1 & 2 & -2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (\text{a})$$

In deriving the upper Hessenberg matrix corresponding to A' by means of Eqs. (6.219), we will treat A' as if it were A , in the sense that we will ignore the prime.

Using Eq. (6.219a) with $r = 1$, we obtain

$$h_{11} = a_{11} = -0.4, \quad h_{21} = a_{21} = 0.1 \quad (\text{b})$$

Then, from Eq. (6.219b) with $r = 1$, we have

$$p_{32} = \frac{a_{31}}{h_{21}} = 10, \quad p_{42} = \frac{a_{41}}{h_{21}} = 0 \quad (\text{c})$$

Next, we let $r = 2$ in Eqs. (6.219a) and (6.219b) and write

$$\begin{aligned} h_{12} &= a_{12} + a_{13}p_{32} + a_{14}p_{42} = -49.8 \\ h_{22} &= a_{22} + a_{23}p_{32} + a_{24}p_{42} = 19.9 \\ h_{32} &= a_{32} + a_{33}p_{32} + a_{34}p_{42} - p_{32}h_{22} = -199 \end{aligned} \quad (\text{d})$$

and

$$p_{43} = \frac{1}{h_{32}} (a_{42} + a_{43}p_{32} + a_{44}p_{42} - p_{42}h_{22}) = -\frac{1}{199} = -0.005025 \quad (\text{e})$$

respectively. At this point, the pattern changes somewhat. Indeed, Eq. (6.219a) with $r = 3$ yields

$$\begin{aligned} h_{13} &= a_{13} + a_{14}p_{43} = -\frac{999}{199} = -5.020101 \\ h_{23} &= a_{23} + a_{24}p_{43} = \frac{400}{199} = 2.010050 \\ h_{33} &= a_{33} + a_{34}p_{43} - h_{23}p_{32} = -\frac{4000}{199} = -20.100503 \\ h_{43} &= a_{43} + a_{44}p_{43} - h_{23}p_{42} - h_{33}p_{43} = -\frac{4000}{199^2} = -0.101008 \end{aligned} \quad (\text{f})$$

Then, skipping to Eq. (6.219c), we obtain

$$\begin{aligned} h_{14} &= a_{14} = 4 \\ h_{24} &= a_{24} = -2 \\ h_{34} &= a_{34} - p_{32}h_{24} = 20 \\ h_{44} &= a_{44} - p_{42}h_{24} - p_{43}h_{34} = \frac{20}{199} = 0.100503 \end{aligned} \quad (\text{g})$$

Hence, using Eqs. (b), (d), (f) and (g), the desired upper Hessenberg matrix is

$$H = \begin{bmatrix} -0.4 & -49.8 & -5.020101 & 4 \\ 0.1 & 19.9 & 2.010050 & -2 \\ 0 & -199 & -20.100503 & 20 \\ 0 & 0 & -0.101008 & 0.100503 \end{bmatrix} \quad (\text{h})$$

Moreover, using Eqs. (c) and (e), the transformation matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 10 & 1 & 0 \\ 0 & 0 & -0.005025 & 1 \end{bmatrix} \quad (\text{i})$$

6.15 THE QR METHOD FOR NONSYMMETRIC EIGENVALUE PROBLEMS

As demonstrated in Sec. 6.8, Givens' method for the computation of the eigenvalues of a real symmetric tridiagonal matrix is very efficient and easy to implement. Hence, the question arises as to whether the method can be extended to nonsymmetric matrices. Whereas general nonsymmetric matrices can be reduced to tridiagonal form, the process is potentially unstable (Ref. 13, p. 404). However, there is a much more serious obstacle preventing the extension of the method to nonsymmetric matrices. Indeed, in the case of symmetric tridiagonal matrices T the polynomials $p_i(\lambda)$ corresponding to the principal minor determinants of the matrix $T - \lambda I$ form a Sturm sequence. Givens' method is based on the Sturm sequence property, which in turn is based on the separation theorem. The Sturm sequence property cannot be demonstrated for nonsymmetric matrices, as the separation theorem holds only for real symmetric matrices.

In contrast with Givens' method for the computation of eigenvalues, the QR method does work for nonsymmetric matrices. In fact, the QR method is the most effective algorithm for computing the eigenvalues of general matrices. Still, before computational efficiencies can be realized, it is necessary to reduce the general matrix to Hessenberg form and to carry out shifts in origin. When all the eigenvalues are real, the QR method reduces the Hessenberg matrix to triangular form, with the eigenvalues lying on the main diagonal. Our interest lies in real matrices, so that if there are complex eigenvalues, then they occur in pairs of complex conjugates. In this case, the matrix differs from a triangular matrix in that there is a 2×2 matrix straddling the main diagonal for every pair of complex conjugate eigenvalues, where the complex pair represents the eigenvalues of the 2×2 matrix.

For real eigenvalues, the QR algorithm for upper Hessenberg matrices is essentially the same as for tridiagonal matrices. In fact, even when complex eigenvalues occur, the algorithm is for the most part the same as that described in Sec. 6.9. Differences begin to arise when the eigenvalues of the 2×2 matrix in the lower right corner turn complex. Indeed, the algorithm of Sec. 6.9 is based on the implicit assumption that the shifts are real, and here we are faced with the problem of complex shifts, an undesirable prospect. To avoid complex shifts, it is necessary to modify the algorithm by considering two complete iteration steps at a time. To this end, we assume that, upon completion of s iteration steps with real shifts, the 2×2 matrix in the lower

right corner of A_s has the eigenvalues μ_s and $\bar{\mu}_s$. Then, using μ_s as a first shift, we can write the next iteration step in the form

$$A_s - \mu_s I = Q_s R_s, \quad A_{s+1} = \mu_s I + R_s Q_s \tag{6.221a, b}$$

where Q_s is an orthogonal matrix, $Q_s^T Q_s = I$, and R_s is an upper triangular matrix. Moreover, using $\bar{\mu}_s$ as a second shift, we can define the following iteration step as

$$A_{s+1} - \bar{\mu}_s I = Q_{s+1} R_{s+1}, \quad A_{s+2} = \bar{\mu}_s I + R_{s+1} Q_{s+1} \tag{6.222a, b}$$

We should note here that A_s , A_{s+1} and A_{s+2} are all Hessenberg matrices.

At this point, we begin to develop an algorithm obviating the problem of complex shifts. To this end, we premultiply Eq. (6.221a) by Q_s^T , postmultiply the result by Q_s and write

$$Q_s^T A_s Q_s - \mu_s Q_s^T Q_s = Q_s^T Q_s R_s Q_s \tag{6.223}$$

so that, recalling that $Q_s^T Q_s = I$ and using Eq. (6.221b), we have

$$A_{s+1} = Q_s^T A_s Q_s \tag{6.224}$$

Next, we premultiply Eq. (6.222a) by Q_s , postmultiply by R_s and use Eqs. (6.221a) and (6.224) to obtain

$$\begin{aligned} Q_s A_{s+1} R_s - \bar{\mu}_s Q_s R_s &= Q_s Q_s^T A_s Q_s R_s - \bar{\mu}_s Q_s R_s \\ &= (A_s - \bar{\mu}_s I)(A_s - \mu_s I) = Q_s Q_{s+1} R_{s+1} R_s \end{aligned} \tag{6.225}$$

Then, introducing the notation

$$Q_s Q_{s+1} = Q, \quad R_{s+1} R_s = R \tag{6.226a, b}$$

we can rewrite Eq. (6.225) in the form

$$R = Q^T (A_s - \bar{\mu}_s I)(A_s - \mu_s I) \tag{6.227}$$

and we observe that R is an upper triangular matrix, Q is an orthogonal matrix and $(A_s - \bar{\mu}_s I)(A_s - \mu_s I) = (A_s - \text{Re } \mu_s I)^2 + (\text{Im } \mu_s I)^2$ is a real matrix. Hence, the algorithm, as described by Eq. (6.227), amounts to the upper triangularization of a real matrix. The actual objective of the algorithm, however, is the matrix Q , and not R , because it is Q that must be used to determine the matrix A_{s+2} required for the next iteration step. Indeed, using Eq. (6.224) with s replaced by $s + 1$ in conjunction with Eq. (6.226a), as well as Eq. (6.224) itself, we can write

$$A_{s+2} = Q_{s+1}^T A_{s+1} Q_{s+1} = Q_{s+1}^T Q_s^T A_s Q_s Q_{s+1} = Q^T A_s Q \tag{6.228}$$

The computation of Q can be carried out by Givens' method. Ordinarily, the triangularization of a matrix would require $n(n-1)/2$ rotations Θ_k , ($k = 1, 2, \dots, n(n-1)/2$) in the planes $(1, 2)$, $(1, 3)$, \dots , $(1, n)$, $(2, 3)$, \dots , $(2, n)$, \dots , $(n-1, n)$, so that

$$Q^T = \Theta_{n(n-1)/2} \dots \Theta_2 \Theta_1 \tag{6.229}$$

We observe, however, that the matrix $(A_s - \bar{\mu}_s I)(A_s - \mu_s I)$ represents the product of a Hessenberg matrix and its complex conjugate and its elements corresponding to the rows $i = 4, 5, \dots, n$ and the columns $j = 1, 2, \dots, i-3$ are zero. It follows that

many of the rotation matrices are identity matrices. For example, $\Theta_3, \Theta_4, \dots, \Theta_n$, corresponding to the planes $(1, 4), (1, 5), \dots, (1, n)$, are all identity matrices. In fact, only $(n-1)(n-2)/2$ rotations are necessary for the computation of Q^T .

The algorithm consists of performing two QR iteration steps at a time, yielding upper Hessenberg matrices A_{s+2}, A_{s+4}, \dots , as can be concluded from Eq. (6.228). Convergence is achieved when the sequence of upper Hessenberg matrices leads to an isolated, constant 2×2 matrix in the lower right corner, which implies that the element $(n-1, n-2)$ has been annihilated and the elements of the 2×2 matrix no longer change. At this point, the pair of complex conjugate eigenvalues is equal to the pair of complex conjugate shifts. Then, the last two rows and columns can be deleted from the matrix and the process continued with an $(n-2) \times (n-2)$ deflated matrix for the next eigenvalue or eigenvalue pair.

Example 6.17

Compute the eigenvalues of the Hessenberg matrix of Example 6.16 by means of the QR method with shifts.

Adopting the notation of this section, the Hessenberg matrix of Example 6.16 is

$$A_1 = A = \begin{bmatrix} -0.4 & -49.8 & -5.020101 & 4 \\ 0.1 & 19.9 & 2.010050 & -2 \\ 0 & -199 & -20.100503 & 20 \\ 0 & 0 & -0.101008 & 0.100503 \end{bmatrix} \quad (a)$$

We begin the process by computing the eigenvalues of the 2×2 matrix in the lower right corner of A_1 . The eigenvalues are 0 and 20, with the eigenvalue closest to a_{44} being 0. Hence, the first iteration stage is carried without a shift, so that the first stage is given by

$$A_1 = Q_1 R_1, \quad A_2 = R_1 Q_1 \quad (b)$$

where Q_1 is obtained by means of three successive Givens rotations in the planes $(1,2)$, $(2,3)$ and $(3,4)$. The process was demonstrated in Example 6.10, so that we dispense with the intermediate steps and list the results of the decomposition of A_1 directly

$$Q_1 = \begin{bmatrix} 0.970143 & -0.008803 & -0.005858 & 0.242305 \\ -0.242536 & -0.035212 & -0.023433 & 0.969220 \\ 0 & 0.999341 & -0.000877 & 0.036285 \\ 0 & 0 & 0.999708 & 0.024170 \end{bmatrix} \quad (c)$$

$$R_1 = \begin{bmatrix} -0.412311 & -53.139555 & -5.357722 & 4.365641 \\ 0 & -199.131207 & -20.113844 & 20.022034 \\ 0 & 0 & -0.101037 & 0.106361 \\ 0 & 0 & 0 & -0.241093 \end{bmatrix} \quad (d)$$

Inserting Eqs. (c) and (d) into the second of Eqs. (b), we obtain

$$A_2 = \begin{bmatrix} 12.488235 & -3.479423 & 5.616691 & -51.692714 \\ 48.296412 & -13.088826 & 24.700036 & -193.247856 \\ 0 & -0.100970 & -0.106418 & -0.001095 \\ 0 & 0 & -0.241023 & -0.005827 \end{bmatrix} \quad (e)$$

The eigenvalues of the 2×2 matrix in the lower right corner of A_2 are 0.108723 and -0.008132 , and they are both real. The eigenvalue closest to the lower right corner element is the negative one, so that we choose as our shift $\mu_2 = -0.001832$. The results of the next iteration stage are

$$Q_2 = \begin{bmatrix} -0.250494 & 0.651485 & 0.669048 & 0.255332 \\ -0.968118 & -0.168567 & -0.173112 & -0.066065 \\ 0 & 0.739697 & -0.628711 & -0.239938 \\ 0 & 0 & 0.356552 & -0.934276 \end{bmatrix} \quad (f)$$

$$R_2 = \begin{bmatrix} -49.886898 & 13.535232 & -25.319500 & 200.035472 \\ 0 & -0.136502 & -0.419696 & -1.102569 \\ 0 & 0 & -0.675983 & -1.129950 \\ 0 & 0 & 0 & -0.433695 \end{bmatrix} \quad (g)$$

and

$$A_3 = R_2 Q_2 = \begin{bmatrix} -0.615469 & -53.510937 & 51.521792 & -194.445077 \\ 0.132150 & -0.295570 & -0.105625 & 1.139822 \\ 0 & -0.500023 & 0.013981 & 1.217879 \\ 0 & 0 & -0.154635 & 0.397058 \end{bmatrix} \quad (i)$$

The eigenvalues of the 2×2 matrix in the lower right corner of A_3 are $0.205520 \pm 0.389409i$, so that they are complex conjugates.

Next, we begin with the algorithm using complex shifts described in this section. Letting $\mu_3 = 0.205520 + 0.389409i$ and $\bar{\mu}_3 = 0.205520 - 0.389409i$ and using Eq. (i), we form

$$(A_3 - \bar{\mu}_3 I)(A_3 - \mu_3 I) = \begin{bmatrix} -6.245835 & 44.983550 & -16.447185 & 124.147732 \\ -0.174713 & -6.615951 & 6.705533 & -26.177482 \\ -0.066078 & 0.346330 & 0.052815 & -0.569937 \\ 0 & 0.077321 & 0 & 0 \end{bmatrix} \quad (j)$$

Then, using the QR decomposition defined by Eq. (6.227), we obtain

$$Q = \begin{bmatrix} 0.999553 & -0.028128 & 0.008647 & -0.005260 \\ 0.027960 & 0.999425 & 0.008489 & -0.017175 \\ 0.010575 & 0.016162 & -0.839774 & 0.542592 \\ 0 & -0.009822 & -0.542800 & -0.839804 \end{bmatrix} \quad (k)$$

and

$$R = \begin{bmatrix} -6.248627 & 44.782125 & -16.251788 & 123.354294 \\ 0 & -7.872595 & 7.165155 & -29.663640 \\ 0 & 0 & -0.129648 & 1.329901 \\ 0 & 0 & 0 & -0.512653 \end{bmatrix} \quad (l)$$

and we note that R , although not needed for future computations, was listed for completeness. Inserting Eqs. (i) and (k) into Eq. (6.228), we obtain

$$A_5 = Q^T A_3 Q = \begin{bmatrix} -1.562554 & -50.711921 & 61.768974 & 192.048645 \\ 0.166495 & 1.106605 & -2.279971 & -6.427138 \\ 0 & -0.008234 & 1.145103 & 2.725010 \\ 0 & 0 & -0.611457 & -1.189155 \end{bmatrix} \quad (m)$$

and we observe that convergence has begun in earnest, as the (3,2) element in A_5 is two orders of magnitude smaller than the (3,2) element in A_3 .

At this point, the procedure is clear, so that we merely list the results of the iterations leading to convergence, as follows:

$$\begin{aligned} \mu_5 &= -0.022026 \pm 0.551395 \\ \bar{\mu}_5 & \end{aligned} \quad (n)$$

$$A_7 = \begin{bmatrix} -2.148580 & -50.602406 & -94.698131 & 178.043230 \\ 0.204670 & 1.703406 & 4.146521 & -8.221022 \\ 0 & -10.659614 \times 10^{-8} & -1.110470 & 2.796046 \\ 0 & 0 & -0.526066 & 1.055644 \end{bmatrix}$$

$$\begin{aligned} \mu_7 &= -0.027413 \pm 0.545795i \\ \bar{\mu}_7 & \end{aligned} \quad (o)$$

$$A_9 = \begin{bmatrix} -2.871503 & -50.536982 & -7.648625 & 201.362541 \\ 0.270420 & 2.426711 & 0.259732 & -12.100907 \\ 0 & -3.1 \times 10^{-13} & 0.191138 & 3.214531 \\ 0 & 0 & -0.107528 & -0.245946 \end{bmatrix}$$

$$\begin{aligned} \mu_9 &= -0.027404 \pm 0.545795i \\ \bar{\mu}_9 & \end{aligned} \quad (p)$$

$$A_{11} = \begin{bmatrix} -3.874946 & -50.410919 & -7.641901 & 201.080236 \\ 0.396483 & 3.429754 & 0.412657 & -16.125867 \\ 0 & 0 & 0.191138 & 3.214531 \\ 0 & 0 & -0.107528 & -0.245946 \end{bmatrix}$$

$$\begin{aligned} \mu_{11} &= -0.027404 \pm 0.545795i \\ \bar{\mu}_{11} & \end{aligned} \quad (q)$$

$$A_{13} = \begin{bmatrix} -5.450540 & -50.129583 & -7.625002 & 200.468774 \\ 0.677820 & 5.005348 & 0.654432 & -22.485010 \\ 0 & 0 & 0.191138 & 3.214531 \\ 0 & 0 & -0.107528 & -0.245946 \end{bmatrix}$$

At this point, the shifts and the lower right corner 2×2 matrix have reached constant values, so that convergence to the first pair of complex conjugate eigenvalues has been achieved. Of course, the eigenvalues are equal to the shifts, or

$$\begin{aligned} \lambda_1 \\ \bar{\lambda}_1 \end{aligned} = -0.027404 \pm 0.545795i \quad (r)$$

We note that these are the same values as those obtained in Example 6.15 by the power method, except that there they were labeled λ_3 and $\bar{\lambda}_3$, because they represent the subdominant pair.

In general, the iteration process continues with the $(n-2) \times (n-2)$ upper left corner matrix obtained by deleting the last two rows and columns from A_{13} . In this particular case, the upper left corner matrix is 2×2 , so that no further iterations

are needed. Hence, the eigenvalues of this 2×2 matrix are simply the other pair of eigenvalues, or

$$\frac{\lambda_3}{\bar{\lambda}_3} = -0.222596 \pm 2.578257i \quad (s)$$

which are precisely the values for the other pair of eigenvalues obtained in Example 6.15, where they were labeled λ_1 and $\bar{\lambda}_1$.

6.16 INVERSE ITERATION FOR COMPLEX EIGENSOLUTIONS

In Sec. 6.15, we have shown how to compute the eigenvalues of a real nonsymmetric matrix A by the QR method, with special emphasis being placed on the case in which some, or all the eigenvalues are complex. The problem of computing the eigenvectors of A remains. As demonstrated in Sec. 6.10, inverse iteration is able to produce the eigenvector \mathbf{x}_r belonging to a known eigenvalue λ_r with extreme efficiency. Indeed, convergence to \mathbf{x}_r is remarkably fast, quite often in two or three iteration steps. But, the algorithm described in Sec. 6.10 is predicated upon the eigensolutions being real, so that the question arises as to what happens when the eigensolutions are complex. From our experience, iterations with complex quantities are to be avoided, and inverse iteration is no exception. In this section, we develop an inverse iteration algorithm capable of producing eigenvectors belonging to given complex eigenvalues working with real quantities alone.

We consider a real nonsymmetric $2n \times 2n$ matrix A and introduce the notation

$$\lambda = \alpha + i\beta, \quad \mathbf{x} = \mathbf{u} + i\mathbf{v} \quad (6.230)$$

Inserting Eqs. (6.230) into Eq. (6.183), we can write

$$A(\mathbf{u} + i\mathbf{v}) = (\alpha + i\beta)(\mathbf{u} + i\mathbf{v}) \quad (6.231)$$

so that, separating the real and imaginary parts, we obtain

$$\begin{aligned} A\mathbf{u} &= \alpha\mathbf{u} - \beta\mathbf{v} \\ A\mathbf{v} &= \alpha\mathbf{v} + \beta\mathbf{u} \end{aligned} \quad (6.232)$$

Equations (6.232) can be written in the compact form

$$B\mathbf{w} = \mathbf{0} \quad (6.233)$$

where $\mathbf{w} = [\mathbf{u}^T \ \mathbf{v}^T]^T$ is a real expanded $4n$ -vector and

$$B = \begin{bmatrix} A - \alpha I & \beta I \\ -\beta I & A - \alpha I \end{bmatrix} \quad (6.234)$$

is a real expanded $4n \times 4n$ coefficient matrix.

Equation (6.233) forms the basis for the inverse iteration algorithm for complex eigenvalues using real quantities. By analogy with the ordinary inverse iteration described in Sec. 6.10, we express the iteration process in the form

$$B\mathbf{w}^{(p)*} = \mathbf{w}^{(p-1)}, \quad \mathbf{w}^{(p)} = c^{(p)}\mathbf{w}^{(p)*}, \quad p = 1, 2, \dots \quad (6.235a, b)$$

where $c^{(p)}$ is a scaling factor. For convenience, we choose

$$c^{(p)} = \frac{1}{\max_i |w_i^{(p)*}|}, \quad p = 1, 2, \dots \quad (6.236)$$

in which $\max_i |w_i^{(p)*}|$ denotes the component of $\mathbf{w}^{(p)*}$ of largest magnitude. The effect of the adopted scaling is to render the component of $\mathbf{w}^{(p)}$ of largest magnitude equal to 1 or to -1 , and is designed to prevent the iterates from becoming too large. In this regard, it should be pointed out that the matrix B is close to being singular, so that $\mathbf{w}^{(p)*}$ is likely to be several orders of magnitude larger than $\mathbf{w}^{(p-1)}$, which is also the reason for the fast convergence. The process begins with an arbitrary initial choice $\mathbf{w}^{(0)}$ and solving Eq. (6.235a) for $\mathbf{w}^{(1)*}$ by means of Gaussian elimination with back-substitution. Then, $\mathbf{w}^{(1)*}$ is normalized to $\mathbf{w}^{(1)}$ according to Eqs. (6.235b) and (6.236) and the process is repeated. If λ is close to an eigenvalue, say $\lambda = \lambda_r = \alpha_r + i\beta_r$, then

$$\lim_{p \rightarrow \infty} (\mathbf{u}^{(p)} + i\mathbf{v}^{(p)}) = \mathbf{x}_r \quad (6.237)$$

where $\mathbf{u}^{(p)}$ and $\mathbf{v}^{(p)}$ are the upper half and lower half of the iterate $\mathbf{w}^{(p)}$, respectively. In practice, only a few iteration steps are necessary.

Equation (6.237) only indicates in a qualitative manner that the process converges, but provides no clues as to when convergence occurs. Hence, we must develop a quantitative convergence criterion. To this end, we recall from Sec. 6.13 that two complex vectors represent the same eigenvector when the magnitude ratios of homologous components of the two vectors are the same and the phase angle difference between any pair of components of one vector is the same as the phase angle difference between homologous components of the other vector. To quantify this statement, we introduce the notation

$$\mathbf{x}^{(p)} = \mathbf{u}^{(p)} + i\mathbf{v}^{(p)} = \begin{bmatrix} |x_1^{(p)}| / \angle \psi_1^{(p)} \\ |x_2^{(p)}| / \angle \psi_2^{(p)} \\ \dots \dots \dots \\ |x_{2n}^{(p)}| / \angle \psi_{2n}^{(p)} \end{bmatrix}, \quad p = 1, 2, \dots \quad (6.238)$$

where $|x_i^{(p)}|$ denotes the magnitude of the i th component of $\mathbf{x}^{(p)}$ and $\angle \psi_i^{(p)}$ denotes the corresponding phase angle ($i = 1, 2, \dots, 2n$). Then, the convergence criterion can be stated in the form

$$\lim_{p \rightarrow \infty} \frac{|x_i^{(p)}|}{|x_{2n}^{(p)}|} = r_i = \text{constant}, \quad i = 1, 2, \dots, 2n - 1 \quad (6.239)$$

$$\lim_{p \rightarrow \infty} (\psi_i^{(p)} - \psi_{2n}^{(p)}) = \Delta \psi_i^{(p)} = \text{constant},$$

The procedure described above yields the right eigenvector \mathbf{x}_r belonging to the known eigenvalue λ_r . The same procedure, but with A replaced by A^T , can be used to obtain the left eigenvector \mathbf{y}_r .

Example 6.18

Consider the damped two-degree-of-freedom system of Example 6.15, $n = 2$, and compute the eigenvector \mathbf{x}_1 belonging to the dominant eigenvalue λ_1 by means of inverse iteration.

From Example 6.15, the coefficient matrix is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & 4 & -0.4 & 0.2 \\ 2 & -2 & 0.1 & -0.1 \end{bmatrix} \quad (a)$$

Also from Example 6.15, the dominant eigenvalue is

$$\lambda_1 = \alpha_1 + i\beta_1 = -0.222596 + 2.578257i \quad (b)$$

Hence, inserting Eqs. (a) and (b) into Eq. (6.234), we can write the expanded matrix

$$B = \begin{bmatrix} 0.222596 & 0 & 1 & 0 & 2.578257 & 0 & 0 & 0 \\ 0 & 0.222596 & 0 & 1 & 0 & 2.578257 & 0 & 0 \\ -5 & 4 & -0.177404 & 0.2 & 0 & 0 & 2.578257 & 0 \\ 2 & -2 & 0.1 & 0.122596 & 0 & 0 & 0 & 2.578257 \\ -2.578257 & 0 & 0 & 0 & 0.222596 & 0 & 1 & 0 \\ 0 & -2.578257 & 0 & 0 & 0 & 0.222596 & 0 & 1 \\ 0 & 0 & -2.578257 & 0 & -5 & 4 & -0.177404 & 0.2 \\ 0 & 0 & 0 & -2.578257 & 2 & -2 & 0.1 & 0.122596 \end{bmatrix} \quad (c)$$

We begin the iteration with the vector

$$\mathbf{w}^{(0)} = [1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1]^T \quad (d)$$

Introducing Eqs. (c) and (d) into Eq. (6.235a) with $p = 1$, using Gaussian elimination with back-substitution and normalizing according to Eqs. (6.235b) and (6.236), we obtain the normalized iterate

$$\mathbf{w}^{(1)} = [-0.230142 \ 0.086898 \ -1 \ 0.440708 \ 0.407729 \ -0.178433 \ -0.684124 \ 0.263765]^T \quad (e)$$

so that, using Eq. (6.238) with $p = 1$, the corresponding complex vector is

$$\mathbf{x}^{(1)} = \begin{bmatrix} -0.230142 + 0.407729i \\ 0.086898 - 0.178433i \\ -1.000000 - 0.684124i \\ 0.440708 + 0.263765i \end{bmatrix} = \begin{bmatrix} 0.468197 / 119.4425^\circ \\ 0.198468 / 295.9664^\circ \\ 1.211621 / 214.3770^\circ \\ 0.513606 / 30.9009^\circ \end{bmatrix} \quad (f)$$

which permits us to compute the magnitude ratios and phase angle differences

$$\begin{aligned} |x_1^{(1)}|/|x_4^{(1)}| &= 0.911588, & \Delta\psi_1^{(1)} &= \psi_1^{(1)} - \psi_4^{(1)} = 88.5416^\circ \\ |x_2^{(1)}|/|x_4^{(1)}| &= 0.386421, & \Delta\psi_2^{(1)} &= \psi_2^{(1)} - \psi_4^{(1)} = 265.0655^\circ \\ |x_3^{(1)}|/|x_4^{(1)}| &= 2.359048, & \Delta\psi_3^{(1)} &= \psi_3^{(1)} - \psi_4^{(1)} = 183.4761^\circ \end{aligned} \quad (g)$$

Following the same pattern, we obtain for $p = 2$

$$\mathbf{x}^{(2)} = \begin{bmatrix} -0.413050 - 0.291785i \\ 0.182270 + 0.112843i \\ 0.844239 - 1.000000i \\ -0.331512 + 0.444819i \end{bmatrix} = \begin{bmatrix} 0.505716 / 215.2380^\circ \\ 0.214373 / 31.7616^\circ \\ 1.308717 / 310.1724^\circ \\ 0.554765 / 126.6962^\circ \end{bmatrix} \quad (\text{h})$$

and

$$\begin{aligned} |x_1^{(2)}|/|x_4^{(2)}| &= 0.911586, & \Delta\psi_1^{(2)} &= \psi_1^{(2)} - \psi_4^{(2)} = 88.5418^\circ \\ |x_2^{(2)}|/|x_4^{(2)}| &= 0.386421, & \Delta\psi_2^{(2)} &= \psi_2^{(2)} - \psi_4^{(2)} = 265.0654^\circ \\ |x_3^{(2)}|/|x_4^{(2)}| &= 2.359048, & \Delta\psi_3^{(2)} &= \psi_3^{(2)} - \psi_4^{(2)} = 183.4762^\circ \end{aligned} \quad (\text{i})$$

Comparing Eqs. (g) and (i), we conclude that convergence has been virtually achieved, so that we accept $\mathbf{x}_1 = \mathbf{x}^{(2)}$ as the eigenvector belonging to λ_1 .

6.17 SYNOPSIS

In this chapter, we presented a large variety of iterative algorithms for the real symmetric eigenvalue problem. The power method has the advantage of simplicity, but convergence can be slow if the eigenvalues are not well spaced. It should be used only when a small number of dominant eigensolutions are desired. The Jacobi method has a certain air of elegance and is easy to understand. It is not particularly fast, however, and should be used only for moderate size problems. The two most attractive algorithms require that the matrix be in tridiagonal form. As tridiagonalization procedures, we single out Givens' method, Householder's method and Lanczos' method. The two algorithms for solving eigenvalue problems for matrices in symmetric tridiagonal form are Givens' method and the QR method, and both can produce only eigenvalues. Givens' method is based on the separation theorem and has the ability to target individual eigenvalues in a given range, but convergence is only linear. By contrast, the QR method has better than cubic convergence. However, before this remarkable convergence can be achieved, it is necessary to carry out shifts using the proper strategy. As far as the computation of eigenvectors belonging to known eigenvalues is concerned, inverse iteration has no peers. Another method with superior convergence characteristics is Rayleigh's quotient iteration. It also targets individual eigensolutions. Before it can be used, however, one must have a good guess of the eigenvector targeted, which for all practical purposes confines the usefulness of the method to the lowest vibration mode. Finally, simultaneous iteration, which can be regarded as an extension of the power method, iterates to several eigensolutions at a time. For a well-populated real symmetric matrix A , the most indicated approach to the full solution of the eigenvalue problem is to tridiagonalize A by means of Householder's method, use the QR method with shifts to compute the eigenvalues and inverse iteration to compute the eigenvectors.

In the case of nonsymmetric matrices, the choice of algorithms for solving the eigenvalue problem is significantly more limited than is the case of symmetric matrices. Here too, the power method has the advantage of simplicity, but should

be considered only if a small number of dominant eigensolutions is required. The method of choice is once again the QR method, provided the matrix A is first reduced to Hessenberg form and the shifting strategy of Sec. 6.15, which obviates the problem of working with complex numbers, is used. Then, the eigenvectors are to be obtained by a version of inverse iteration capable of producing complex eigenvectors working with real iterates.

PROBLEMS

6.1 Solve the set of algebraic equations

$$\begin{aligned} 6x_1 + 5x_2 - x_3 + 3x_4 &= 2 \\ -3x_1 + x_2 + 3x_3 - 2x_4 &= -5.5 \\ 2x_1 - 2x_2 + x_3 - 6x_4 &= -7.5 \\ 4x_1 + x_2 - 2x_3 + 5x_4 &= 11.5 \end{aligned}$$

by Gaussian elimination with back-substitution.

6.2 Solve the set of algebraic equations

$$\begin{aligned} 1.2x_1 + 4.7x_2 + x_3 - 6x_4 &= 4.1 \\ 5.2x_1 + x_2 - 3x_3 + 2x_4 &= 16.1 \\ x_1 - x_2 + 4x_3 - 2x_4 &= -10.35 \\ 1.3x_1 + 2.2x_2 - x_3 - 5.5x_4 &= 3.625 \end{aligned}$$

by Gaussian elimination with back-substitution.

6.3 Solve Problem 6.1 by the Gauss-Jordan method.

6.4 Solve Problem 6.2 by the Gauss-Jordan method.

6.5 Verify that the real symmetric matrix

$$A = \begin{bmatrix} 1.44 & -2.76 & 0 & 0 \\ & 25.54 & 12.15 & 0 \\ \text{Symm} & & 20.25 & -2.88 \\ & & & 2.89 \end{bmatrix}$$

is positive definite. Then, carry out the Cholesky decomposition of A .

6.6 Verify that the real symmetric matrix

$$A = \begin{bmatrix} 12.25 & 4.2 & -7.525 & 2.87 \\ & 23.53 & 4.752 & -10.296 \\ \text{Symm} & & 35.1461 & -14.252 \\ & & & 23.5949 \end{bmatrix}$$

is positive definite. Then, carry out the Cholesky decomposition of A .

6.7 The mass and stiffness matrices of a three-degree-of-freedom system are

$$M = m \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad K = k \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Compute the natural frequencies and modes of vibration by means of the power method. Use the formulation given by Eqs. (6.72) and (6.73)

6.8 The mass and stiffness matrices of a four-degree-of-freedom system are

$$M = m \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad K = k \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

Compute the natural frequencies and modes of vibration by means of the power method. Use the formulation given by Eqs. (6.72) and (6.73).

6.9 The mass and stiffness matrices of a four-degree-of-freedom system are

$$M = m \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \quad K = k \begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 3 & -2 \\ 0 & 0 & -2 & 5 \end{bmatrix}$$

Compute the natural frequencies and modes of vibration by means of the power method. Use the formulation given by Eqs. (6.72) and (6.73).

- 6.10 Solve Problem 6.7 by means of the threshold serial Jacobi method using a threshold value of 10^{-6} .
- 6.11 Solve Problem 6.8 by means of the threshold serial Jacobi method using a threshold value of 10^{-6} .
- 6.12 Solve Problem 6.9 by means of the threshold serial Jacobi method using a threshold value of 10^{-6} .
- 6.13 Use Givens' method to tridiagonalize the matrix $A = M^{1/2}K^{-1}M^{1/2}$, where M and K are as in Problem 6.8.
- 6.14 Use Givens' method to tridiagonalize the matrix $A = M^{1/2}K^{-1}M^{1/2}$, where M and K are as in Problem 6.9.
- 6.15 Tridiagonalize the matrix A of Problem 6.13 by means of Householder's method.
- 6.16 Tridiagonalize the matrix A of Problem 6.14 by means of Householder's method.
- 6.17 Tridiagonalize the matrix A of Problem 6.13 by means of Lanczos' method.
- 6.18 Tridiagonalize the matrix A of Problem 6.14 by means of Lanczos' method.
- 6.19 Use Givens' method (Sec. 6.8) to compute the eigenvalues of the tridiagonal matrix $M^{-1/2}KM^{-1/2}$, where M and K are as in Problem 6.7.
- 6.20 Solve Problem 6.19 with M and K as in Problem 6.8.
- 6.21 Solve Problem 6.19 with M and K as in Problem 6.9.
- 6.22 Compute the eigenvalues of the tridiagonal matrix obtained in Problem 6.13 (or Problem 6.15) by means of Givens' method (Sec. 6.8). Compare results with those obtained in Problem 6.20 and draw conclusions.
- 6.23 Compute the eigenvalues of the tridiagonal matrix obtained in Problem 6.14 (or Problem 6.16) by means of Givens' method (Sec. 6.8). Compare results with those obtained in Problem 6.9 and draw conclusions.
- 6.24 Solve Problem 6.19 by the QR method.
- 6.25 Solve Problem 6.20 by the QR method.
- 6.26 Solve Problem 6.21 by the QR method.
- 6.27 Solve Problem 6.22 by the QR method.
- 6.28 Solve Problem 6.23 by the QR method.

- 6.29** Use inverse iteration to compute the eigenvectors belonging to the eigenvalues obtained in Problem 6.24. Then, determine the actual modal vectors.
- 6.30** Use inverse iteration to compute the eigenvectors belonging to the eigenvalues obtained in Problem 6.27. Then, determine the actual modal vectors.
- 6.31** Use inverse iteration to compute the eigenvectors belonging to the eigenvalues obtained in Problem 6.28. Then, determine the actual modal vectors.
- 6.32** Compute the two lowest modes of vibration for the system of Problem 6.7 by means of Rayleigh's quotient iteration.
- 6.33** Compute the two lowest modes of vibration for the system of Problem 6.8 by means of Rayleigh's quotient iteration.
- 6.34** Compute the two lowest modes of vibration for the system of Problem 6.9 by means of Rayleigh's quotient iteration.
- 6.35** Solve Problem 6.32 by means of simultaneous iteration.
- 6.36** Solve Problem 6.33 by means of simultaneous iteration.
- 6.37** Solve Problem 6.34 by means of simultaneous iteration.
- 6.38** Solve Problem 6.32 by means of subspace iteration.
- 6.39** Solve Problem 6.33 by means of subspace iteration.
- 6.40** Solve Problem 6.34 by means of subspace iteration.
- 6.41** A damped three-degree-of-freedom system is defined by the mass, damping and stiffness matrices

$$M = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.4 & -0.2 & 0 \\ -0.2 & 0.3 & -0.1 \\ 0 & -0.1 & 0.1 \end{bmatrix}, \quad K = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Solve the eigenvalue problem by the power method (Sec. 6.13).

- 6.42** A damped three-degree-of-freedom system is defined by the mass, damping and stiffness matrices

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0.4 & -0.2 & 0 \\ -0.2 & 0.3 & -0.1 \\ 0 & -0.1 & 2.1 \end{bmatrix}, \quad K = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Solve the eigenvalue problem by the power method (Sec. 6.13).

- 6.43** Reduce the matrix A from Problem 6.41 to upper Hessenberg form.
- 6.44** Reduce the matrix A from Problem 6.42 to upper Hessenberg form.
- 6.45** Compute the eigenvalues of the upper Hessenberg matrix from Problem 6.43 by the QR method.
- 6.46** Compute the eigenvalues of the upper Hessenberg matrix from Problem 6.44 by the QR method.
- 6.47** Use inverse iteration (Sec. 6.16) to compute the right and left eigenvectors belonging to the eigenvalues obtained in Problem 6.45. Work with the matrix A from Problem 6.41.
- 6.48** Use inverse iteration (Sec. 6.16) to compute the right and left eigenvectors belonging to the eigenvalues obtained in Problem 6.46. Work with the matrix A from Problem 6.42.

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DISTRIBUTED-PARAMETER SYSTEMS

Mathematical models of vibrating systems are commonly divided into two broad classes, discrete, or lumped-parameter models, and continuous, or distributed-parameter models. In real life, systems can contain both lumped and distributed parameters. Until now, our study has been confined to discrete systems. In this chapter, our attention turns to systems with parameters distributed throughout the domain and in some cases with lumped parameters at boundaries. The emphasis is on theoretical developments and exact solutions. Because exact solutions are possible in only a limited number of cases, quite often the interest lies in approximate solutions. Such solutions are generally obtained through spatial discretization, which amounts to approximating distributed-parameter systems by discrete ones. Discretization methods are presented in Chapters 8 and 9. In this regard, it should be pointed out that systems containing both distributed parameters and lumped parameters dispersed throughout the system can be treated by some of the techniques developed in Chapters 8 and 9. The theory presented in this chapter not only provides a great deal of insight into the behavior of vibrating systems but the theory is essential to the approximate techniques developed in Chapters 8 and 9.

Discrete systems consist of aggregates of discrete components, such as masses and springs, with the masses assumed to be rigid and the springs assumed to be flexible but massless. The masses and the spring stiffnesses represent the system parameters, with the masses being concentrated at given points and connected by the springs, which explains why the parameters are referred to as lumped. The spatial position of each mass is identified by an index, and in general the number of masses coincides with the number of degrees of freedom of the system. In contrast, at each point of a continuous system there is both mass and stiffness, and these parameters are

distributed over the entire system. The position of a point in a distributed-parameter system is identified by one, two, or three spatial coordinates, with the set of interior points defining a domain D and the set of points on the exterior of D defining the boundary S . Because there is an infinity of points in D , a distributed system can be regarded as having an infinite number of degrees of freedom.

As can be expected, the mathematical formalism for distributed systems differs significantly from the formalism for discrete systems. For n -degree-of-freedom discrete systems, the motion is governed by n simultaneous ordinary differential equations. To solve these equations, it is necessary to solve the associated algebraic eigenvalue problem. The solution consists of n eigenvalues and eigenvectors, where the eigenvectors possess the orthogonality property. The orthogonal eigenvectors form a basis for an n -dimensional vector space, which can be used in conjunction with an expansion theorem to decouple the equations of motion into n independent second-order equations. The independent equations resemble the equation of motion for a single-degree-of-freedom system and can be solved with relative ease. In contrast, the motion of distributed-parameter systems is governed by boundary-value problems consisting of one, or several partial differential equations to be satisfied over D and an appropriate number of boundary conditions to be satisfied at every point of S . The solution of a boundary-value problem requires the solution of an associated differential eigenvalue problem, where the latter solution consists of an infinite set of eigenvalues and eigenfunctions. The eigenfunctions are orthogonal and can be used as a basis for an infinite-dimensional function space in conjunction with an expansion theorem to transform the boundary-value problem into an infinite set of independent second-order ordinary differential equations resembling entirely the independent equations for discrete systems, so that they can be solved with the same ease. Hence, whereas the mathematical formalism and the methods of solution for distributed systems are different from those for discrete systems, many concepts are entirely analogous.

In this chapter, we begin with the derivation of boundary-value problems for a variety of elastic members, such as strings in transverse vibration, rods in axial vibration, shafts in torsion and beams in bending. Subsequently, a generic boundary-value problem consisting of a Lagrange partial differential equation and suitable boundary conditions is derived. The free vibration problem leads naturally to the differential eigenvalue problem, which can be cast conveniently in differential operator form. Here, the concept of operator self-adjointness, which represents the counterpart of matrix symmetry in discrete systems, proves quite powerful in generalizing the developments to a large class of systems, covering most systems of interest to our study. In fact, the various elastic members mentioned in the beginning of this paragraph represent mere special cases of this general theory. In extending our discussion to two-dimensional systems, and in particular to membranes and plates, we encounter new concepts such as the shape of the boundary and degeneracy. Subjects such as the variational formulation and the integral formulation of the eigenvalue problem have certain implications in approximate methods of solution. Other topics of interest are distributed gyroscopic systems and distributed damped systems. The chapter concludes with an extensive discussion of systems with nonhomogeneous boundary conditions.

7.1 THE BOUNDARY-VALUE PROBLEM FOR STRINGS, RODS AND SHAFTS

Our interest lies in the vibration of systems with distributed parameters. The motion of such systems depends not only on time but also on the spatial position, which is defined by one, two, or three coordinates. Consistent with this, the domain of extension of the distributed-parameter system is one-, two-, or three-dimensional, respectively. Distributed-parameter systems are governed by boundary-value problems, which consist of differential equations of motion to be satisfied over all interior points of the domain and boundary conditions to be satisfied at points bounding the domain. Because there are at least two independent variables, the equations of motion are partial differential equations. Unlike discrete systems, for which the equations of motion tend to have the same form, the boundary-value problem tends to differ from one type of distributed system to another. Using operator notation, the various boundary-value problems for large classes of systems can be reduced to the same form, which enables us to draw general conclusions concerning all systems in a given class. In this section, we derive the boundary-value problem for a particular distributed system, and later in this chapter we generalize the formulation to large classes of systems.

We consider the string in transverse vibration shown in Fig. 7.1a. This is a one-dimensional distributed-parameter system with the domain of extension $D : 0 < x < L$ and the boundary $S : x = 0, L$, where x is the spatial variable and L the length of the string. We denote the displacement in the transverse direction

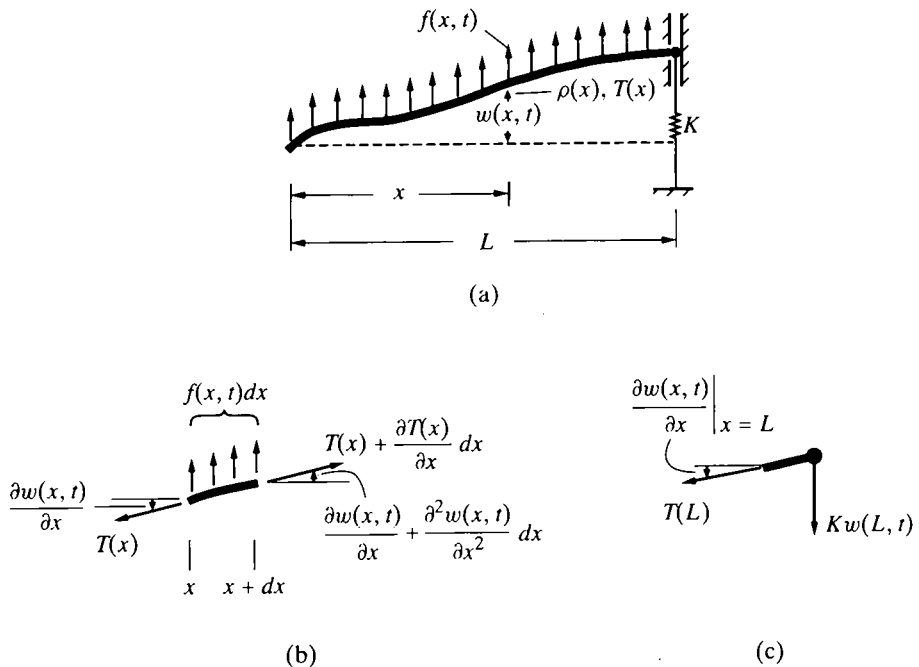


Figure 7.1 (a) String in transverse vibration (b) Free-body diagram for a differential element of string (c) Force diagram at $x = L$

of a typical point x of the string by the dependent variable $w(x, t)$, the force per unit length by $f(x, t)$, the mass per unit length of string by $\rho(x)$ and the tension by $T(x)$. The left end of the string is fixed and the right end is attached to a spring of stiffness K . For simplicity, we assume that the displacement is measured from the equilibrium position $w_{eq}(x)$, in which case a pretension in the string balances the force due to gravity. In all future discussions, we omit gravitational forces on the basis of this assumption. We propose to derive the boundary-value problem in two ways, first by means of Newton's second law and then by means of the extended Hamilton's principle.

To derive the boundary-value problem by means of Newton's second law, we refer to the free-body diagram of Fig. 7.1b and assume that the displacement $w(x, t)$ is sufficiently small that the sine and the tangent of the angle made by the string with respect to the x -axis is approximately equal to the slope of the displacement curve. Hence, summing up forces in the transverse direction, we obtain

$$\begin{aligned} \left[T(x) + \frac{\partial T(x)}{\partial x} dx \right] \left[\frac{\partial w(x, t)}{\partial x} + \frac{\partial^2 w(x, t)}{\partial x^2} dx \right] + f(x, t) dx - T(x) \frac{\partial w(x, t)}{\partial x} \\ = \rho(x) dx \frac{\partial^2 w(x, t)}{\partial t^2}, \quad 0 < x < L \end{aligned} \quad (7.1)$$

which, upon ignoring second-order terms in dx and dividing through by dx , can be reduced to

$$\frac{\partial}{\partial x} \left[T(x) \frac{\partial w(x, t)}{\partial x} \right] + f(x, t) = \rho(x) \frac{\partial^2 w(x, t)}{\partial t^2}, \quad 0 < x < L \quad (7.2)$$

Because the string is fixed at $x = 0$, the displacement must satisfy

$$w(x, t) = 0, \quad x = 0 \quad (7.3a)$$

On the other hand, from Fig. 7.1c, we conclude that the transverse force balance at the right end of the string requires that

$$T(x) \frac{\partial w(x, t)}{\partial x} + K w(x, t) = 0, \quad x = L \quad (7.3b)$$

Equations (7.2) and (7.3) represent the *boundary-value problem*, in which Eq. (7.2) is the *partial differential equation of motion* and Eqs. (7.3) are the *boundary conditions*.

Next, we wish to derive the boundary-value problem by means of the extended Hamilton's principle, which can be expressed in the form

$$\int_{t_1}^{t_2} (\delta T - \delta V + \overline{\delta W}_{nc}) dt = 0, \quad \delta w(x, t) = 0, \quad t = t_1, t_2 \quad (7.4)$$

where

$$T(t) = \frac{1}{2} \int_0^L \rho(x) \left[\frac{\partial w(x, t)}{\partial t} \right]^2 dx \quad (7.5)$$

is the kinetic energy. The potential energy arises from the restoring forces due to the tension in the string and the elongation of the spring at $x = L$. To determine the potential energy due to the tension in the string, we denote the length of the

differential element dx in displaced position by ds . Then, the potential energy is simply the sum of the work that must be performed by the tensile force to restore the string to the original horizontal position and the potential energy due to the end spring, or

$$V(t) = \int_0^L T(x)(ds - dx) + \frac{1}{2}Kw^2(L, t) \quad (7.6)$$

But, recognizing from Fig. 7.2 that $\partial w/\partial x \ll 1$, we can write

$$ds = \left[(dx)^2 + \left(\frac{\partial w}{\partial x} dx \right)^2 \right]^{1/2} = \left[1 + \left(\frac{\partial w}{\partial x} \right)^2 \right]^{1/2} dx \cong \left[1 + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] dx \quad (7.7)$$

where we retained two terms only in the binomial expansion. Hence, inserting Eq. (7.7) into Eq. (7.6), we obtain the potential energy of the system

$$V(t) = \frac{1}{2} \int_0^L T(x) \left[\frac{\partial w(x, t)}{\partial x} \right]^2 dx + \frac{1}{2}Kw^2(L, t) \quad (7.8)$$

Moreover, the virtual work due to the nonconservative distributed force is simply

$$\overline{\delta W}_{nc}(t) = \int_0^L f(x, t) \delta w(x, t) dx \quad (7.9)$$

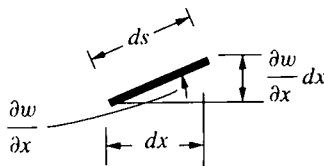


Figure 7.2 Differential element of string in displaced position

From Eq. (7.5), the variation in the kinetic energy is

$$\delta T(t) = \int_0^L \rho(x) \frac{\partial w(x, t)}{\partial t} \delta \left[\frac{\partial w(x, t)}{\partial t} \right] dx \quad (7.10)$$

Similarly, from Eq. (7.8), the variation in the potential energy has the form

$$\delta V(t) = \int_0^L T(x) \frac{\partial w(x, t)}{\partial x} \delta \left[\frac{\partial w(x, t)}{\partial x} \right] dx + Kw(L, t) \delta w(L, t) \quad (7.11)$$

Equation (7.10) contains the variation in the velocity, and we must transform the equation into one in terms of the virtual displacement. To this end, we assume that

variations and differentiations with respect to time are interchangeable and carry out the following integration by parts with respect to t :

$$\begin{aligned}
 \int_{t_1}^{t_2} \delta T dt &= \int_{t_1}^{t_2} \int_0^L \rho(x) \frac{\partial w(x, t)}{\partial t} \delta \left[\frac{\partial w(x, t)}{\partial t} \right] dx dt \\
 &= \int_0^L \left[\int_{t_1}^{t_2} \rho(x) \frac{\partial w(x, t)}{\partial t} \frac{\partial \delta w(x, t)}{\partial t} dt \right] dx \\
 &= \int_0^L \left[\rho(x) \frac{\partial w(x, t)}{\partial t} \delta w(x, t) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \rho(x) \frac{\partial^2 w(x, t)}{\partial t^2} \delta w(x, t) dt \right] dx \\
 &= - \int_{t_1}^{t_2} \int_0^L \rho(x) \frac{\partial^2 w(x, t)}{\partial t^2} \delta w(x, t) dx dt \tag{7.12}
 \end{aligned}$$

in which we used the end conditions on time in Eq. (7.4). Similarly, we perform an integration by parts with respect to x and rewrite Eq. (7.11) in the form

$$\begin{aligned}
 \delta V(t) &= \int_0^L T(x) \frac{\partial w(x, t)}{\partial x} \delta \left[\frac{\partial w(x, t)}{\partial x} \right] dx + Kw(L, t) \delta w(L, t) \\
 &= \int_0^L T(x) \frac{\partial w(x, t)}{\partial x} \frac{\partial \delta w(x, t)}{\partial x} dx + Kw(L, t) \delta w(L, t) \\
 &= T(x) \frac{\partial w(x, t)}{\partial x} \delta w(x, t) \Big|_0^L \\
 &\quad - \int_0^L \frac{\partial}{\partial x} \left[T(x) \frac{\partial w(x, t)}{\partial x} \right] \delta w(x, t) dx + Kw(L, t) \delta w(L, t) \\
 &= - \int_0^L \frac{\partial}{\partial x} \left[T(x) \frac{\partial w(x, t)}{\partial x} \right] \delta w(x, t) dx \\
 &\quad + \left[T(x) \frac{\partial w(x, t)}{\partial x} + Kw(x, t) \right] \delta w(x, t) \Big|_{x=L} \\
 &\quad - T(x) \frac{\partial w(x, t)}{\partial x} \delta w(x, t) \Big|_{x=0} \tag{7.13}
 \end{aligned}$$

Hence, inserting Eqs. (7.9), (7.12) and (7.13) into Eq. (7.4), we obtain

$$\begin{aligned}
 \int_{t_1}^{t_2} \left\{ - \int_0^L \left[\rho(x) \frac{\partial^2 w(x, t)}{\partial t^2} - \frac{\partial}{\partial x} \left[T(x) \frac{\partial w(x, t)}{\partial x} \right] - f(x, t) \right] \delta w(x, t) dx \right. \\
 \left. - \left[T(x) \frac{\partial w(x, t)}{\partial x} + Kw(x, t) \right] \delta w(x, t) \Big|_{x=L} \right.
 \end{aligned}$$

$$+ T(x) \frac{\partial w(x, t)}{\partial x} \delta w(x, t) \Big|_{x=0} \Big) dt = 0 \quad (7.14)$$

But, the virtual displacements are arbitrary, and hence they can be assigned values at will. We assume that either δw or $T\partial w/\partial x$ is zero at $x = 0$, that either δw or $(T\partial w/\partial x) + Kw$ is zero at $x = L$ and that δw is completely arbitrary at every point of the domain $0 < x < L$. Under these circumstances, we conclude that Eq. (7.14) can be satisfied if and only if the displacement $w(x, t)$ satisfies

$$\frac{\partial}{\partial x} \left[T(x) \frac{\partial w(x, t)}{\partial x} \right] + f(x, t) = \rho(x) \frac{\partial^2 w(x, t)}{\partial t^2}, \quad 0 < x < L \quad (7.15)$$

and is such that

$$T(x) \frac{\partial w(x, t)}{\partial x} \delta w(x, t) = 0 \quad \text{at } x = 0 \quad (7.16a)$$

$$\left[T(x) \frac{\partial w(x, t)}{\partial x} + Kw(x, t) \right] \delta w(x, t) = 0 \quad \text{at } x = L \quad (7.16b)$$

Conditions (7.16) can be satisfied in two ways each, where we note that the coefficient of $\delta w(x, t)$ in both Eq. (7.16a) and Eq. (7.16b) represents the transverse component of force. At $x = 0$ the transverse component of force cannot be zero for all t , so that $\delta w(x, t)|_{x=0}$ must be zero. Hence, Eq. (7.16a) is satisfied by

$$w(x, t) = 0, \quad x = 0 \quad (7.17a)$$

On the other hand, $\delta w(x, t) \neq 0$ at $x = L$, so that $\delta w(x, t)|_{x=L}$ is arbitrary. It follows that Eq. (7.16b) can be satisfied if and only if

$$T(x) \frac{\partial w(x, t)}{\partial x} + Kw(x, t) = 0, \quad x = L \quad (7.17b)$$

Equation (7.15) represents the partial differential equation and Eqs. (7.17) are the boundary conditions. Equations (7.15) and (7.17) constitute the boundary-value problem for the transverse vibration of the string shown in Fig. 7.1a, and they coincide with Eqs. (7.2) and (7.3) derived earlier by Newton's second law, respectively.

At this point, we wish to examine the question of boundary conditions more closely. Boundary condition (7.3a), or (7.17a), is geometric in nature and it indicates that the solution $w(x, t)$ of the differential equation, Eq. (7.2), or Eq. (7.15), must be zero at $x = 0$. Boundary conditions resulting from pure geometric compatibility are called *geometric, essential, or imposed boundary conditions*. On the other hand, boundary condition (7.3b), or (7.17b), states that the vertical component of the tension in the string must balance the spring force at $x = L$. Boundary conditions resulting from force balance are known as *natural, dynamic, or additional boundary conditions*. It is the satisfaction of the boundary conditions that renders the solution of the differential equation unique. It should be noted that geometric boundary conditions tend to be simpler than natural ones.

Boundary-value problems are classified according to order, which is determined by the order of the highest derivative with respect to the spatial coordinate in the

differential equation of motion. The boundary-value problem defined by Eqs. (7.15) and (7.17) is of order two, and it represents one of the simplest boundary-value problems in vibrations. Because the order is two, there are two boundary conditions, one at each end. In the case of the string of Fig. 7.1a, one boundary condition is geometric and the other is natural. In general, there are other possibilities. For example, in the case of a string fixed at both ends, both boundary conditions are geometric. In the case of a free end, the slope to the displacement curve must be zero, provided the tension is not zero. Although this appears as a geometric boundary condition, it is really a natural one, as the transverse component of force, which is equal to $T(x)\partial w(x, t)/\partial x$, must be zero.

Other elastic members, such as shafts in torsion and rods in axial vibration, are also defined by boundary-value problems of order two. In fact, they are governed by boundary-value problems mathematically equivalent to that for strings in transverse vibration. The only difference lies in the nature of the displacement, excitation and parameters. Indeed, the structure of the differential equation is exactly the same, but the transverse displacement $w(x, t)$ must be replaced by the angular displacement $\theta(x, t)$ in the case of a shaft and by the axial displacement $u(x, t)$ in the case of a rod, the transverse force density $f(x, t)$ must be replaced by the distributed torque $m(x, t)$ and by the distributed axial force $f(x, t)$, the tension $T(x)$ must be replaced by the torsional stiffness $GJ(x)$, in which G is the shear modulus and $J(x)$ the polar area moment of inertia, and by the axial stiffness $EA(x)$, where E is the modulus of elasticity and $A(x)$ the cross-sectional area, and the mass density $\rho(x)$ must be replaced by the polar mass moment of inertia density $I(x)$ and by the mass per unit length $m(x)$, respectively.

Before turning our attention to other types of systems, we should point out that the reason for introducing second-order boundary-value problems via strings in transverse vibration, instead of shafts in torsional vibration or rods in axial vibration, is simply that transverse displacements are easier to visualize than angular or axial displacements. This point is made abundantly clear by the fact that angular displacements and axial displacement are commonly plotted as if they were transverse displacements, which at times can lead to confusion.

7.2 THE BOUNDARY-VALUE PROBLEM FOR BEAMS IN BENDING

An elastic member used most frequently in structures is the beam. Figure 7.3a shows a beam in bending vibration under the distributed transverse force $f(x, t)$. In addition, we assume that the beam is subjected to the axial force $P(x)$, as shown in Fig. 7.3b. We propose to derive the boundary-value problem by means of the extended Hamilton's principle, Eq. (7.4). To this end, we assume that the kinetic energy is due entirely to translation. Hence, denoting the transverse displacement by $w(x, t)$, we can write the kinetic energy expression in the form

$$T(t) = \frac{1}{2} \int_0^L m(x) \left[\frac{\partial w(x, t)}{\partial t} \right]^2 dx \quad (7.18)$$

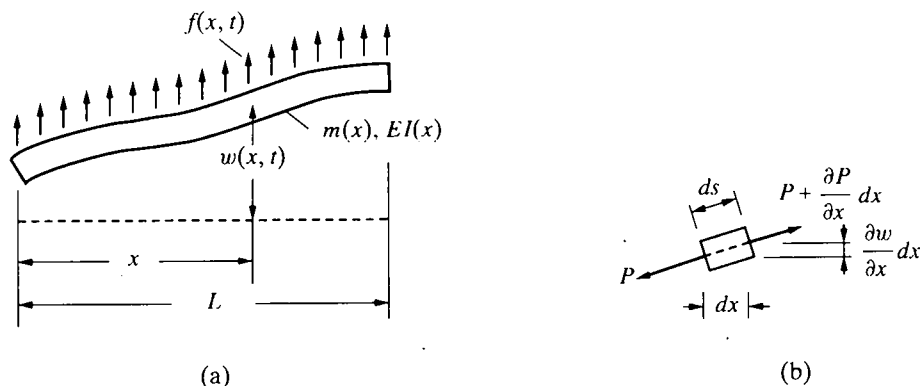


Figure 7.3 (a) Beam in bending vibration (b) Axial forces acting on a beam differential element

where $m(x)$ is the mass per unit length of beam. The potential energy consists of two parts, one due to bending and one due to the axial force, where the latter is entirely analogous to the potential energy in a string, the integral in Eq. (7.8). The potential energy of a beam in bending can be found in any textbook on mechanics of materials. Moreover, using the analogy with the string, the combined potential energy can be shown to have the form

$$V(t) = \frac{1}{2} \int_0^L \left\{ EI(x) \left[\frac{\partial^2 w(x, t)}{\partial x^2} \right]^2 + P(x) \left[\frac{\partial w(x, t)}{\partial x} \right]^2 \right\} dx \quad (7.19)$$

where $EI(x)$ is the *bending stiffness*, or *flexural rigidity*, in which E is the Young's modulus of elasticity and $I(x)$ is the area moment of inertia about an axis normal to the plane defined by x and w . It should be noted here that the contribution of the gravitational forces to the potential energy can be ignored by measuring displacements from the equilibrium position. Finally, the virtual work due to nonconservative forces is simply

$$\overline{\delta W}_{nc} = \int_0^L f(x, t) \delta w(x, t) dx \quad (7.20)$$

We carry out the operations involved in the extended Hamilton's principle, Eq. (7.4), term by term. To this end, we assume that variations and differentiations are interchangeable, use Eq. (7.18) and write the variation in the kinetic energy in the form

$$\delta T = \int_0^L m \frac{\partial w}{\partial t} \delta \frac{\partial w}{\partial t} dx = \int_0^L m \frac{\partial w}{\partial t} \frac{\partial}{\partial t} \delta w dx \quad (7.21)$$

so that, integrating by parts, the first term in Eq. (7.4) yields

$$\begin{aligned} \int_{t_1}^{t_2} \delta T dt &= \int_{t_1}^{t_2} \int_0^L m \frac{\partial w}{\partial t} \frac{\partial}{\partial t} \delta w dx dt = \int_0^L \int_{t_1}^{t_2} m \frac{\partial w}{\partial t} \frac{\partial}{\partial t} \delta w dt dx \\ &= \int_0^L m \frac{\partial w}{\partial t} \delta w \Big|_{t_1}^{t_2} dx - \int_0^L \int_{t_1}^{t_2} m \frac{\partial^2 w}{\partial t^2} \delta w dt dx \end{aligned}$$

$$= - \int_{t_1}^{t_2} \int_0^L m \frac{\partial^2 w}{\partial t^2} \delta w \, dx \, dt \quad (7.22)$$

where we considered the fact that $\delta w = 0$ at $t = t_1, t_2$. Moreover, we take the variation in the potential energy, Eq. (7.19), integrate by parts and obtain

$$\begin{aligned} \delta V &= \int_0^L \left(EI \frac{\partial^2 w}{\partial x^2} \delta \frac{\partial^2 w}{\partial x^2} + P \frac{\partial w}{\partial x} \delta \frac{\partial w}{\partial x} \right) dx \\ &= \int_0^L \left(EI \frac{\partial^2 w}{\partial x^2} \frac{\partial^2}{\partial x^2} \delta w + P \frac{\partial w}{\partial x} \frac{\partial}{\partial x} \delta w \right) dx \\ &= EI \frac{\partial^2 w}{\partial x^2} \frac{\partial}{\partial x} \delta w \Big|_0^L - \frac{\partial}{\partial x} \left(EI \frac{\partial^2 w}{\partial x^2} \right) \delta w \Big|_0^L + P \frac{\partial w}{\partial x} \delta w \Big|_0^L \\ &\quad + \int_0^L \left[\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 w}{\partial x^2} \right) - \frac{\partial}{\partial x} \left(P \frac{\partial w}{\partial x} \right) \right] \delta w \, dx \end{aligned} \quad (7.23)$$

Finally, inserting Eqs. (7.20), (7.22) and (7.23) into Eq. (7.4) and grouping the terms in appropriate fashion, we obtain

$$\begin{aligned} & - \int_{t_1}^{t_2} \left\{ \left[- \frac{\partial}{\partial x} \left(EI \frac{\partial^2 w}{\partial x^2} \right) + P \frac{\partial w}{\partial x} \right] \delta w \Big|_0^L + EI \frac{\partial^2 w}{\partial x^2} \delta \frac{\partial w}{\partial x} \Big|_0^L \right. \\ & \quad \left. + \int_0^L \left[m \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 w}{\partial x^2} \right) - \frac{\partial}{\partial x} \left(P \frac{\partial w}{\partial x} \right) - f \right] \delta w \, dx \right\} dt = 0 \end{aligned} \quad (7.24)$$

At this point, we invoke the arbitrariness of the virtual displacements in a judicious manner. In particular, we assume that either δw or its coefficient in the boundary term is zero at $x = 0$ and $x = L$, that either $\delta(\partial w/\partial x)$ or its coefficient in the boundary term is zero at $x = 0$ and $x = L$ and that δw is entirely arbitrary over the domain $0 < x < L$. It follows that Eq. (7.24) can be satisfied if and only if

$$m \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 w}{\partial x^2} \right) - \frac{\partial}{\partial x} \left(P \frac{\partial w}{\partial x} \right) - f = 0, \quad 0 < x < L \quad (7.25)$$

and, in addition, either

$$- \frac{\partial}{\partial x} \left(EI \frac{\partial^2 w}{\partial x^2} \right) + P \frac{\partial w}{\partial x} = 0 \quad \text{at } x = 0, L \quad (7.26a)$$

or

$$w = 0 \quad \text{at } x = 0, L \quad (7.26b)$$

and either

$$EI \frac{\partial^2 w}{\partial x^2} = 0 \quad \text{at } x = 0, L \quad (7.27a)$$

or

$$\frac{\partial w}{\partial x} = 0 \quad \text{at } x = 0, L \quad (7.27b)$$

Equation (7.25) represents the equation of motion, a fourth-order partial differential equation to be satisfied at every point of the domain, and Eqs. (7.26) and (7.27) represent boundary conditions. We note that two boundary conditions must be satisfied at $x = 0$ and $x = L$, one from Eqs. (7.26) and one from Eqs. (7.27). The choice as to which of the two equations must be satisfied depends on the nature of the boundary. For example, we know that the displacement is zero at a pinned end, but the slope of the displacement curve must be different from zero. Hence, at a *pinned end* we must retain Eqs. (7.26b) and (7.27a) as the boundary conditions, or

$$w = 0; \quad M = EI \frac{\partial^2 w}{\partial x^2} = 0 \quad (7.28)$$

where M is identified as the bending moment. At a *free end*, we know that the displacement and slope must be different from zero, so that from Eqs. (7.26a) and (7.27a) the boundary conditions are

$$Q = -\frac{\partial}{\partial x} \left(EI \frac{\partial^2 w}{\partial x^2} \right) + P \frac{\partial w}{\partial x} = 0, \quad M = EI \frac{\partial^2 w}{\partial x^2} = 0 \quad (7.29)$$

where Q is recognized as a shearing force. Finally, at a *clamped end* the bending moment and shearing force are not zero, so that from Eqs. (7.26b) and (7.27b) we obtain the boundary conditions

$$w = 0, \quad \partial w / \partial x = 0 \quad (7.30)$$

Other boundary conditions than those described by Eqs. (7.28)–(7.30) are possible, but they require modifications in the formulation. For example, if an end is supported by a linear spring of stiffness K , then the term $\frac{1}{2} K w^2$ must be added to the potential energy, and if an end is supported by a torsional spring of stiffness K_T , the term $\frac{1}{2} K_T (\partial w / \partial x)^2$ must be added. Clearly, the effect of such terms will be reflected in the boundary conditions, as shown in the next section.

The model of a beam in bending vibration considered in this section is the simplest possible and is known as an *Euler-Bernoulli beam*.

7.3 LAGRANGE'S EQUATION FOR DISTRIBUTED SYSTEMS. THE BOUNDARY-VALUE PROBLEM

In deriving boundary-value problems by means of the extended Hamilton's principle there are several steps that must be repeated time and again. In this regard, we recall from Chapter 2 that a similar situation exists for discrete systems, in which case it is possible to avoid the repetition by using the extended Hamilton's principle to derive Lagrange's equations and then derive the equations of motion by means of Lagrange's equations. Hence, it is only natural to seek a similar approach for distributed systems. Of course, the situation is more complicated in the case of distributed systems, because there are at least two independent variables instead of

one. In view of this, the resulting formulation consists of a single Lagrange's equation, a partial differential equation, and associated boundary conditions. This formulation is quite general and covers a relatively large class of systems.

For convenience, we rewrite the extended Hamilton's principle, Eq. (7.4), as

$$\int_{t_1}^{t_2} (\delta L + \delta \overline{W}_{nc}) dt = 0, \quad \delta w = 0, \quad t = t_1, t_2 \quad (7.31)$$

where $L = T - V$ is the Lagrangian. Moreover, we can simplify the notation by denoting derivatives with respect to the spatial variable x by primes and derivatives with respect to the time t by overdots whenever appropriate. This permits us to write the kinetic energy in the general functional form

$$T = \int_0^L \hat{T}(\dot{w}) dx \quad (7.32)$$

in which the overcaret denotes a kinetic energy density. Similarly, the potential energy is assumed to have the form

$$V = V_0 [w(0, t), w'(0, t)] + V_L [w(L, t), w'(L, t)] + \int_0^L \hat{V}(w, w', w'') dx \quad (7.33)$$

where the subscripts 0 and L refer to potential energy due to springs at the ends $x = 0$ and $x = L$, respectively, and the overcaret denotes a potential energy density. Hence, the Lagrangian can be expressed as

$$L = L_0 [w(0, t), w'(0, t)] + L_L [w(L, t), w'(L, t)] + \int_0^L \hat{L}(w, w', w'', \dot{w}) dx \quad (7.34)$$

in which L_0 and L_L are boundary Lagrangians and \hat{L} is a Lagrangian density. Moreover, the virtual work is simply

$$\delta \overline{W}_{nc} = \int_0^L f \delta w dx \quad (7.35)$$

where $f = f(x, t)$ is the distributed force, and note that concentrated forces can be expressed as distributed by means of spatial Dirac delta functions.

The extended Hamilton's principle, Eq. (7.31), calls for the variation in the Lagrangian, which can be expressed in the form

$$\delta L = \delta L_0 + \delta L_L + \int_0^L \delta \hat{L} dx \quad (7.36)$$

where

$$\delta L_0 = \frac{\partial L_0}{\partial w(0, t)} \delta w(0, t) + \frac{\partial L_0}{\partial w'(0, t)} \delta w'(0, t) \quad (7.37a)$$

$$\delta L_L = \frac{\partial L_L}{\partial w(L, t)} \delta w(L, t) + \frac{\partial L_L}{\partial w'(L, t)} \delta w'(L, t) \quad (7.37b)$$

$$\delta \hat{L} = \frac{\partial \hat{L}}{\partial w} \delta w + \frac{\partial \hat{L}}{\partial w'} \delta w' + \frac{\partial \hat{L}}{\partial w''} \delta w'' + \frac{\partial \hat{L}}{\partial \dot{w}} \delta \dot{w} \quad (7.37c)$$

From Secs. 7.1 and 7.2, we recall that, before we can invoke the arbitrariness of the virtual displacements, it is necessary to carry out a number of integrations by parts with respect to x and t . This can be conveniently done term by term. First, we carry out integrations with respect to t , or

$$\int_{t_1}^{t_2} \frac{\partial \hat{L}}{\partial \dot{w}} \delta \dot{w} dt = \frac{\partial \hat{L}}{\partial \dot{w}} \delta w \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}} \right) \delta w dt = - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}} \right) \delta w dt \quad (7.38)$$

where we considered the fact that δw is zero at $t = t_1, t_2$. Next, we carry out integrations with respect to x , which involve terms in \hat{L} . First, we have

$$\int_0^L \frac{\partial \hat{L}}{\partial w'} \delta w' dx = \int_0^L \frac{\partial \hat{L}}{\partial w'} \frac{\partial}{\partial x} \delta w dx = \frac{\partial \hat{L}}{\partial w'} \delta w \Big|_0^L - \int_0^L \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w'} \right) \delta w dx \quad (7.39a)$$

Similarly,

$$\int_0^L \frac{\partial \hat{L}}{\partial w''} \delta w'' dx = \frac{\partial \hat{L}}{\partial w''} \delta w' \Big|_0^L - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w''} \right) \delta w \Big|_0^L + \int_0^L \frac{\partial^2}{\partial x^2} \left(\frac{\partial \hat{L}}{\partial w''} \right) \delta w dx \quad (7.39b)$$

Introducing Eqs. (7.35)–(7.37) into Eq. (7.31), considering Eqs. (7.38) and (7.39) and collecting terms involving $\delta w(x, t)$, $\delta w(0, t)$, $\delta w(L, t)$, $\delta w'(0, t)$ and $\delta w'(L, t)$, we obtain

$$\begin{aligned} & \int_{t_1}^{t_2} \left\{ \frac{\partial L_0}{\partial w(0, t)} \delta w(0, t) + \frac{\partial L_0}{\partial w'(0, t)} \delta w'(0, t) \right. \\ & \quad + \frac{\partial L_L}{\partial w(L, t)} \delta w(L, t) + \frac{\partial L_L}{\partial w'(L, t)} \delta w'(L, t) \\ & \quad + \int_0^L \left[\frac{\partial \hat{L}}{\partial w} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w'} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial \hat{L}}{\partial w''} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}} \right) + f \right] \delta w dx \\ & \quad \left. + \left[\frac{\partial \hat{L}}{\partial w'} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w''} \right) \right] \delta w \Big|_0^L + \frac{\partial \hat{L}}{\partial w''} \delta w' \Big|_0^L \right\} dt \\ & = \int_{t_1}^{t_2} \left\{ \int_0^L \left[\frac{\partial \hat{L}}{\partial w} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w'} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial \hat{L}}{\partial w''} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}} \right) + f \right] \delta w dx \right. \\ & \quad \left. + \left\{ \frac{\partial L_0}{\partial w(0, t)} - \left[\frac{\partial \hat{L}}{\partial w'} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w''} \right) \right] \Big|_{x=0} \right\} \delta w(0, t) \right. \end{aligned}$$

$$\begin{aligned}
& + \left\{ \frac{\partial L_L}{\partial w(L, t)} + \left[\frac{\partial \hat{L}}{\partial w'} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w''} \right) \right] \Big|_{x=L} \right\} \delta w(L, t) \\
& + \left[\frac{\partial L_0}{\partial w'(0, t)} - \frac{\partial \hat{L}}{\partial w''} \Big|_{x=0} \right] \delta w'(0, t) \\
& + \left[\frac{\partial L_L}{\partial w'(L, t)} + \frac{\partial \hat{L}}{\partial w''} \Big|_{x=L} \right] \delta w'(L, t) \Big) dt = 0 \tag{7.40}
\end{aligned}$$

At this point, we invoke the arbitrariness of the virtual displacements. If we let $\delta w(0, t) = \delta w(L, t) = 0$ and $\delta w'(0, t) = \delta w'(L, t) = 0$, we conclude that Eq. (7.40) can be satisfied for all values of δw in the open domain $0 < x < L$ if and only if the coefficient of δw is zero, or

$$\frac{\partial \hat{L}}{\partial w} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w'} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial \hat{L}}{\partial w''} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}} \right) + f = 0, \quad 0 < x < L \tag{7.41}$$

Moreover, by writing

$$\left\{ \frac{\partial L_0}{\partial w(0, t)} - \left[\frac{\partial \hat{L}}{\partial w'} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w''} \right) \right] \Big|_{x=0} \right\} \delta w(0, t) = 0 \tag{7.42a}$$

$$\left[\frac{\partial L_0}{\partial w'(0, t)} - \frac{\partial \hat{L}}{\partial w''} \Big|_{x=0} \right] \delta w'(0, t) = 0 \tag{7.42b}$$

$$\left\{ \frac{\partial L_L}{\partial w(L, t)} + \left[\frac{\partial \hat{L}}{\partial w'} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w''} \right) \right] \Big|_{x=L} \right\} \delta w(L, t) = 0 \tag{7.43a}$$

$$\left[\frac{\partial L_L}{\partial w'(L, t)} + \frac{\partial \hat{L}}{\partial w''} \Big|_{x=L} \right] \delta w'(L, t) = 0 \tag{7.43b}$$

we take into account that either $\delta w(0, t)$ or its coefficient is zero and either $\delta w'(0, t)$ or its coefficient is zero, and similar statements can be made about conditions at $x = L$.

Equation (7.41) represents the *Lagrange differential equation of motion* for the fourth-order distributed-parameter system with the Lagrangian given by Eq. (7.34). It is a partial differential equation to be satisfied at every point of the open domain $0 < x < L$. Moreover, from Eqs. (7.42), we conclude that at $x = 0$ the displacement must be such that either

$$\frac{\partial L_0}{\partial w(0, t)} - \left[\frac{\partial \hat{L}}{\partial w'} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w''} \right) \right] \Big|_{x=0} = 0 \tag{7.44a}$$

or

$$w(0, t) = 0 \tag{7.44b}$$

and either

$$\frac{\partial L_0}{\partial w'(0, t)} - \frac{\partial \hat{L}}{\partial w''} \Big|_{x=0} = 0 \quad (7.45a)$$

or

$$w'(0, t) = 0 \quad (7.45b)$$

In addition, at $x = L$ either

$$\frac{\partial L_L}{\partial w(L, t)} + \left[\frac{\partial \hat{L}}{\partial w'} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w''} \right) \right] \Big|_{x=L} = 0 \quad (7.46a)$$

or

$$w(L, t) = 0 \quad (7.46b)$$

and either

$$\frac{\partial L_L}{\partial w'(L, t)} + \frac{\partial \hat{L}}{\partial w''} \Big|_{x=L} = 0 \quad (7.47a)$$

or

$$w'(L, t) = 0 \quad (7.47b)$$

For a given system, the solution $w(x, t)$ of the Lagrange equation, Eq. (7.41), must satisfy one of each of Eqs. (7.44)–(7.47), for a total of two equations at each end. The four equations represent the *boundary conditions*, and the choice of boundary conditions at each end is dictated by the physics of the problem.

It should be pointed out here that, although the formulation is for fourth-order systems, the formulation can be used also for second-order systems by simply deleting terms that do not apply.

Example 7.1

Derive the Lagrange equation and boundary conditions for the string of Fig. 7.1a.

This is only a second-order system, so that deleting from Eq. (7.41) terms involving derivatives higher than two the Lagrange equation reduces to

$$\frac{\partial \hat{L}}{\partial w} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w'} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}} \right) + f = 0, \quad 0 < x < L \quad (a)$$

Moreover, the physics of the problem dictates the boundary conditions

$$w(0, t) = 0 \quad (b)$$

$$\frac{\partial L_L}{\partial w(L, t)} + \frac{\partial \hat{L}}{\partial w'} \Big|_{x=L} = 0 \quad (c)$$

From Eqs. (7.5) and (7.8), we can write the Lagrangian density

$$\hat{L} = \hat{T} - \hat{V} = \frac{1}{2} \rho \dot{w}^2 - \frac{1}{2} T (w')^2 \quad (d)$$

and the boundary Lagrangians

$$L_0 = 0, \quad L_L = -\frac{1}{2} K w^2(L, t) \quad (e)$$

Inserting Eq. (d) into Eq. (a), we obtain the partial differential equation of motion

$$\frac{\partial}{\partial x} \left(T \frac{\partial w}{\partial x} \right) - \rho \frac{\partial^2 w}{\partial t^2} + f = 0, \quad 0 < x < L \quad (f)$$

where we recognized that $\rho = \rho(x)$ does not depend on t . Moreover, introducing Eq. (d) and the second of Eqs. (e) into Eq. (c), we obtain the boundary condition at $x = L$

$$Kw(L, t) + T \frac{\partial w}{\partial x} \Big|_{x=L} = 0 \quad (g)$$

The boundary condition at $x = 0$ remains in the form of Eq. (b).

It is easy to verify that the boundary-value problem given by Eqs. (f), (b) and (g) is identical to that obtained in Sec. 7.1

Example 7.2

Derive the Lagrange equation and boundary conditions for the rotating cantilever beam shown in Fig. 7.4a.

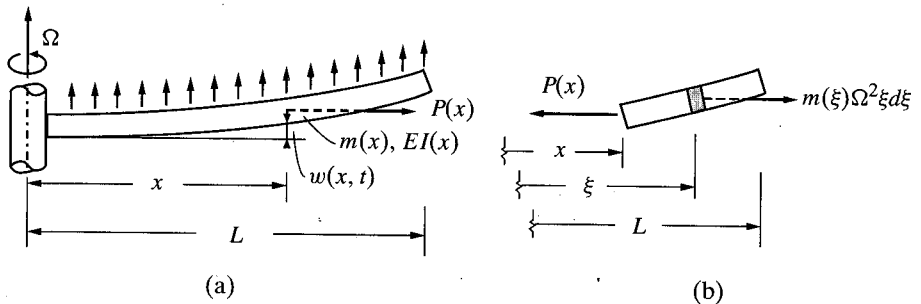


Figure 7.4 (a) Rotating cantilever beam (b) Axial force due to centrifugal effects

From Eqs. (7.18) and (7.19), the Lagrangian density has the expression

$$\hat{L} = \hat{T} - \hat{V} = \frac{1}{2}m\dot{w}^2 - \frac{1}{2}EI(w'')^2 - \frac{1}{2}P(w')^2 \quad (a)$$

where, from Fig. 7.4b, the axial force has the form

$$P(x) = \int_x^L m\Omega^2 \xi d\xi \quad (b)$$

Note that in the case at hand $L_0 = L_L = 0$.

The Lagrange differential equation for a fourth-order system is given in general form by Eq. (7.41). To obtain its explicit expression, we write

$$\frac{\partial \hat{L}}{\partial w} = 0, \quad \frac{\partial \hat{L}}{\partial w'} = -Pw', \quad \frac{\partial \hat{L}}{\partial w''} = -EIw'', \quad \frac{\partial \hat{L}}{\partial \dot{w}} = m\dot{w} \quad (c)$$

Inserting Eqs. (c) into Eq. (7.41), we obtain the Lagrange equation in the explicit form

$$\frac{\partial}{\partial x} \left(P \frac{\partial w}{\partial x} \right) - \frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 w}{\partial x^2} \right) - m \frac{\partial^2 w}{\partial t^2} + f = 0, \quad 0 < x < L \quad (d)$$

where P is given by Eq. (b).

To derive the boundary conditions, we observe that the displacement and rotation are zero at a clamped end. Hence, on physical grounds, we choose Eqs. (7.44b) and (7.45b) as the boundary conditions, so that

$$w = 0, \quad w' = 0, \quad x = 0 \quad (e)$$

On the other hand, the displacement and rotation at $x = L$ are not zero, so that we must choose Eqs. (7.46a) and (7.47a) as the boundary conditions. Observing from Eq. (b) that $P(L) = 0$, the boundary conditions at $x = L$ are

$$\frac{\partial}{\partial x} \left(EI \frac{\partial^2 w}{\partial x^2} \right) = 0, \quad EI \frac{\partial^2 w}{\partial x^2} = 0, \quad x = L \quad (f)$$

7.4 FREE VIBRATION OF CONSERVATIVE SYSTEMS. THE DIFFERENTIAL EIGENVALUE PROBLEM

In the absence of external forces, $f(x, t) = 0$, the boundary-value problem for the vibrating string of Fig. 7.1a, Eqs. (7.2) and (7.3), reduces to the partial differential equation of motion

$$\frac{\partial}{\partial x} \left[T(x) \frac{\partial w(x, t)}{\partial x} \right] = \rho(x) \frac{\partial^2 w(x, t)}{\partial t^2}, \quad 0 < x < L \quad (7.48)$$

and the boundary conditions

$$w(x, t) = 0, \quad x = 0 \quad (7.49a)$$

$$T(x) \frac{\partial w(x, t)}{\partial x} + Kw(x) = 0, \quad x = L \quad (7.49b)$$

Equations (7.48) and (7.49) describe the free vibration of conservative second-order systems.

Our interest lies in the solution of Eqs. (7.48) and (7.49). In particular, we wish to explore the existence of solutions whereby the system executes *synchronous motions*, defined as motions in which every point performs the same motion in time. The physical implication is that synchronous motions are characterized by the fact that the ratio of the displacements corresponding to two different points of the string is constant. Mathematically, synchronous motions imply that the solution of Eqs. (7.48) and (7.49) is separable in the spatial variable and time, and hence it has the form

$$w(x, t) = W(x)F(t) \quad (7.50)$$

where $W(x)$ depends on the spatial position alone and $F(t)$ depends on time alone. Introducing Eq. (7.50) into Eqs. (7.48) and (7.49), we can write

$$\frac{d}{dx} \left[T(x) \frac{dW(x)}{dx} \right] F(t) = \rho(x)W(x) \frac{d^2 F(t)}{dt^2}, \quad 0 < x < L \quad (7.51)$$

$$W(0)F(t) = 0, \quad \left[T(x) \frac{dW(x)}{dx} + KW(x) \right] \Big|_{x=L} F(t) = 0 \quad (7.52a, b)$$

Next, we divide Eq. (7.51) by ρWF and obtain

$$\frac{1}{\rho(x)W(x)} \frac{d}{dx} \left[T(x) \frac{dW(x)}{dx} \right] = \frac{1}{F(t)} \frac{d^2 F(t)}{dt^2}, \quad 0 < x < L \quad (7.53)$$

Observing that the left side of Eq. (7.53) depends on x alone and the right side on time alone and that F can be simply omitted from boundary conditions (7.52), we conclude that the solution is indeed separable in x and t . But, because the left side of Eq. (7.53) depends only on x and the right side only on t and, moreover, x and t are independent variables, the two sides of the equation must be equal to a constant, the same constant. In addition, the two sides are real, so that the constant must be real. For reasons that will become obvious shortly, we denote the constant by $-\lambda$, where λ is a positive constant, so that the right side of Eq. (7.53) yields

$$\ddot{F}(t) + \lambda F(t) = 0 \quad (7.54)$$

The solution of Eq. (7.54) has the exponential form

$$F(t) = Ae^{st} \quad (7.55)$$

Introducing Eq. (7.55) into Eq. (7.54) and dividing through by Ae^{st} , we obtain the *characteristic equation*

$$s^2 + \lambda = 0 \quad (7.56)$$

which has the roots

$$\begin{aligned} s_1 &= \pm\sqrt{-\lambda} = \pm i\sqrt{\lambda} = \pm i\omega \\ s_2 & \end{aligned} \quad (7.57)$$

It follows that the solution of Eq. (7.54) has the harmonic form

$$F(t) = A_1 e^{i\omega t} + A_2 e^{-i\omega t} = A_1 e^{i\omega t} + \bar{A}_1 e^{-i\omega t} = C \cos(\omega t - \phi) \quad (7.58)$$

where we let A_2 be equal to the complex conjugate \bar{A}_1 of A_1 in recognition of the fact that $F(t)$ must be real, and we note that the constants A_1 and \bar{A}_1 are related to the *amplitude* C and *phase angle* ϕ by

$$A_1 + \bar{A}_1 = C \cos \phi, \quad i(A_1 - \bar{A}_1) = C \sin \phi \quad (7.59)$$

At this point our earlier choice of sign for the constant used in Eq. (7.53) is made clear by the fact that $F(t)$, as given by Eq. (7.58), represents harmonic oscillation, which is typical of the response of stable conservative systems. The choice will be justified mathematically in Sec. 7.5.

The question remains as to the displacement pattern, as well as to the frequency of oscillation. To answer these questions, we set the left side of Eq. (7.53) equal to $-\lambda$ and obtain the differential equation

$$-\frac{d}{dx} \left[T(x) \frac{dW(x)}{dx} \right] = \lambda \rho(x)W(x), \quad \lambda = \omega^2, \quad 0 < x < L \quad (7.60)$$

Moreover, we eliminate F from Eqs. (7.52) and obtain the boundary conditions

$$W(0) = 0; \quad T(x) \frac{dW(x)}{dx} + KW(x) = 0, \quad x = L \quad (7.61a, b)$$

Equations (7.60) and (7.61) represent the *differential eigenvalue problem* for the string shown in Fig. 7.1a. It can be described in words as *the problem of determining the constant λ so that the differential equation (7.60) admits nontrivial solutions satisfying boundary conditions (7.61)*. Second-order problems of this type are known as *Sturm-Liouville problems*.

Next, we wish to derive the differential eigenvalue problem for the beam in bending discussed in Sec. 7.2. Letting $f(x, t) = 0$ in Eq. (7.25), assuming that the solution has the form given by Eq. (7.50) and following the approach used earlier in this section, we conclude once again that $F(t)$ is harmonic, as indicated by Eq. (7.58), and $W(x)$ satisfies the differential equation

$$\frac{d^2}{dx^2} \left[EI(x) \frac{d^2 W(x)}{dx^2} \right] - \frac{d}{dx} \left[P(x) \frac{dW(x)}{dx} \right] = \lambda m(x) W(x), \quad \lambda = \omega^2, \\ 0 < x < L \quad (7.62)$$

Moreover, from Eqs. (7.26) and (7.27), $W(x)$ must satisfy either

$$-\frac{d}{dx} \left[EI(x) \frac{d^2 W(x)}{dx^2} \right] + P(x) \frac{dW(x)}{dx} = 0 \quad \text{at } x = 0, L \quad (7.63a)$$

or

$$W(x) = 0 \quad \text{at } x = 0, L \quad (7.63b)$$

and either

$$EI(x) \frac{d^2 W(x)}{dx^2} = 0 \quad \text{at } x = 0, L \quad (7.64a)$$

or

$$\frac{dW(x)}{dx} = 0 \quad \text{at } x = 0, L \quad (7.64b)$$

The differential eigenvalue problem consists of the differential equation, Eq. (7.62), to be satisfied over the domain $0 < x < L$ and two boundary conditions, one from Eqs. (7.63) and one from Eqs. (7.64), to be satisfied at $x = 0$ and $x = L$.

The differential eigenvalue problem is of vital importance in the vibration of distributed-parameter systems. Indeed, as in the case of discrete systems, its solution not only provides a great deal of information concerning the system characteristics but also can be used to derive the system response.

7.5 THE DIFFERENTIAL EIGENVALUE PROBLEM FOR SELF-ADJOINT SYSTEMS

In Secs. 7.3 and 7.4, we derived the differential eigenvalue problem for second-order and fourth-order systems. A cursory examination of the eigenvalue problem formulations permits us to conclude that the form differs from system to system, depending on the order of the system and the nature of the boundaries. In spite of the difference in appearance, various systems have many things in common. Hence, instead of discussing the individual systems separately, it is convenient to formulate

the differential eigenvalue problem so as to apply to large classes of systems. This formulation can be best achieved by resorting to operator notation.

Let w be a function of one or two independent spatial variables x or x and y , so that in essence we confine ourselves to one- or two-dimensional problems, and consider the *homogeneous differential expression*

$$Lw = a_1w + a_2\frac{\partial w}{\partial x} + a_3\frac{\partial w}{\partial y} + a_4\frac{\partial^2 w}{\partial x^2} + a_5\frac{\partial^2 w}{\partial x\partial y} + \dots \quad (7.65)$$

where the coefficients a_1, a_2, \dots are known functions of the spatial variables x and y . We assume that Lw involves derivatives of w through order $2p$, where p is an integer, so that the differential expression is said to be of order $2p$. We can then define the *homogeneous differential operator* associated with Eq. (7.65) in the form

$$L = a_1 + a_2\frac{\partial}{\partial x} + a_3\frac{\partial}{\partial y} + a_4\frac{\partial^2}{\partial x^2} + a_5\frac{\partial^2}{\partial x\partial y} + \dots \quad (7.66)$$

and refer to L as being of order $2p$. If for the functions w_1 and w_2 the relation

$$L(c_1w_1 + c_2w_2) = c_1Lw_1 + c_2Lw_2 \quad (7.67)$$

holds, where c_1 and c_2 are constants, then the homogeneous differential operator L is said to be *linear*.

Next, we consider a generic eigenvalue problem and express the differential equation in the operator form

$$Lw = \lambda mw, \quad x, y \text{ in } D \quad (7.68)$$

where L is a linear homogeneous differential operator of order $2p$, λ a parameter, m the mass density and D the domain of definition of Eq. (7.68). The operator L , referred to as *stiffness operator*, is of the type (7.66). Associated with the differential equation (7.68) there are p boundary conditions to be satisfied by the solution w at every point of the boundary S of the domain D . The boundary conditions are also expressed in operator form, as follows:

$$B_i w = 0, \quad x, y \text{ in } S, \quad i = 1, 2, \dots, p \quad (7.69)$$

where B_i are linear homogeneous differential operators of maximum order $2p - 1$. They are referred to as *boundary operators*. In the one-dimensional case, the domain D is a segment of the real line and the boundary S consists of the two points bounding D . In the two-dimensional case, the domain D is a plane and the boundary S consists of one or more closed curves bounding D . *The eigenvalue problem is defined as the problem of determining the values of the parameter λ for which there are nontrivial functions w satisfying the differential equation (7.68) and the boundary conditions (7.69).* Such parameters are called *eigenvalues* and the corresponding functions are called *eigenfunctions*. The eigenvalue problem defined by Eqs. (7.68) and (7.69) admits a *denumerably, or countably infinite set of eigenvalues $\lambda_1, \lambda_2, \dots$ and associated eigenfunctions w_1, w_2, \dots* . The implication is that *the eigenfunction w_r belongs to the eigenvalue λ_r ($r = 1, 2, \dots$).*

One of the most important problems in the vibration of distributed-parameter systems involves the expansion of the response in terms of known functions, and in

particular the expansion in terms of the system eigenfunctions. This expansion is based on some remarkable properties of the eigenfunctions. Before proceeding with the investigation of these properties, it will prove useful to introduce certain pertinent definitions concerning functions in general. In the first place, we assume that the functions considered are real and *piecewise smooth*, i.e., that they are piecewise continuous and possess piecewise continuous first derivatives in a given domain D . Then, for two such functions f and g , we define the *inner product of the two functions f and g over the domain D* as

$$(f, g) = \int_D fg \, dD \quad (7.70)$$

If the inner product vanishes, then the two functions f and g are said to be *orthogonal* over D . The square root of the inner product of a function f with itself is known as the *norm* of f , defined as

$$\|f\| = (f, f)^{1/2} = \left(\int_D f^2 \, dD \right)^{1/2} \quad (7.71)$$

Orthogonal functions with unit norm are said to be *orthonormal*. The existence of the norm simply implies that $\|f\| < \infty$. A function whose norm exists is said to be *square summable* in D , which implies that f^2 is integrable in the Lebesgue sense (Ref. 1, p. 108). Functions f such that $\|f\| < \infty$ are said to have *finite energy*, and the space of such functions is denoted by \mathcal{K}^0 , where the superscript indicates the order of the derivative required for finite energy, in this case zero.

A property of functions intimately related to orthogonality is *linear independence*. To define the concept, we consider a set of n functions $\phi_1, \phi_2, \dots, \phi_n$. Then, if a homogeneous linear relation with constant coefficients of the type

$$\sum_{i=1}^n c_i \phi_i = 0 \quad (7.72)$$

exists without all the coefficients c_i ($i = 1, 2, \dots, n$) being identically zero, the set of functions $\phi_1, \phi_2, \dots, \phi_n$ is said to be *linearly dependent*. If Eq. (7.72) is satisfied only when all the coefficients c_i ($i = 1, 2, \dots, n$) are identically zero, then the set of functions is said to be *linearly independent*. To explore the connection between orthogonality and linear independence, we assume that the set of functions $\phi_1, \phi_2, \dots, \phi_n$ is orthogonal, multiply Eq. (7.72) by ϕ_j , integrate over the domain D and obtain

$$\sum_{i=1}^n c_i \int_D \phi_i \phi_j \, dD = \sum_{i=1}^n c_i \|\phi_j\|^2 \delta_{ij} = c_j \|\phi_j\|^2 = 0, \quad j = 1, 2, \dots, n \quad (7.73)$$

Because the norms $\|\phi_j\|$ cannot be zero, it follows from Eq. (7.73) that all the coefficients c_j ($j = 1, 2, \dots, n$) must be zero. But, this is precisely the condition for the functions $\phi_1, \phi_2, \dots, \phi_n$ to be linearly independent, which proves that *orthogonal functions are linearly independent*. The converse is not true, as *independent functions are not necessarily orthogonal*. This is a matter of semantics, however, because

independent functions can be rendered orthogonal. The procedure for rendering independent functions orthogonal is known as the *Gram-Schmidt orthogonalization process* (Ref. 1). It is common practice to normalize the functions during the process, so that the result is a set of orthonormal functions.

Next, we consider the problem of expanding any given function f in terms of known functions. To this end, we let ϕ_1, ϕ_2, \dots be an orthonormal system and express f as the *linear combination*

$$f = \sum_{r=1}^{\infty} c_r \phi_r \quad (7.74)$$

where the coefficients

$$c_r = (f, \phi_r) = \int_D f \phi_r dD, \quad r = 1, 2, \dots \quad (7.75)$$

are known as *components of f with respect to the orthonormal system ϕ_1, ϕ_2, \dots* . This expansion is not unlike the expansion of a periodic function in terms of a Fourier series, and in fact it represents a generalization of the Fourier series expansion.

In vibrations there is considerable interest in approximating the given function f by means of *finite series* of the type

$$f = \sum_{r=1}^n d_r \phi_r \quad (7.76)$$

where ϕ_r are orthonormal functions, d_r are constant coefficients and n is fixed. The objective is to produce the "best" approximation of f , in the sense that the *mean square error*

$$M = \int_D \left(f - \sum_{r=1}^n d_r \phi_r \right)^2 dD \quad (7.77)$$

is as small as possible. To this end, we expand the right side of Eq. (7.77), consider Eqs. (7.71) and (7.75), as well as the orthonormality of ϕ_r ($r = 1, 2, \dots, n$), and write

$$\begin{aligned} M &= \int_D f^2 dD - 2 \sum_{r=1}^n d_r \int_D f \phi_r dD + \sum_{r=1}^n \sum_{s=1}^n d_r d_s \int_D \phi_r \phi_s dD \\ &= \|f\|^2 - 2 \sum_{r=1}^n d_r c_r + \sum_{r=1}^n d_r^2 = \|f\|^2 + \sum_{r=1}^n (d_r - c_r)^2 - \sum_{r=1}^n c_r^2 \end{aligned} \quad (7.78)$$

It is clear from Eq. (7.78) that M takes the smallest value when $d_r = c_r$ ($r = 1, 2, \dots, n$). An approximation of the type

$$f_n = \sum_{r=1}^n c_r \phi_r, \quad n = 1, 2, \dots \quad (7.79)$$

where ϕ_1, ϕ_2, \dots are orthonormal functions, is known as a *least squares approximation*, or an *approximation in the mean*. If, by choosing n sufficiently large, the mean square error satisfies the inequality

$$\int_D \left(f - \sum_{r=1}^n c_r \phi_r \right)^2 dD = \|f - f_n\|^2 < \epsilon \quad (7.80)$$

where ϵ is any arbitrarily small positive number, the set of functions ϕ_1, ϕ_2, \dots is said to be *complete*. Moreover, if

$$\lim_{n \rightarrow \infty} \|f - f_n\| = 0 \quad (7.81)$$

then the sequence f_1, f_2, \dots *converges in the mean* to f .

The completeness of a set of functions does not require that the functions be orthonormal, so that a system of functions is complete if every piecewise continuous function can be approximated in the mean to any desired degree of accuracy by means of a linear combination of the functions of the system. It should be pointed out that, whereas completeness is necessary for convergence, it gives no indication concerning the rate of convergence. The rate of convergence is often as important as convergence itself, as a given function can be approximated by more than one complete set. Clearly, *our interest lies in the set of functions capable of providing an accurate approximation with a number n of terms as small as possible*.

Conservative distributed structures represent a very large and important class of systems, namely, the class of *self-adjoint systems*. The eigenvalues and eigenfunctions of all systems belonging to this class possess very interesting and useful properties. To demonstrate these properties, we consider two $2p$ times differentiable trial functions u and v satisfying all the boundary conditions of the problem, Eq. (7.69), and introduce the inner product

$$(u, Lv) = \int_D uLv dD \quad (7.82)$$

Then, we say that the differential operator L is *self-adjoint* if

$$(u, Lv) = (v, Lu) \quad (7.83)$$

Self-adjointness implies certain mathematical symmetry and can be ascertained through integrations by parts with due consideration to the boundary conditions. This mathematical symmetry can be used to simplify the test for self-adjointness, Eq. (7.83). Indeed, if the left (right) side of Eq. (7.83) can be reduced to a symmetric form in u and v and their derivatives through p integrations by parts, then the operator L is self-adjoint, and the test can be regarded as successfully completed. *If the stiffness operator L is self-adjoint, then the system is said to be self-adjoint*.

The concept of self-adjointness can perhaps be best explained by invoking the analogy between distributed and discrete systems. Indeed, the stiffness operator L being self-adjoint corresponds to the stiffness matrix K being symmetric. In the particular case in which the eigenvalue problem is defined by Eqs. (7.68) and (7.69), the mass operator m is a mere function, which is self-adjoint by definition. It corresponds to the mass matrix M being symmetric. Hence, the system self-adjointness,

which is implied by the self-adjointness of L , corresponds to the symmetry of the stiffness matrix K and mass matrix M . Because the mass matrix is positive definite, it further corresponds to the symmetry of the system matrix A . In this regard, we recall that in Sec. 4.8 we referred to an algebraic eigenvalue problem as self-adjoint if it is defined by a single real symmetric matrix A .

We denote the symmetric form in u and v resulting from integrations by parts of (u, Lv) by $[u, v]$ and refer to it as an *energy inner product*. For one-dimensional domains, the energy inner product has the general expression

$$[u, v] = \int_0^L \sum_{k=0}^p a_k \frac{d^k u}{dx^k} \frac{d^k v}{dx^k} dx + \sum_{\ell=0}^{p-1} b_\ell \left. \frac{d^\ell u}{dx^\ell} \frac{d^\ell v}{dx^\ell} \right|_0^L = [v, u] \quad (7.84)$$

where a_k ($k = 0, 1, \dots, p$) and b_ℓ ($\ell = 0, 1, \dots, p-1$) are in general functions of x . It is clear from Eq. (7.84) that the mathematical symmetry consists of the fact that the functions u and v can trade positions without altering the result. Energy inner products for two-dimensional domains are discussed later in this chapter. The reason for the term "energy inner product" will become evident shortly.

If for any $2p$ times differentiable function u satisfying all the boundary conditions of the problem, Eqs. (7.69), the inequality

$$\int_D u L u dD \geq 0 \quad (7.85)$$

is true and the equality sign holds if and only if $u \equiv 0$, then the operator L is said to be *positive definite*. If the expression can be zero without u being identically zero, then the operator L is only *positive semidefinite*. If the operator L is positive definite (semidefinite), then the system is positive definite (semidefinite).

For $v = u$, Eq. (7.84) reduces to

$$[u, u] = \int_0^L \sum_{k=0}^p a_k \left(\frac{d^k u}{dx^k} \right)^2 dx + \sum_{\ell=0}^{p-1} b_\ell \left(\frac{d^\ell u}{dx^\ell} \right)^2 \Big|_0^L \quad (7.86)$$

and we note that $[u, u]$ is a measure of the potential energy. Indeed, if the system executes harmonic motion with frequency ω , then $[u, u]$ is equal to twice the maximum potential energy, which explains why we referred earlier to $[u, v]$ as an energy inner product. Equation (7.86) can be used to define the *energy norm*

$$\|u\|_E = [u, u]^{1/2} \quad (7.87)$$

Next, we introduce the sequence of approximations

$$u_n = \sum_{r=1}^n c_r \phi_r, \quad n = 1, 2, \dots \quad (7.88)$$

where ϕ_1, ϕ_2, \dots are given functions from an independent set. Then, if by choosing n sufficiently large,

$$\|u - u_n\|_E < \epsilon \quad (7.89)$$

in which ϵ is an arbitrarily small positive number, the set of functions ϕ_1, ϕ_2, \dots is said to be *complete in energy*. Moreover, if

$$\lim_{n \rightarrow \infty} \|u - u_n\|_E = 0 \quad (7.90)$$

the sequence of approximations u_1, u_2, \dots is said to *converge in energy* to u .

In our study of vibrations, it is convenient to define two classes of functions. One class consists of functions that are $2p$ times differentiable and satisfy all the boundary conditions of the problem. This class was introduced earlier in conjunction with the self-adjointness definition. We refer to it as the class of *comparison functions* and denote it by \mathcal{K}_B^{2p} . It should be noted that the eigenfunctions are by definition comparison functions, but they represent only a very small subset of the class of comparison functions, as the comparison functions need not satisfy the differential equation. Examining Eq. (7.84), we conclude that the energy inner product is defined for functions outside the space \mathcal{K}_B^{2p} . Indeed, Eq. (7.84) is defined for functions that are only p times differentiable. Moreover, in integrating the left side of Eq. (7.83) by parts to obtain the energy inner product, Eq. (7.84), due consideration was given to the natural boundary conditions, in the sense that the higher-order derivatives in the natural boundary conditions were eliminated in favor of lower-order derivatives, such as those arising in geometric boundary conditions. As a result, the energy inner product is defined for functions that are only p times differentiable and satisfy only the geometric boundary conditions. We refer to p times differentiable functions satisfying only the geometric boundary conditions of the problem as *admissible functions*, and we denote this class of functions by \mathcal{K}_G^p . The comparison functions are by definition admissible functions, and in fact they constitute a small subset of the much larger class of admissible functions.

At this point, we turn our attention to the properties of the eigenvalues and eigenfunctions. To this end, we assume that the problem is self-adjoint and consider two distinct solutions λ_r, w_r and λ_s, w_s of the eigenvalue problem, Eqs. (7.68) and (7.69). Inserting these solutions into Eq. (7.68), we can write

$$Lw_r = \lambda_r m w_r, \quad Lw_s = \lambda_s m w_s \quad (7.91a, b)$$

Multiplying Eq. (7.91a) by w_s and Eq. (7.91b) by w_r , subtracting the second from the first, integrating over domain D and considering Eqs. (7.82) and (7.83), we obtain

$$\int_D (w_s Lw_r - w_r Lw_s) dD = (\lambda_r - \lambda_s) \int_D m w_r w_s dD = 0 \quad (7.92)$$

But, by assumption, the eigenvalues λ_r and λ_s are distinct. Hence, Eq. (7.92) can be satisfied if and only if

$$\int_D m w_r w_s dD = 0, \quad \lambda_r \neq \lambda_s, \quad r, s = 1, 2, \dots \quad (7.93)$$

Equation (7.93) represents the *orthogonality relation* for the eigenfunctions of distributed-parameter systems described by the eigenvalue problem given by Eqs. (7.68)

and (7.69). Multiplying Eq. (7.91b) by w_r , integrating over D and using Eq. (7.93), it is easy to see that the eigenfunctions satisfy a *second orthogonality relation*, namely,

$$\int_D w_r L w_s dD = 0, \quad \lambda_r \neq \lambda_s, \quad r, s = 1, 2, \dots \quad (7.94)$$

It should be stressed here that *the orthogonality of the eigenfunctions is a direct consequence of the system being self-adjoint*.

In the case of repeated eigenvalues, there are as many eigenfunctions belonging to the repeated eigenvalue as the multiplicity of the repeated eigenvalue, and these eigenfunctions are generally not orthogonal to one another, although they are independent and orthogonal to the remaining eigenfunctions of the system. But, as pointed out earlier in this section, independent functions can be orthogonalized by grouping them in proper linear combinations. Hence, *all the eigenfunctions of a self-adjoint system can be regarded as orthogonal, regardless of whether there are repeated eigenvalues or not*.

Because the eigenvalue problem, Eqs. (7.68) and (7.69), is homogeneous, *only the shape of the eigenfunctions is unique, and the amplitude is arbitrary*. This arbitrariness can be removed through normalization. A mathematically convenient normalization scheme is given by

$$\int_D m w_r^2 dD = 1, \quad r = 1, 2, \dots \quad (7.95a)$$

which implies that

$$\int_r w_r L w_r dD = \lambda_r, \quad r = 1, 2, \dots \quad (7.95b)$$

Then, Eqs. (7.93)–(7.95) can be combined into the *orthonormality relations*

$$\int_D m w_r w_s dD = \delta_{rs}, \quad \int_D w_r L w_s dD = \lambda_r \delta_{rs}, \quad r, s = 1, 2, \dots \quad (7.96a, b)$$

where δ_{rs} is the Kronecker delta.

In Sec. 7.4 and in this section, we assumed on physical grounds that the eigenvalues and eigenfunctions are real. We propose to prove here mathematically that this is indeed so, provided the system is self-adjoint. To this end, we consider a complex solution λ , w of the eigenvalue problem, Eqs. (7.68) and (7.69). Because the eigenvalue problem is real, if the pair λ , w is a complex solution, then the complex conjugate pair $\bar{\lambda}$, \bar{w} must also be a solution, so that Eq. (7.68) yields

$$Lw = \lambda w, \quad L\bar{w} = \bar{\lambda} \bar{w} \quad (7.97a, b)$$

Multiplying Eq. (7.97a) by \bar{w} and Eq. (7.97b) by w , subtracting the second from the first, integrating over D and invoking the self-adjointness of L , we obtain

$$\int_D (\bar{w} L w - w L \bar{w}) dD = (\lambda - \bar{\lambda}) \int_D m \bar{w} w dD = 0 \quad (7.98)$$

Introducing the notation

$$\frac{\lambda}{\bar{\lambda}} = \alpha \pm i\beta, \quad \frac{w}{\bar{w}} = \text{Re } w \pm i\text{Im } w \quad (7.99)$$

we conclude that the integral on the right side of Eq. (7.98) is real and positive, so that the only alternative is

$$\lambda - \bar{\lambda} = \alpha + i\beta - (\alpha - i\beta) = 2i\beta = 0 \quad (7.100)$$

It follows that *the eigenvalues of a self-adjoint system are real*. As a corollary, *the eigenfunctions of a self-adjoint system are real*. Moreover, contrasting Eq. (7.95b) with inequality (7.85), we conclude that, *if the operator L is positive definite, all the eigenvalues are positive*. On the other hand, *if the operator L is only positive semidefinite, all the eigenvalues are nonnegative*, i.e., some are zero and the rest are positive.

The eigenfunctions w_r ($r = 1, 2, \dots$) of a self-adjoint system constitute a *complete orthonormal set* of infinite dimension (see, for example, Ref. 1). The implication is that the eigenfunctions can be used as a *basis* for a *function space*, sometimes referred to as a *Hilbert space*. This fact can be stated formally as the following *expansion theorem for self-adjoint systems*: *Every function w with continuous Lw and satisfying the boundary conditions of the system can be expanded in an absolutely and uniformly convergent series in the eigenfunctions in the form*

$$w = \sum_{r=1}^{\infty} c_r w_r \quad (7.101)$$

where the coefficients c_r are such that

$$c_r = \int_D m w_r w \, dD, \quad \lambda_r c_r = \int_D w_r Lw \, dD, \quad r = 1, 2, \dots \quad (7.102a, b)$$

The expansion theorem for self-adjoint distributed systems, Eqs. (7.101) and (7.102), represents the counterpart of the expansion theorem for conservative discrete systems, Eqs. (4.129) and (4.130), defined by symmetric matrices. The expansion theorem is made possible by the orthogonality of the eigenfunctions. Of course, there are many infinite sets of orthogonal functions, such as trigonometric functions, Bessel functions, etc., but none of these can be used as a basis for an expansion theorem for self-adjoint distributed systems, unless they happen to represent the system eigenfunctions. What is so remarkable about the eigenfunctions is that they are orthogonal not only with respect to the mass density m but also with respect to the stiffness operator L , as indicated by Eqs. (7.93) and (7.94).

The expansion theorem plays a pivotal role in the vibration of self-adjoint systems, as it permits the solution of the boundary-value problem by transforming it into an infinite set of *modal equations*, which are second-order ordinary differential equations for the time-dependent modal coordinates. The solution process is known as *modal analysis* and is entirely analogous to the one for discrete systems. In fact, the second-order differential equations look exactly like the modal equations for

discrete systems, and can be solved by the techniques discussed in Chapter 3. The only difference is that in distributed systems the set is infinite and in discrete systems the set is finite.

The class of self-adjoint systems is extremely large and it includes essentially all the vibrating conservative systems discussed in this text. If it can be demonstrated that an individual system belongs to this class, then it can be assumed automatically that the system possesses the remarkable properties of self-adjoint systems discussed in this section. In effect, the various conservative systems considered in this text can be regarded as special cases. It should be pointed out that, provided a system is self-adjoint, the general properties hold, regardless of whether a closed-form solution to the differential eigenvalue problem exists or not. This fact can be of great value in developing approximate solutions.

Example 7.3

Consider the eigenvalue problem for the string in transverse vibration shown in Fig. 7.1a, demonstrate that the problem fits one of the generic formulations of this section, check whether the problem is self-adjoint and positive definite and draw conclusions.

From Sec. 7.4, Eqs. (7.60) and (7.61), the eigenvalue problem is given by the differential equation

$$-\frac{d}{dx} \left[T(x) \frac{dW(x)}{dx} \right] = \lambda \rho(x) W(x), \quad 0 < x < L \quad (a)$$

and the boundary conditions

$$W = 0, \quad x = 0 \quad (b)$$

$$T(x) \frac{dW(x)}{dx} + KW(x) = 0, \quad x = L \quad (c)$$

The eigenvalue problem fits the generic formulation given by Eqs. (7.68) and (7.69), so that, comparing Eq. (a) with Eq. (7.68), we conclude that

$$L = -\frac{d}{dx} \left[T(x) \frac{d}{dx} \right], \quad m = \rho \quad (d)$$

Because L is of order 2, $p = 1$. Moreover, comparing Eqs. (b) and (c) with Eqs. (7.69) we can write

$$B_1 = 1, \quad x = 0 \quad (e)$$

$$B_1 = T(x) \frac{d}{dx} + K, \quad x = L \quad (f)$$

To check for self-adjointness, we write

$$(u, Lv) = \int_0^L uLv \, dx = -\int_0^L u \frac{d}{dx} \left(T \frac{dv}{dx} \right) dx = -uT \frac{dv}{dx} \Big|_0^L + \int_0^L \frac{du}{dx} T \frac{dv}{dx} dx \quad (g)$$

Upon considering boundary conditions (b) and (c), Eq. (g) reduces to the energy inner product

$$[u, v] = Ku(L)v(L) + \int_0^L T \frac{du}{dx} \frac{dv}{dx} dx = [v, u] \quad (h)$$

which is clearly symmetric, so that *the operator L , and hence the system, is self-adjoint.* We conclude immediately that *the eigenvalues are real and the eigenfunctions are orthogonal with respect to the mass density m and the stiffness operator L .*

Finally, we let $v = u$ in Eq. (h) and obtain the energy norm squared

$$[u, u] = \|u\|_E^2 = Ku^2(L) + \int_0^L T \left(\frac{du}{dx} \right)^2 dx > 0 \quad (i)$$

which, in view of Eq. (7.8), is recognized as twice the maximum potential energy. Clearly, the energy norm cannot be zero, except in the trivial case, so that *the operator L , and hence the system, is positive definite.* It follows that *all the eigenvalues are positive.*

Example 7.4

Consider the rotating cantilever beam in bending vibration shown in Fig. 7.4a and demonstrate that the eigenvalue problem fits one of the generic formulations of this section. Then, check whether the system is self-adjoint and positive definite and draw conclusions.

The boundary-value problem for a rotating cantilever beam was derived in Example 7.2. Letting $f = 0$ and using the procedure presented in Sec. 7.4, Eqs. (d)–(f) of Example 7.2 yield the eigenvalue problem defined by the differential equation

$$\frac{d^2}{dx^2} \left(EI \frac{d^2 W}{dx^2} \right) - \frac{d}{dx} \left(P \frac{dW}{dx} \right) = \lambda m W, \quad 0 < x < L \quad (a)$$

and the boundary conditions

$$W = 0, \quad \frac{dW}{dx} = 0, \quad x = 0 \quad (b)$$

$$EI \frac{d^2 W}{dx^2} = 0, \quad \frac{d}{dx} \left(EI \frac{d^2 W}{dx^2} \right) = 0, \quad x = L \quad (c)$$

Comparing Eq. (a) with Eq. (7.68), we conclude that

$$L = \frac{d^2}{dx^2} \left(EI \frac{d^2}{dx^2} \right) - \frac{d}{dx} \left(P \frac{d}{dx} \right) \quad (d)$$

Moreover, comparing Eqs. (b) and (c) with Eqs. (7.69), we can write

$$B_1 = 1, \quad B_2 = \frac{d}{dx}, \quad x = 0 \quad (e)$$

$$B_1 = EI \frac{d^2}{dx^2}, \quad B_2 = \frac{d}{dx} \left(EI \frac{d^2}{dx^2} \right), \quad x = L$$

Next, we consider

$$\begin{aligned} (u, Lv) &= \int_0^L uLv \, dx = \int_0^L u \left[\frac{d^2}{dx^2} \left(EI \frac{d^2 v}{dx^2} \right) - \frac{d}{dx} \left(P \frac{dv}{dx} \right) \right] dx \\ &= u \frac{d}{dx} \left(EI \frac{d^2 v}{dx^2} \right) \Big|_0^L - \frac{du}{dx} EI \frac{d^2 v}{dx^2} \Big|_0^L + \int_0^L \frac{d^2 u}{dx^2} EI \frac{d^2 v}{dx^2} dx \\ &\quad - uP \frac{dv}{dx} \Big|_0^L + \int_0^L \frac{du}{dx} P \frac{dv}{dx} dx \end{aligned} \quad (f)$$

and use boundary conditions (b) and (c) to obtain the symmetric energy inner product

$$[u, v] = \int_0^L \left(EI \frac{d^2 u}{dx^2} \frac{d^2 v}{dx^2} + P \frac{du}{dx} \frac{dv}{dx} \right) dx = [v, u] \quad (g)$$

It follows that *the operator L, and hence the system, is self-adjoint*, so that *the eigenvalues are real and the eigenfunctions are orthogonal with respect to the mass density m and the stiffness operator L*. Then, letting $v = u$ in Eq. (g), we obtain the energy norm squared

$$[u, u] = \|u\|_E^2 = \int_0^L \left[EI \left(\frac{d^2 u}{dx^2} \right)^2 + P \left(\frac{du}{dx} \right)^2 \right] dx \geq 0 \quad (h)$$

The norm reduces to zero for $u = \text{constant}$. In view of the first of boundary conditions (b), however, this constant must be zero. Because $u = \text{constant} \neq 0$ is not a solution of the eigenvalue problem, the energy norm is positive definite, so that *the operator L, and hence the system, is positive definite*, which implies that *all the eigenvalues are positive*.

7.6 SOLUTION OF THE EIGENVALUE PROBLEM FOR STRINGS, RODS AND SHAFTS

As indicated at the end of Sec. 7.1, the behavior of strings in transverse vibration, rods in axial vibration and shafts in torsional vibration is described by mathematically equivalent second-order boundary-value problems, the difference between the various types lying in the nature of the parameters. This implies that the solution obtained for one of the three types of systems applies equally well for the remaining two. To emphasize this point, we shall solve the eigenvalue problem for all three types interchangeably using the formulation derived in Sec. 7.4 for strings.

i. Strings in transverse vibration

A problem of considerable interest in vibrations is that of *a string fixed at both ends*. From Sec. 7.4, the eigenvalue problem for a vibrating string is defined by the differential equation

$$-\frac{d}{dx} \left[T(x) \frac{dW(x)}{dx} \right] = \lambda \rho(x) W(x), \quad \lambda = \omega^2, \quad 0 < x < L \quad (7.103)$$

In the case at hand, the boundary conditions are

$$W(0) = 0, \quad W(L) = 0 \quad (7.104a, b)$$

Using the method of Sec. 7.5, it is not difficult to verify that the system is self-adjoint and positive definite, so that the eigenvalues are real and positive and the eigenfunctions are real and orthogonal. Although this is one of the simplest examples of a distributed-parameter systems, no closed-form solution exists in the general case in which the tension $T(x)$ and the mass density $\rho(x)$ are arbitrary functions of the spatial position x . A closed-form solution does exist in the frequently encountered case in which *the tension is constant*, $T(x) = T = \text{constant}$, and *the mass density is*

uniform, $\rho(x) = \rho = \text{constant}$. In this case, the differential equation, Eq. (7.103), can be rewritten as

$$\frac{d^2 W(x)}{dx^2} + \beta^2 W(x) = 0, \quad \beta^2 = \frac{\omega^2 \rho}{T}, \quad 0 < x < L \quad (7.105)$$

The boundary conditions, Eqs. (7.104), do not depend on the system parameters, so that they remain the same. The solution of Eq. (7.105) is simply

$$W(x) = C_1 \sin \beta x + C_2 \cos \beta x \quad (7.106)$$

where C_1 and C_2 are constants. Solution (7.106) holds true for all strings with constant tension and uniform mass distribution, and it reduces to the eigenfunctions of a particular system only after the boundary conditions have been enforced.

Inserting solution (7.106) into boundary condition (7.104a), we obtain

$$W(0) = C_2 = 0 \quad (7.107)$$

Then, introducing solution (7.106) with $C_2 = 0$ into boundary condition (7.104b), we have

$$W(L) = C_1 \sin \beta L = 0 \quad (7.108)$$

Equation (7.108) can be satisfied in two ways. The first alternative is to set $C_1 = 0$, which corresponds to the trivial solution, so that it must be ruled out. Hence, we must opt for the second alternative, namely,

$$\sin \beta L = 0 \quad (7.109)$$

Equation (7.109) is known as the *characteristic equation*, or *frequency equation*, and has the denumerably infinite set of solutions

$$\beta_r = \frac{r\pi}{L}, \quad r = 1, 2, \dots \quad (7.110)$$

which represent the system *eigenvalues*. It should be pointed out that the term "eigenvalues" is used somewhat loosely here, as strictly speaking the eigenvalues of the system are λ_r , which are related to β_r by

$$\lambda_r = \beta_r^2 T / \rho, \quad r = 1, 2, \dots \quad (7.111)$$

Moreover, recalling that $\lambda = \omega^2$, Eqs. (7.110) and (7.111) can be combined to obtain the *natural frequencies*

$$\omega_r = \sqrt{\lambda_r} = \beta_r \sqrt{\frac{T}{\rho}} = r\pi \sqrt{\frac{T}{\rho L^2}}, \quad r = 1, 2, \dots \quad (7.112)$$

The frequency ω_1 is called the *fundamental frequency* and the higher frequencies are known as *overtones*. Overtones that are *integer multiples* of the fundamental frequency are called *higher harmonics*, in which case the fundamental frequency represents the *fundamental harmonic*. There are only a few vibrating systems with harmonic overtones, and most of them are used in musical instruments because they tend to produce pleasant sounds. In this regard, it should be mentioned that in a

symphonic orchestra the group of instruments including the violin, viola, cello, etc., is commonly referred to as the "string section".

We conclude from the above that the two boundary conditions can be used to determine one of the constants and to derive the characteristic equation. The second constant of integration, C_1 in the case at hand, cannot be determined uniquely, so that the amplitude of the solution is arbitrary. This is consistent with the fact that the eigenvalue problem, Eqs. (7.103) and (7.104), is a homogeneous problem. In view of Eq. (7.110), the *eigenfunction*, or *natural mode*, belonging to β_r can be written in the form

$$W_r(x) = A_r \sin \frac{r\pi x}{L}, \quad r = 1, 2, \dots \quad (7.113)$$

It is easy to verify that *the eigenfunctions are orthogonal* both with respect to the mass density ρ and with respect to the operator L , which in the case of constant tension reduces to

$$L = -T \frac{d^2}{dx^2} \quad (7.114)$$

The amplitudes A_r can be rendered unique through normalization. A convenient normalization process is given by

$$\int_0^L \rho W_r^2(x) dx = 1, \quad r = 1, 2, \dots \quad (7.115)$$

which yields the *orthonormal set of eigenfunctions*, or *normal modes*

$$W_r(x) = \sqrt{\frac{2}{\rho L}} \sin \frac{r\pi x}{L}, \quad r = 1, 2, \dots \quad (7.116)$$

The first three modes and natural frequencies are displayed in Fig. 7.5. We observe that there are points at which the displacement is zero. These points are referred to as *nodes* and they form a certain pattern. Indeed, excluding the end points, the mode $W_r(x)$ has $r - 1$ equidistant nodes occurring at the points $x_i = iL/r$ ($i = 1, 2, \dots, r - 1$).

The term "denumerably infinite set" introduced in Sec. 7.5 implies that the eigenvalues λ_r ($r = 1, 2, \dots$) assume an infinite set of discrete values. Under these circumstances, the string is said to possess a *discrete frequency spectrum*. Whereas the shape of the modes is independent of the system parameters, the natural frequencies are proportional to the square root of the tension T , inversely proportional to the square root of the mass density ρ and inversely proportional to the length L , as can be observed from Eq. (7.112). In many string instruments, such as the violin, there are four strings, all differing in density. For all practical purposes, the density of each of the strings can be regarded as constant. Hence, the frequency spectrum for each string can be altered by changing the tension and the length. The tension T is generally held constant. In fact, the process of tuning a violin consists of adjusting the tension so as to ensure a certain fundamental frequency. During performance, the violinist alters the length of the string so as to produce the notes demanded by the score. It should be pointed out that pleasing sounds are produced by enriching the fundamental harmonic with certain higher harmonics.

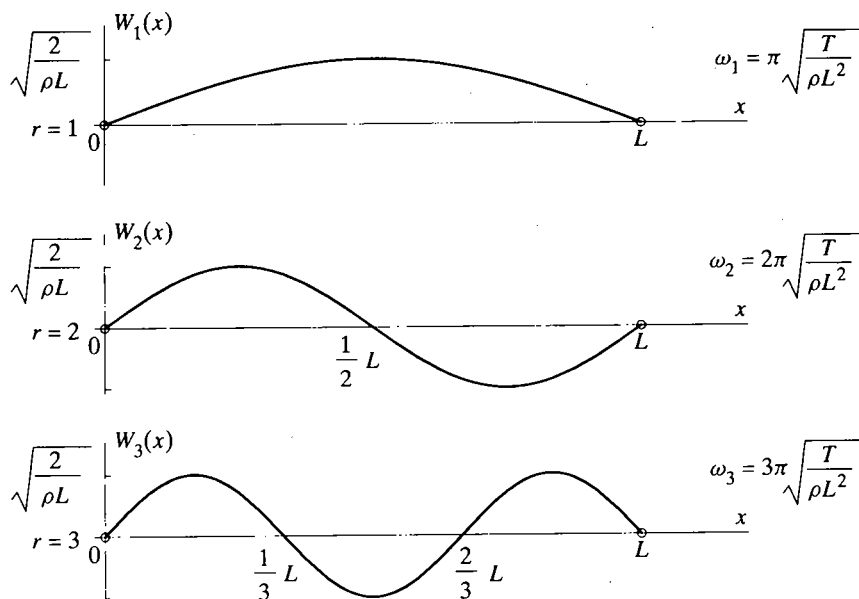


Figure 7.5 First three modes of vibration of a uniform string fixed at both ends

As the length of a string increases, the natural frequencies draw closer and closer together. In fact, as L approaches infinity, we obtain a *continuous frequency spectrum*. At this point it is no longer meaningful to speak of natural frequencies and natural modes, and a different point of view must be adopted. Indeed, for infinitely long strings the motion can be regarded as consisting of *traveling waves*. The wave description of motion applies also to strings of finite length, except that in this case the waves are reflected from the boundaries, and the combination of incident and reflected waves gives rise to *standing waves*. It can be shown that the natural modes description of vibration is mathematically equivalent to the standing waves description (see Ref. 8, Sec. 8-2).

ii. Rods in axial vibration

Invoking the analogy discussed at the end of Sec. 7.1, the eigenvalue problem for a rod in axial vibration can be described by the differential equation

$$-\frac{d}{dx} \left[EA(x) \frac{dU(x)}{dx} \right] = \lambda m(x) U(x), \quad \lambda = \omega^2, \quad 0 < x < L \quad (7.117)$$

where $EA(x)$ is the axial stiffness, in which E is Young's modulus and $A(x)$ is the cross-sectional area, and $m(x)$ is the mass density. The solution $U(x)$ is subject to given boundary conditions. We consider a rod *fixed at $x = 0$ and free at $x = L$* , as shown in Fig. 7.6, so that the boundary conditions are

$$U(0) = 0, \quad EA(x) \frac{dU(x)}{dx} \Big|_{x=L} = 0 \quad (7.118a, b)$$

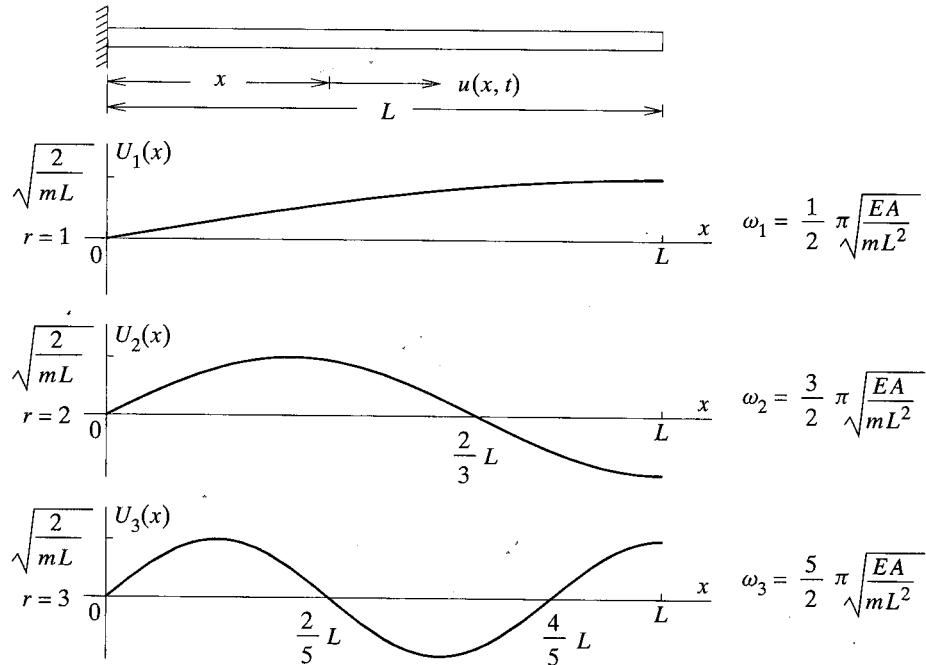


Figure 7.6 Rod in axial vibration fixed at $x = 0$ and free at $x = L$, and the first three modes of vibration

There is no difficulty in demonstrating that the problem is self-adjoint and positive definite.

For a *uniform* rod, $EA(x) = EA = \text{constant}$, $m(x) = m = \text{constant}$, the eigenvalue problem reduces to the differential equation

$$\frac{d^2 U(x)}{dx^2} + \beta^2 U(x) = 0, \quad \beta^2 = \frac{\omega^2 m}{EA}, \quad 0 < x < L \quad (7.119)$$

and the boundary conditions

$$U(0) = 0, \quad \left. \frac{dU(x)}{dx} \right|_{x=L} = 0 \quad (7.120a, b)$$

The differential equation is essentially the same as that for a uniform string, Eq. (7.105), so that the solution is

$$U(x) = C_1 \sin \beta x + C_2 \cos \beta x \quad (7.121)$$

Using boundary condition (7.120a), we conclude that $C_2 = 0$. Moreover, use of boundary condition (7.120b) yields the characteristic equation

$$\cos \beta L = 0 \quad (7.122)$$

Its solutions consist of the eigenvalues

$$\beta_r = \frac{(2r - 1)\pi}{2L}, \quad r = 1, 2, \dots \quad (7.123)$$

so that the natural frequencies are

$$\omega_r = \beta_r \sqrt{\frac{EA}{m}} = \frac{(2r-1)\pi}{2} \sqrt{\frac{EA}{mL^2}}, \quad r = 1, 2, \dots \quad (7.124)$$

The eigenfunctions belonging to β_r are

$$U_r(x) = A_r \sin \frac{(2r-1)\pi x}{2L}, \quad r = 1, 2, \dots \quad (7.125)$$

and they are orthogonal. The coefficients A_r are arbitrary, and we propose to normalize them so as to satisfy

$$\int_0^L m U_r^2(x) dx = 1, \quad r = 1, 2, \dots \quad (7.126)$$

so that the eigenfunctions reduce to the orthonormal set

$$U_r(x) = \sqrt{\frac{2}{mL}} \sin \frac{(2r-1)\pi x}{2L}, \quad r = 1, 2, \dots \quad (7.127)$$

The first three modes are shown in Fig. 7.6. Note that, as customary, displacements have been plotted vertically, when they are in fact in the axial direction. Excluding the point $x = 0$, the r th mode, $U_r(x)$, has nodes at the points $x_i = 2iL/(2r-1)$ ($i = 1, 2, \dots, r-1$).

It will prove instructive to investigate a different case, namely, that in which *both ends* of the rod are free (Fig. 7.7). Of course, the differential equation remains the same, but the new boundary conditions are

$$EA(x) \frac{dU(x)}{dx} = 0, \quad x = 0, L. \quad (7.128)$$

Once again it can be verified that the problem is self-adjoint. To check for positive definiteness, we carry out an integration by parts with due consideration to the boundary conditions and obtain

$$\begin{aligned} \int_0^L U_r L U_r dx &= - \int_0^L U_r \frac{d}{dx} \left(EA \frac{dU_r}{dx} \right) dx \\ &= -U_r \left(EA \frac{dU_r}{dx} \right) \Big|_0^L + \int_0^L \frac{dU_r}{dx} \left(EA \frac{dU_r}{dx} \right) dx \\ &= \int_0^L EA \left(\frac{dU_r}{dx} \right)^2 dx \geq 0 \end{aligned} \quad (7.129)$$

The last integral in (7.129) is equal to zero if U_r is constant. However, unlike our earlier experience in which end restraints required that the constant be zero, in the case at hand a nonzero constant solution is possible. It follows that *for a free-free rod in axial vibration the operator L is only positive semidefinite*, so that *the system is only positive semidefinite*.

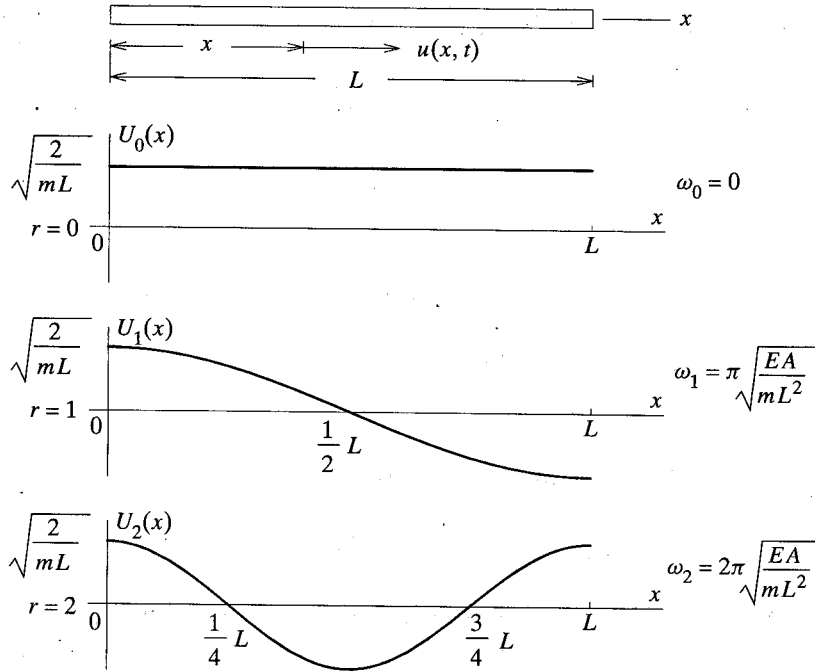


Figure 7.7 Rod in axial vibration free at both ends, the rigid-body mode and the first two elastic modes

In view of the fact that the system is positive semidefinite, *zero eigenvalues*, and hence *zero natural frequencies are possible*. The eigenfunctions belonging to zero eigenvalues represent *rigid-body modes*. To examine the question of rigid-body modes in more detail, we let $\lambda = \lambda_0 = 0$, $U = U_0$ in Eq. (7.117) and write

$$\frac{d}{dx} \left[EA(x) \frac{dU_0(x)}{dx} \right] = 0, \quad 0 < x < L \quad (7.130)$$

Integrating with respect to x once and considering boundary conditions (7.128), we have

$$EA(x) \frac{dU_0(x)}{dx} = 0, \quad 0 < x < L \quad (7.131)$$

Ignoring $EA(x)$ and integrating once more, we obtain

$$U_0(x) = A_0 = \frac{1}{\sqrt{mL}} \quad (7.132)$$

which represents the rigid-body mode with the zero natural frequency, $\omega_0 = 0$, and note that the mode has been normalized so that $\int_0^L mU_0^2 dx = 1$. Clearly, in the case under consideration *there is only one rigid-body mode*. Physically, the rigid-body mode represents displacement of the body as a whole, without elastic deformations. Rigid-body modes are typical of unrestrained systems, for which there are no forces

or moments exerted by the supports. In the case at hand, we are concerned with forces in the longitudinal direction alone, and not with moments.

Next, we assume that the external excitations are zero and consider the vibration of the rod in the r th mode. Using Newton's second law, the equation of motion in the axial direction is

$$\begin{aligned} F(t) &= \int_0^L m(x) \frac{\partial^2 u_r(x, t)}{\partial t^2} dx \\ &= - \left[\int_0^L m(x) U_r(x) dx \right] c_r \omega_r^2 \cos(\omega_r t - \phi_r) = 0, \\ & \qquad \qquad \qquad r = 1, 2, \dots \end{aligned} \quad (7.133)$$

which can be interpreted as the orthogonality of the rigid-body mode to the elastic modes and rewritten in the form

$$\int_0^L m(x) U_0 U_r(x) dx = 0, \quad r = 1, 2, \dots \quad (7.134)$$

Recalling that the system is self-adjoint and normalizing the elastic modes, we can extend the orthonormality relations so as to include the rigid-body mode, or

$$\int_0^L m(x) U_r(x) U_s(x) dx = \delta_{rs}, \quad r, s = 0, 1, 2, \dots \quad (7.135a)$$

$$\int_0^L U_r(x) L U_s(x) dx = \lambda_r \delta_{rs}, \quad r, s = 0, 1, 2, \dots \quad (7.135b)$$

At this point, we return to the solution of the eigenvalue problem, Eqs. (7.117) and (7.128). For a *uniform rod*, the differential equation is given once again by Eq. (7.119) and its solution by Eq. (7.121). Then, we conclude that, in contrast with the fixed-free case, in the free-free case the first of boundary conditions (7.128) yields $C_1 = 0$, whereas the second gives the characteristic equation

$$\sin \beta L = 0 \quad (7.136)$$

which leads to the eigenvalues

$$\beta_r = \frac{r\pi}{L}, \quad r = 0, 1, 2, \dots \quad (7.137)$$

and we note that $\beta_0 = 0$ is also an eigenvalue, corroborating our discussion of the rigid-body mode. Upon normalization, the eigenfunctions of a *free-free* rod are

$$\begin{aligned} U_0(x) &= A_0 = \frac{1}{\sqrt{mL}} \\ U_r(x) &= A_r \cos \frac{r\pi x}{L} = \sqrt{\frac{2}{mL}} \cos \frac{r\pi x}{L}, \quad r = 1, 2, \dots \end{aligned} \quad (7.138)$$

The first three modes are plotted in Fig. 7.7. We observe that the modes have nodes at the points $x_i = (2i - 1)L/2r$ ($i = 1, 2, \dots, r$).

iii. Shafts in torsional vibration

Using once again the analogy discussed at the end of Sec. 7.1, we express the eigenvalue problem for a shaft in torsional vibration by means of the differential equation

$$-\frac{d}{dx} \left[GJ(x) \frac{d\Theta(x)}{dx} \right] = \lambda I(x)\Theta(x), \quad \lambda = \omega^2, \quad 0 < x < L \quad (7.139)$$

where $GJ(x)$ is the torsional stiffness, in which G is the shear modulus and $J(x)$ is the area polar moment of inertia, and $I(x)$ is the mass polar moment of inertia density. The solution $\Theta(x)$ must satisfy boundary conditions yet to be specified. We consider a shaft *clamped at* $x = 0$ and *supported by a torsional spring of stiffness* K_T *at* $x = L$ (Fig. 7.8). A system analogous in both the differential equation and the boundary conditions was considered in Example 7.3 in the form of a string in transverse vibration. Hence, using the analogy with the string of Example 7.3, the boundary conditions are

$$\Theta(0) = 0; \quad GJ(x) \frac{d\Theta(x)}{dx} + K_T \Theta(x) = 0, \quad x = L \quad (7.140a, b)$$

The system was shown in Example 7.3 to be self-adjoint and positive definite, so that the eigenvalues are real and positive and the eigenfunctions are real and orthogonal.

Under the assumption that *the shaft is uniform*, $GJ(x) = GJ = \text{constant}$, $I(x) = I = \text{constant}$, the solution once again has the form given by Eq. (7.121), except that $\Theta(x)$ replaces $U(x)$. Moreover, use of boundary condition (7.140a) results once again in $C_2 = 0$, so that

$$\Theta(x) = C_1 \sin \beta x \quad (7.141)$$

On the other hand, boundary condition (7.140b) yields

$$GJ\beta C_1 \cos \beta L + K_T C_1 \sin \beta L = 0 \quad (7.142)$$

The solution $C_1 = 0$ must be ruled out as representing the trivial solution. Hence, dividing through by C_1 and rearranging, we obtain the characteristic equation

$$\tan \beta L = -\frac{GJ}{K_T L} \beta L \quad (7.143)$$

which is a transcendental equation in βL ; its solutions consist of a denumerably infinite set of eigenvalues $\beta_r L$ ($r = 1, 2, \dots$). The natural frequencies are related to the eigenvalues by

$$\omega_r = \beta_r L \sqrt{\frac{GJ}{IL^2}}, \quad r = 1, 2, \dots \quad (7.144)$$

Belonging to the eigenvalues $\beta_r L$ are the eigenfunctions

$$\Theta_r(x) = A_r \sin \beta_r x, \quad r = 1, 2, \dots \quad (7.145)$$

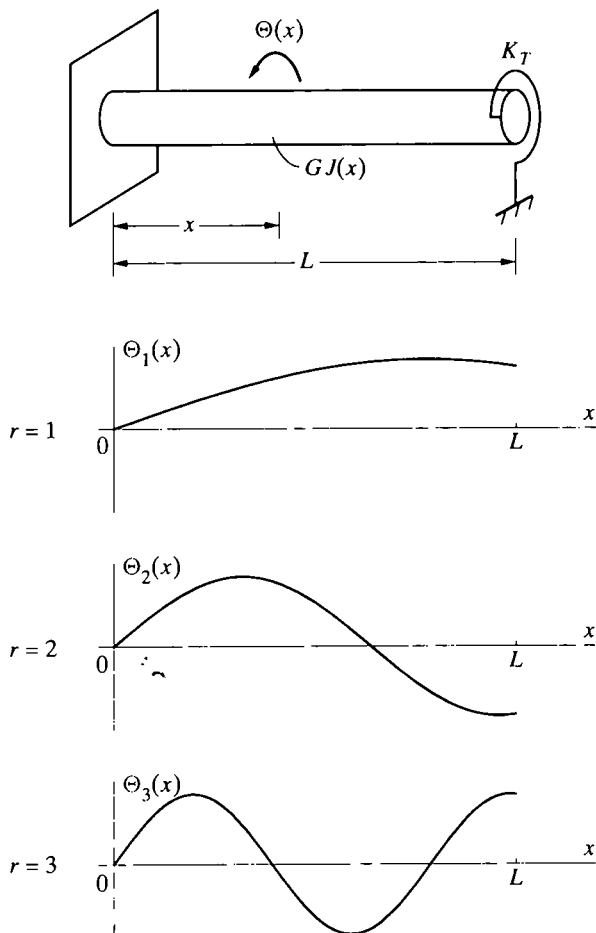


Figure 7.8 Shaft in torsional vibration clamped at $x = 0$ and supported by a spring at $x = L$, and the first three modes of vibration

where A_r are arbitrary amplitudes. The eigenfunctions are orthogonal and can be normalized so as to satisfy $\int_0^L I \Theta_r^2 dx = 1$, in which case the coefficients A_r can be shown to have the values

$$A_r = 2 \sqrt{\frac{\beta_r}{I (2\beta_r L - \sin 2\beta_r L)}}, \quad r = 1, 2, \dots \quad (7.146)$$

The first three eigenfunctions for a ratio $GJ/K_T L = 1$ are plotted in Fig. 7.8.

The solution of the characteristic equation, Eq. (7.143) must be obtained numerically for a given ratio $GJ/K_T L$ of parameters. If the eigenvalues need not be very accurate, a solution can also be obtained graphically, as shown in Fig. 7.9. We observe from the figure that, as $r \rightarrow \infty$, the eigenvalues $\beta_r L$ approach odd multiples of $\pi/2$ and the amplitudes of the eigenfunctions approach $\sqrt{2/IL}$, both

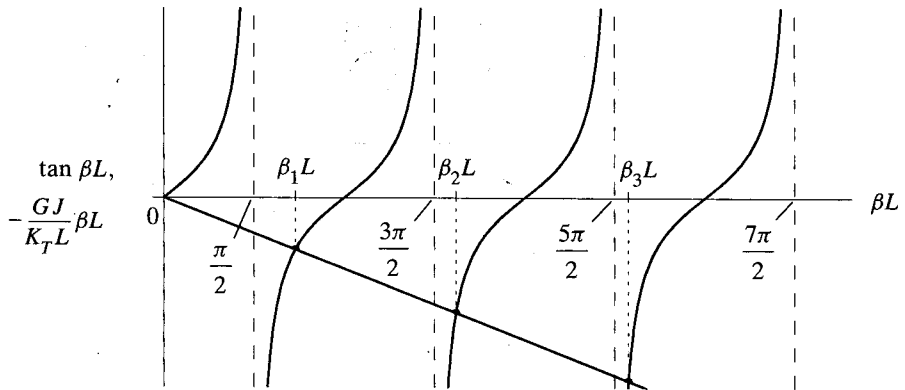


Figure 7.9 Graphical solution of the characteristic equation, Eq. (7.143).

eigenvalues and eigenfunctions being typical of a clamped-free system. Hence, the effect of the end spring K_T tends to diminish as the mode number increases.

In the fixed-fixed string, fixed-free rod and free-free rod discussed earlier in this section, the orthogonality of the modes was guaranteed by the system self-adjointness, and the same can be said about the fixed-spring supported shaft at hand. But, whereas in the first three cases orthogonality can be verified by inspection, this is not true in the present case. Verification of the orthogonality of the eigenfunctions given by Eq. (7.145) can be carried out by showing that the integral $\int_0^L \sin \beta_r x \sin \beta_s x dx$ is zero for $r \neq s$, which requires the use of the characteristic equation, Eq. (7.143). Of course, the fact that the system is self-adjoint makes this verification unnecessary.

7.7 SOLUTION OF THE EIGENVALUE PROBLEM FOR BEAMS IN BENDING

The differential eigenvalue problem for beams in bending was derived in Sec. 7.4. In this section, we wish to solve the problem for a number of cases lending themselves to closed-form solution.

We consider first the simplest case, namely, that of a *uniform beam hinged at both ends and with no axial force*. Under these circumstances, the differential equation, Eq. (7.62), can be rewritten in the form

$$\frac{d^4 W(x)}{dx^4} - \beta^4 W(x) = 0, \quad \beta^4 = \frac{\omega^2 m}{EI}, \quad 0 < x < L \quad (7.147)$$

and the boundary conditions, Eqs. (7.63b) and (7.64a), reduce to

$$W(0) = 0, \quad W(L) = 0, \quad \left. \frac{d^2 W(x)}{dx^2} \right|_{x=0} = 0, \quad \left. \frac{d^2 W(x)}{dx^2} \right|_{x=L} = 0 \quad (7.148a-d)$$

The solution of Eq. (7.147) is

$$W(x) = C_1 \sin \beta x + C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x \quad (7.149)$$

and we note that solution (7.149) is valid for all uniform beams. Differences in the solution begin to appear only when the boundary conditions are enforced. Using boundary conditions (7.148a) and (7.148c), we obtain

$$W(0) = C_2 + C_4 = 0 \quad (7.150a)$$

$$\left. \frac{d^2 W(x)}{dx^2} \right|_{x=0} = -\beta^2 (C_2 - C_4) = 0 \quad (7.150b)$$

which yield

$$C_2 = C_4 = 0 \quad (7.151)$$

On the other hand, using boundary conditions (7.148b) and (7.148d), we have

$$W(L) = C_1 \sin \beta L + C_3 \sinh \beta L = 0 \quad (7.152a)$$

$$\left. \frac{d^2 W(x)}{dx^2} \right|_{x=L} = -\beta^2 (C_1 \sin \beta L - C_3 \sinh \beta L) = 0 \quad (7.152b)$$

Equations (7.152) have nontrivial solutions provided

$$C_3 = 0 \quad (7.153)$$

and

$$\sin \beta L = 0 \quad (7.154)$$

where the latter is recognized as the characteristic equation. Its solutions are the eigenvalues

$$\beta_r L = r\pi, \quad r = 1, 2, \dots \quad (7.155)$$

Belonging to these eigenvalues are the eigenfunctions

$$W_r(x) = \sqrt{\frac{2}{mL}} \sin \frac{r\pi x}{L}, \quad r = 1, 2, \dots \quad (7.156)$$

which were normalized so as to satisfy $\int_0^L m W_r^2 dx = 1$. We observe that the eigenvalues and eigenfunctions are the same as for a uniform fixed-fixed string, so that the first three modes have the same shape as in Fig. 7.5. However, from Eq. (7.147), the natural frequencies are

$$\omega_r = (r\pi)^2 \sqrt{\frac{EI}{mL^4}}, \quad r = 1, 2, \dots \quad (7.157)$$

which are different from the natural frequencies of the fixed-fixed string.

A somewhat more involved case is the *uniform cantilever beam*, namely, a beam with one end clamped and the other end free, as shown in Fig. 7.10. In this case, the boundary conditions are

$$W(0) = 0, \quad \left. \frac{dW(x)}{dx} \right|_{x=0} = 0, \quad \left. \frac{d^2 W(x)}{dx^2} \right|_{x=L} = 0, \quad \left. \frac{d^3 W(x)}{dx^3} \right|_{x=L} = 0 \quad (7.158a-d)$$

Inserting solution (7.149) into boundary conditions (7.158a) and (7.158b), we obtain

$$W(0) = C_2 + C_4 = 0, \quad \left. \frac{dW(x)}{dx} \right|_{x=0} = \beta (C_1 + C_3) = 0 \quad (7.159)$$

so that the solution reduces to

$$W(x) = C_1 (\sin \beta x - \sinh \beta x) + C_2 (\cos \beta x - \cosh \beta x) \quad (7.160)$$

Then, using boundary conditions (7.158c) and (7.158d), we arrive at the two simultaneous homogeneous equations

$$-\beta^2 [C_1 (\sin \beta L + \sinh \beta L) + C_2 (\cos \beta L + \cosh \beta L)] = 0 \quad (7.161a)$$

$$-\beta^3 [C_1 (\cos \beta L + \cosh \beta L) - C_2 (\sin \beta L - \sinh \beta L)] = 0 \quad (7.161b)$$

Equating the determinant of the coefficients to zero, we obtain the characteristic equation

$$\cos \beta L \cosh \beta L = -1 \quad (7.162)$$

The solutions, obtained numerically, are $\beta_1 L = 1.875$, $\beta_2 L = 4.694$, $\beta_3 L = 7.855$, ... In addition, solving Eq. (7.161b) for C_2 in terms of C_1 and substituting into Eq. (7.160), we obtain the corresponding eigenfunctions

$$W_r(x) = A_r [(\sin \beta_r L - \sinh \beta_r L) (\sin \beta_r x - \sinh \beta_r x) + (\cos \beta_r L + \cosh \beta_r L) (\cos \beta_r x - \cosh \beta_r x)], \quad r = 1, 2, \dots \quad (7.163)$$

where we introduced the notation $A_r = C_1 / (\sin \beta_r L - \sinh \beta_r L)$. The system can be verified to be self-adjoint and positive definite, with the usual positivity of the eigenvalues and orthogonality of the eigenfunctions. The normalization ordinarily used in this text is not feasible. The first three natural modes and natural frequencies are displayed in Fig. 7.10. The mode $W_r(x)$ has $r - 1$ nodes ($r = 1, 2, \dots, n$), but their location can no longer be expressed as a rational fraction of L .

Another case of interest is the *free-free beam* (Fig. 7.11). From Sec. 7.4, the eigenvalue problem is defined by the differential equation

$$\frac{d^2}{dx^2} \left[EI(x) \frac{d^2 W(x)}{dx^2} \right] = \lambda m(x) W(x), \quad \lambda = \omega^2, \quad 0 < x < L \quad (7.164)$$

and the boundary conditions

$$EI(x) \frac{d^2 W(x)}{dx^2} \Big|_{x=0} = 0, \quad \frac{d}{dx} \left[EI(x) \frac{d^2 W(x)}{dx^2} \right] \Big|_{x=0} = 0 \quad (7.165a, b)$$

$$EI(x) \frac{d^2 W(x)}{dx^2} \Big|_{x=L} = 0, \quad \frac{d}{dx} \left[EI(x) \frac{d^2 W(x)}{dx^2} \right] \Big|_{x=L} = 0 \quad (7.165c, d)$$

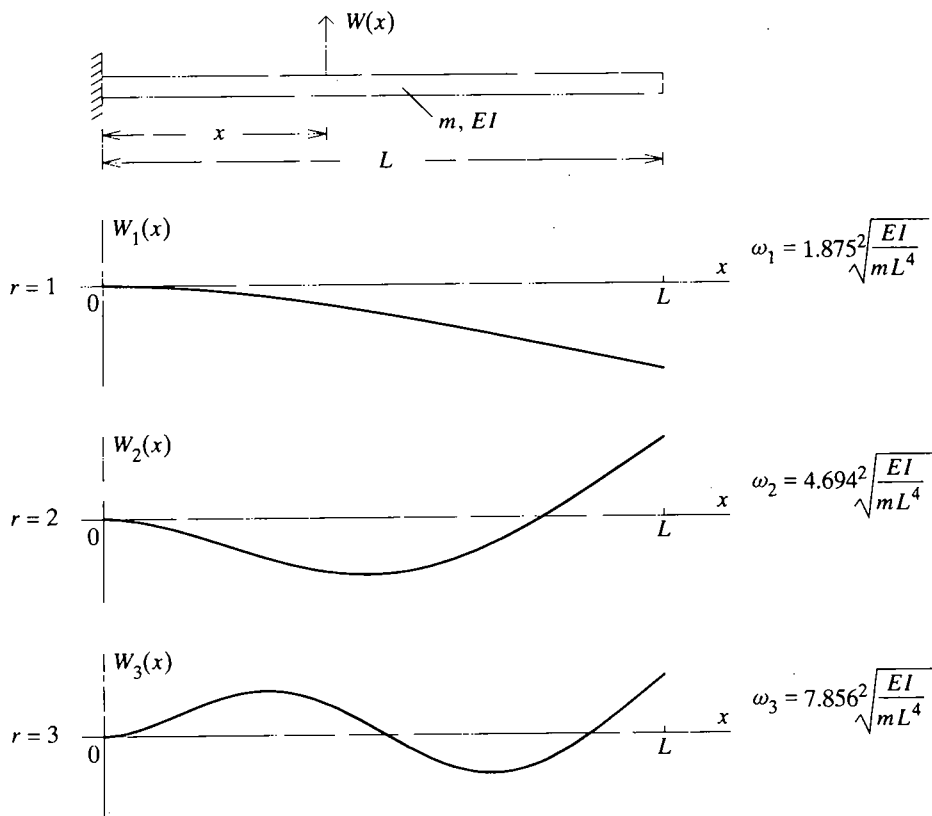


Figure 7.10 Cantilever beam and the first three modes of vibration

Using results obtained in Example 7.4, it is not difficult to show that the system is self-adjoint, so that the eigenfunctions are real and orthogonal. Moreover, the energy norm squared is given by

$$\|W\|_E^2 = \int_0^L EI(x) \left[\frac{d^2 W(x)}{dx^2} \right]^2 dx \geq 0 \quad (7.166)$$

and it is easy to see that the integral is equal to zero if W is either a constant or a linear function of x . We observe that boundary conditions (7.165) permit such solutions. Under these circumstances, *the system is only positive semidefinite*, so that *the system admits eigensolutions in the form of rigid-body modes with zero natural frequencies*. To examine the nature of the rigid-body modes, we let $\lambda = \lambda_0 = 0$ in Eq. (7.164) and write

$$\frac{d^2}{dx^2} \left[EI(x) \frac{d^2 W(x)}{dx^2} \right] = 0, \quad 0 < x < L \quad (7.167)$$

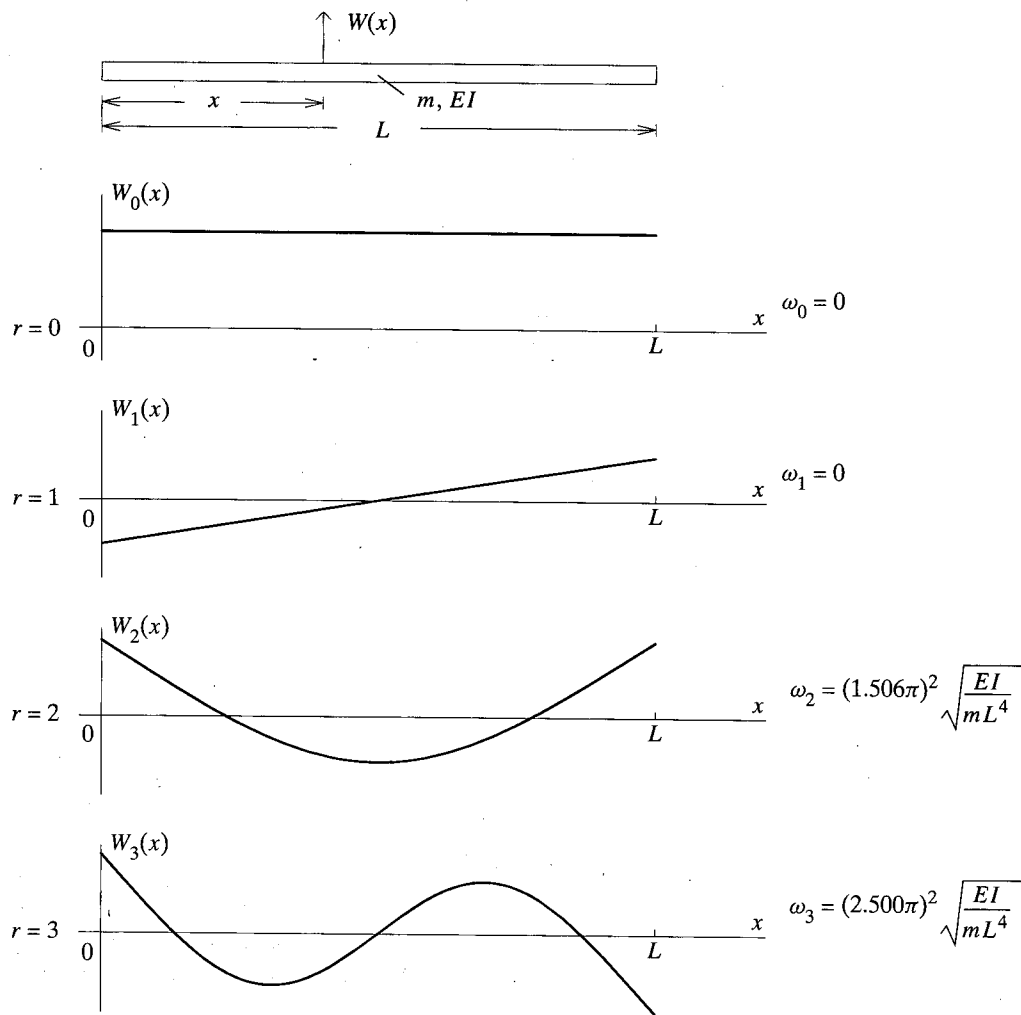


Figure 7.11 Free-free beam, the two rigid-body modes and the first two elastic modes

Integrating Eq. (7.167) once and using boundary conditions (7.165b) and (7.165d), we obtain

$$\frac{d}{dx} \left[EI(x) \frac{d^2 W(x)}{dx^2} \right] = 0, \quad 0 < x < L \quad (7.168)$$

One more integration in conjunction with boundary conditions (7.165a) and (7.165c) yields

$$EI(x) \frac{d^2 W(x)}{dx^2} = 0, \quad 0 < x < L \quad (7.169)$$

Then, dividing by $EI(x)$ and integrating twice, we can write

$$W(x) = D_1 + D_2 x \quad (7.170)$$

Because $W(x)$ contains two independent constants of integration, we conclude that there are two rigid-body modes. It is convenient to identify them as the transverse translation of the mass center C and rotation about C . Upon the usual normalization, the two rigid-body modes can be shown to have the form

$$W_0(x) = A_0 = \frac{1}{\sqrt{mL}} \quad (7.171a)$$

$$W_1(x) = A_1(x - x_C) = \frac{1}{\sqrt{I_C}}(x - x_C) \quad (7.171b)$$

where I_C is the mass moment of inertia of the beam about C and x_C is the distance between the left end and the mass center C .

Following the pattern established in Sec. 7.6, for zero resultants of external forces and moments about C , Newton's second law for the motion in the r th mode can be written as

$$\begin{aligned} F(t) &= \int_0^L m(x) \frac{\partial^2 w_r(x, t)}{\partial t^2} dx \\ &= - \left[\int_0^L m(x) W_r(x) dx \right] c_r \omega_r^2 \cos(\omega_r t - \phi_r) = 0, \\ & \quad r = 2, 3, \dots \end{aligned} \quad (7.172a)$$

$$\begin{aligned} M_C(t) &= \int_0^L m(x)(x - x_C) \frac{\partial^2 w_r(x, t)}{\partial t^2} dx \\ &= - \left[\int_0^L m(x)(x - x_C) W_r(x) dx \right] c_r \omega_r^2 \cos(\omega_r t - \phi_r) = 0, \\ & \quad r = 2, 3, \dots \end{aligned} \quad (7.172b)$$

In view of Eqs. (7.171), Eqs. (7.172) amount to the orthogonality of the rigid-body modes to the elastic modes. Upon normalization, the orthonormality relations for all modes are

$$\int_0^L m(x) W_r(x) W_s(x) dx = \delta_{rs}, \quad r = 0, 1, 2, \dots \quad (7.173a)$$

$$\int_0^L W_r(x) \frac{d^2}{dx^2} \left[EI(x) \frac{d^2 W_s(x)}{dx^2} \right] dx = \lambda_r \delta_{rs}, \quad r = 0, 1, 2, \dots \quad (7.173b)$$

Next, we wish to solve the eigenvalue problem for the elastic modes of the *uniform free-free beam*. The solution is once again given by Eq. (7.149), whereas the boundary conditions, Eqs. (7.165), reduce to

$$\frac{d^2 W(x)}{dx^2} = 0, \quad \frac{d^3 W(x)}{dx^3} = 0, \quad x = 0, L \quad (7.174)$$

Inserting Eq. (7.149) into Eqs. (7.174) and using the same pattern as earlier in this section, we obtain the characteristic equation

$$\cos \beta L \cosh \beta L = 1 \quad (7.175)$$

which has the roots $\beta_0 L = \beta_1 L = 0$, $\beta_2 L = 1.506\pi$, $\beta_3 L = 2.500\pi, \dots$, and note that for large r the eigenvalues approach $(2r - 1)\pi/2$. The eigenfunctions belonging to $\beta_r L$ ($r = 2, 3, \dots$) are

$$\begin{aligned} W_r(x) = A_r [& (\cos \beta_r L - \cosh \beta_r L) (\sin \beta_r x + \sinh \beta_r x) \\ & - (\sin \beta_r L - \sinh \beta_r L) (\cos \beta_r x + \cosh \beta_r x)] , \\ & r = 2, 3, \dots \end{aligned} \quad (7.176)$$

The first four natural modes and natural frequencies are displayed in Fig. 7.11.

7.8 EXTENSIONS OF LAGRANGE'S EQUATION FOR DISTRIBUTED SYSTEMS

The Lagrange equation for distributed systems derived in Sec. 7.3 is somewhat limited, as it excludes certain effects that cannot always be ignored. The reason for this is mostly pedagogical, as the inclusion of these effects tends to raise the level of difficulty of the formulation. In this section, we propose to extend the Lagrange equation so as to include these effects. In carrying out the extension we follow a parallel path to the one of Sec. 7.3.

As in Sec. 7.3, we wish to derive the Lagrange equation by means of the extended Hamilton's principle, Eq. (7.31). The additional effects to be included here can all be accounted for in the kinetic energy, so that we replace Eq. (7.32) by

$$T = T_0 [\dot{w}(0, t), \dot{w}'(0, t)] + T_L [\dot{w}(L, t), \dot{w}'(L, t)] + \int_0^L \hat{T}(\dot{w}, \dot{w}') dx \quad (7.177)$$

and we note that the kinetic energy terms T_0 and T_L are designed to account for the effect of lumped boundary masses in translation and rotation and the kinetic energy density \hat{T} includes the rotation of a differential element of mass, in addition to the translation considered in Eq. (7.32). On the other hand, the potential energy remains in the form given by Eq. (7.33). Combining Eqs. (7.33) and (7.177), the Lagrangian can be expressed as

$$\begin{aligned} L = & L_0 [w(0, t), w'(0, t), \dot{w}(0, t), \dot{w}'(0, t)] \\ & + L_L [w(L, t), w'(L, t), \dot{w}(L, t), \dot{w}'(L, t)] \\ & + \int_0^L \hat{L}(w, w', w'', \dot{w}, \dot{w}') dx \end{aligned} \quad (7.178)$$

The virtual work remains as given by Eq. (7.35).

The extended Hamilton's principle requires the variation in the Lagrangian, which retains the form given by Eq. (7.36), except that the individual terms, Eqs. (7.37), must be augmented as follows:

$$\begin{aligned} \delta L_0 &= \frac{\partial L_0}{\partial w(0, t)} \delta w(0, t) + \frac{\partial L_0}{\partial w'(0, t)} \delta w'(0, t) \\ &\quad + \frac{\partial L_0}{\partial \dot{w}(0, t)} \delta \dot{w}(0, t) + \frac{\partial L_0}{\partial \dot{w}'(L, t)} \delta \dot{w}'(L, t) \end{aligned} \quad (7.179a)$$

$$\begin{aligned} \delta L_L &= \frac{\partial L_L}{\partial w(L, t)} \delta w(L, t) + \frac{\partial L_L}{\partial w'(L, t)} \delta w'(L, t) \\ &\quad + \frac{\partial L_L}{\partial \dot{w}(L, t)} \delta \dot{w}(L, t) + \frac{\partial L_L}{\partial \dot{w}'(L, t)} \delta \dot{w}'(L, t) \end{aligned} \quad (7.179b)$$

$$\delta \hat{L} = \frac{\partial \hat{L}}{\partial w} \delta w + \frac{\partial \hat{L}}{\partial w'} \delta w' + \frac{\partial \hat{L}}{\partial w''} \delta w'' + \frac{\partial \hat{L}}{\partial \dot{w}} \delta \dot{w} + \frac{\partial \hat{L}}{\partial \dot{w}'} \delta \dot{w}' \quad (7.179c)$$

and we note that each of Eqs. (7.179a) and (7.179b) have two extra terms compared to Eqs. (7.37a) and (7.37b), and Eq. (7.179c) has one additional term. The next step in the use of the extended Hamilton's principle is to carry out integrations by parts with respect to x and t so as to produce variations in translational and rotational displacements alone. Many of these steps are given by Eqs. (7.38) and (7.39), so that it is necessary to carry out the integrations by parts only for the additional terms in Eqs. (7.179). To this end, we integrate with respect to t , recall that the variations vanish at $t = t_1, t_2$ and write

$$\begin{aligned} &\int_{t_1}^{t_2} \frac{\partial L_0}{\partial \dot{w}(0, t)} \delta \dot{w}(0, t) dt \\ &= \frac{\partial L_0}{\partial \dot{w}(0, t)} \delta w(0, t) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(\frac{\partial L_0}{\partial \dot{w}(0, t)} \right) \delta w(0, t) dt \\ &= - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(\frac{\partial L_0}{\partial \dot{w}(0, t)} \right) \delta w(0, t) dt \end{aligned} \quad (7.180a)$$

$$\begin{aligned} &\int_{t_1}^{t_2} \frac{\partial L_0}{\partial \dot{w}'(0, t)} \delta \dot{w}'(0, t) dt \\ &= \frac{\partial L_0}{\partial \dot{w}'(0, t)} \delta w'(0, t) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(\frac{\partial L_0}{\partial \dot{w}'(0, t)} \right) \delta w'(0, t) dt \\ &= - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(\frac{\partial L_0}{\partial \dot{w}'(0, t)} \right) \delta w'(0, t) dt \end{aligned} \quad (7.180b)$$

$$\int_{t_1}^{t_2} \frac{\partial L_L}{\partial \dot{w}(L, t)} \delta \dot{w}(L, t) dt$$

$$\begin{aligned}
&= \frac{\partial L_L}{\partial \dot{w}(L, t)} \delta w(L, t) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(\frac{\partial L_L}{\partial \dot{w}(L, t)} \right) \delta w(L, t) dt \\
&= - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(\frac{\partial L_L}{\partial \dot{w}(L, t)} \right) \delta w(L, t) dt \tag{7.180c}
\end{aligned}$$

$$\begin{aligned}
&\int_{t_1}^{t_2} \frac{\partial L_L}{\partial \dot{w}'(L, t)} \delta \dot{w}'(L, t) dt \\
&= \frac{\partial L_L}{\partial \dot{w}'(L, t)} \delta w'(L, t) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(\frac{\partial L_L}{\partial \dot{w}'(L, t)} \right) \delta w'(L, t) dt \\
&= - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(\frac{\partial L_L}{\partial \dot{w}'(L, t)} \right) \delta w'(L, t) dt \tag{7.180d}
\end{aligned}$$

The next term involves integrations both with respect to x and t , as well as changes in the order of these integrations, as follows:

$$\begin{aligned}
&\int_{t_1}^{t_2} \int_0^L \frac{\partial \hat{L}}{\partial \dot{w}'} \delta \dot{w}' dx dt = \int_0^L \left(\int_{t_1}^{t_2} \frac{\partial \hat{L}}{\partial \dot{w}'} \delta \dot{w}' dt \right) dx \\
&= \int_0^L \left[\frac{\partial \hat{L}}{\partial \dot{w}'} \delta w' \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}'} \right) \delta w' dt \right] dx \\
&= - \int_{t_1}^{t_2} \left[\int_0^L \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}'} \right) \delta w' dx \right] dt \\
&= - \int_{t_1}^{t_2} \left[\frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}'} \right) \delta w \Big|_0^L - \int_0^L \frac{\partial^2}{\partial x \partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}'} \right) \delta w dx \right] dt \tag{7.181}
\end{aligned}$$

Introducing Eqs. (7.35) and (7.36) into Eq. (7.31), considering Eqs. (7.38), (7.39), (7.180) and (7.181) and collecting terms involving $\delta w(x, t)$, $\delta w(0, t)$, $\delta w(L, t)$, $\delta \dot{w}'(0, t)$ and $\delta \dot{w}'(L, t)$, we obtain the replacement of Eq. (7.40) in the form

$$\begin{aligned}
&\int_{t_1}^{t_2} \left\{ \int_0^L \left[\frac{\partial \hat{L}}{\partial w} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w'} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial \hat{L}}{\partial w''} \right) \right. \right. \\
&\quad \left. \left. - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}} \right) + \frac{\partial^2}{\partial x \partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}'} \right) + f \right] \delta w dx \right. \\
&\quad \left. + \left[\frac{\partial L_0}{\partial w(0, t)} - \frac{\partial}{\partial t} \left(\frac{\partial L_0}{\partial \dot{w}(0, t)} \right) \right] \delta w(0, t) \right. \\
&\quad \left. + \left[\frac{\partial L_0}{\partial w'(0, t)} - \frac{\partial}{\partial t} \left(\frac{\partial L_0}{\partial \dot{w}'(0, t)} \right) \right] \delta \dot{w}'(0, t) \right.
\end{aligned}$$

$$\begin{aligned}
 & + \left[\frac{\partial L_L}{\partial w(L, t)} - \frac{\partial}{\partial t} \left(\frac{\partial L_L}{\partial \dot{w}(L, t)} \right) \right] \delta w(L, t) \\
 & + \left[\frac{\partial L_L}{\partial w'(L, t)} - \frac{\partial}{\partial t} \left(\frac{\partial L_L}{\partial \dot{w}'(L, t)} \right) \right] \delta w'(L, t) \\
 & + \left[\frac{\partial \hat{L}}{\partial w'} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w''} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}'} \right) \right] \delta w \Big|_0^L + \frac{\partial \hat{L}}{\partial w''} \delta w' \Big|_0^L \Big\} dt \\
 = & \int_{t_1}^{t_2} \left\langle \int_0^L \left[\frac{\partial \hat{L}}{\partial w} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w'} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial \hat{L}}{\partial w''} \right) \right. \right. \\
 & \left. \left. - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}} \right) + \frac{\partial^2}{\partial x \partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}'} \right) + f \right] \delta w \right. \\
 & + \left\{ \frac{\partial L_0}{\partial w(0, t)} - \frac{\partial}{\partial t} \left(\frac{\partial L_0}{\partial \dot{w}(0, t)} \right) \right. \\
 & \left. - \left[\frac{\partial \hat{L}}{\partial w'} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w''} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}'} \right) \right] \Big|_{x=0} \right\} \delta w(0, t) \\
 & + \left\{ \frac{\partial L_L}{\partial w(L, t)} - \frac{\partial}{\partial t} \left(\frac{\partial L_L}{\partial \dot{w}(L, t)} \right) \right. \\
 & \left. + \left[\frac{\partial \hat{L}}{\partial w'} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w''} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}'} \right) \right] \Big|_{x=L} \right\} \delta w(L, t) \\
 & + \left[\frac{\partial L_0}{\partial w'(0, t)} - \frac{\partial}{\partial t} \left(\frac{\partial L_0}{\partial \dot{w}'(0, t)} \right) - \frac{\partial \hat{L}}{\partial w''} \Big|_{x=0} \right] \delta w'(0, t) \\
 & + \left[\frac{\partial L_L}{\partial w'(L, t)} - \frac{\partial}{\partial t} \left(\frac{\partial L_L}{\partial \dot{w}'(L, t)} \right) \right. \\
 & \left. + \frac{\partial \hat{L}}{\partial w''} \Big|_{x=L} \right] \delta w'(L, t) \Big\rangle dt = 0 \tag{7.182}
 \end{aligned}$$

Then, invoking the arbitrariness of the virtual displacements in a manner similar to that in Sec. 7.3, we conclude that Eq. (7.182) can be satisfied for all δw over the open domain $0 < x < L$ if and only if

$$\begin{aligned}
 \frac{\partial \hat{L}}{\partial w} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w'} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\partial \hat{L}}{\partial w''} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}} \right) + \frac{\partial^2}{\partial x \partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}'} \right) + f = 0, \\
 0 < x < L \tag{7.183}
 \end{aligned}$$

Moreover, by writing

$$\left\{ \frac{\partial L_0}{\partial w(0, t)} - \frac{\partial}{\partial t} \left(\frac{\partial L_0}{\partial \dot{w}(0, t)} \right) - \left[\frac{\partial \hat{L}}{\partial w'} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w''} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}'} \right) \right] \Big|_{x=0} \right\} \delta w(0, t) = 0 \quad (7.184a)$$

$$\left[\frac{\partial L_0}{\partial w'(0, t)} - \frac{\partial}{\partial t} \left(\frac{\partial L_0}{\partial \dot{w}'(0, t)} \right) - \frac{\partial \hat{L}}{\partial w''} \Big|_{x=0} \right] \delta w'(0, t) = 0 \quad (7.184b)$$

and

$$\left\{ \frac{\partial L_L}{\partial w(L, t)} - \frac{\partial}{\partial t} \left(\frac{\partial L_L}{\partial \dot{w}(L, t)} \right) + \left[\frac{\partial \hat{L}}{\partial w'} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w''} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}'} \right) \right] \Big|_{x=L} \right\} \delta w(L, t) = 0 \quad (7.185a)$$

$$\left[\frac{\partial L_L}{\partial w'(L, t)} - \frac{\partial}{\partial t} \left(\frac{\partial L_L}{\partial \dot{w}'(L, t)} \right) + \frac{\partial \hat{L}}{\partial w''} \Big|_{x=L} \right] \delta w'(L, t) = 0 \quad (7.185b)$$

we take into account that either $\delta w(0, t)$ or its coefficient is zero and either $\delta w'(0, t)$ or its coefficient is zero, and similar statements can be made about conditions at the end $x = L$.

Equation (7.183) represents *Lagrange's differential equation of motion* corresponding to the extended Lagrangian given by Eq. (7.178). Moreover, Eqs. (7.184) and (7.185) can be used to obtain a variety of possible boundary conditions. Indeed, from Eqs. (7.184) we conclude that at $x = 0$ either

$$\frac{\partial L_0}{\partial w(0, t)} - \frac{\partial}{\partial t} \left(\frac{\partial L_0}{\partial \dot{w}(0, t)} \right) - \left[\frac{\partial \hat{L}}{\partial w'} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w''} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}'} \right) \right] \Big|_{x=0} = 0 \quad (7.186a)$$

or

$$w = 0 \quad (7.186b)$$

and either

$$\frac{\partial L_0}{\partial w'(0, t)} - \frac{\partial}{\partial t} \left(\frac{\partial L_0}{\partial \dot{w}'(0, t)} \right) - \frac{\partial \hat{L}}{\partial w''} \Big|_{x=0} = 0 \quad (7.187a)$$

or

$$w' = 0 \quad (7.187b)$$

In addition, from Eqs. (7.185), at $x = L$ either

$$\frac{\partial L_L}{\partial w(L, t)} - \frac{\partial}{\partial t} \left(\frac{\partial L_L}{\partial \dot{w}(L, t)} \right) + \left[\frac{\partial \hat{L}}{\partial w'} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w''} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}'} \right) \right] \Big|_{x=L} = 0 \quad (7.188a)$$

or

$$w = 0 \quad (7.188b)$$

and either

$$\frac{\partial L_L}{\partial w'(L, t)} - \frac{\partial}{\partial t} \left(\frac{\partial L_L}{\partial \dot{w}'(L, t)} \right) + \frac{\partial \hat{L}}{\partial w''} \Big|_{x=L} = 0 \quad (7.189a)$$

or

$$w' = 0 \quad (7.189b)$$

The Lagrange equation together with appropriate boundary conditions constitute a boundary-value problem. The formulation consisting of Eqs. (7.183)–(7.189) is suitable for fourth-order systems, but it can be used for second-order systems as well by merely omitting terms and boundary conditions that do not apply. In the case of a fourth-order system, the boundary-value problem consists of Lagrange's equation, Eq. (7.183), and two boundary conditions at each end, namely, one from each of Eqs. (7.186)–(7.189). On the other hand, the boundary-value problem for second-order systems consists of the differential equation, Eq. (7.183), with the third and fifth term removed and one boundary condition at each end, one from each of Eqs. (7.186) and Eqs. (7.188), where the fourth and fifth term are deleted from Eqs. (7.186a) and (7.188a).

Example 7.5

Derive the boundary-value problem for a shaft in torsional vibration with the left end clamped and with the right end supporting a disk of mass moment of inertia I_D , as shown in Fig. 7.12.

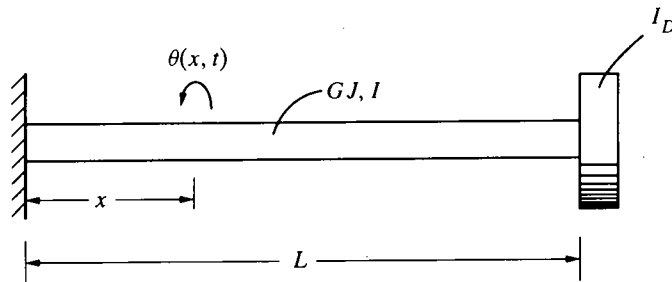


Figure 7.12 Shaft in torsional vibration clamped at $x = 0$ and with a disk at $x = L$

The kinetic energy has the expression

$$T(t) = \frac{1}{2} \int_0^L I(x) \left[\frac{\partial \theta(x, t)}{\partial t} \right]^2 dx + \frac{1}{2} I_D \dot{\theta}^2(L, t) \quad (a)$$

and the potential energy is simply

$$V(t) = \frac{1}{2} \int_0^L GJ(x) \left[\frac{\partial \theta(x, t)}{\partial x} \right]^2 dx \quad (b)$$

so that the Lagrangian can be written in the form

$$L = L_L + \int_0^L \hat{L} dx \quad (c)$$

where the boundary Lagrangian is given by

$$L_L = \frac{1}{2} I_D \dot{\theta}^2(L, t) \quad (d)$$

and the Lagrangian density by

$$\hat{L} = \hat{T} - \hat{V} = \frac{1}{2} I(x) \dot{\theta}^2(x, t) - \frac{1}{2} GJ(x) [\theta'(x, t)]^2 \quad (e)$$

This being a second-order system, Lagrange's equation, Eq. (7.183), reduces to

$$\frac{\partial \hat{L}}{\partial \theta} - \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial \theta'} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{\theta}} \right) + m = 0, \quad 0 < x < L \quad (f)$$

in which $m = m(x, t)$ is a distributed torque. Inserting Eq. (e) into Eq. (f) and recognizing that $I(x)$ does not depend on t , we obtain the partial differential equation of motion

$$\frac{\partial}{\partial x} \left(GJ \frac{\partial \theta}{\partial x} \right) - I \frac{\partial^2 \theta}{\partial t^2} + m = 0, \quad 0 < x < L \quad (g)$$

In view of the fact that the left end is clamped, according to Eq. (7.186b), the boundary condition at $x = 0$ is

$$\theta(0, t) = 0 \quad (h)$$

Moreover, the boundary condition at $x = L$ is given by Eq. (7.188a) with appropriate deletions, or

$$\frac{\partial L_L}{\partial \theta(L, t)} - \frac{\partial}{\partial t} \left(\frac{\partial L_L}{\partial \dot{\theta}(L, t)} \right) + \frac{\partial \hat{L}}{\partial \theta'} \Big|_{x=L} = 0 \quad (i)$$

Hence, inserting Eqs. (d) and (e) into Eq. (i), the boundary condition at $x = L$ is

$$I_D \ddot{\theta}(L, t) + GJ(x) \theta'(x, t) \Big|_{x=L} = 0 \quad (j)$$

We observe that, whereas there is nothing unusual about the differential equation, Eq. (g), and the boundary condition at $x = 0$, Eq. (h), the boundary condition at $x = L$, Eq. (j), depends on the acceleration $\ddot{\theta}(L, t)$. As a result, the associated differential eigenvalue problem does not fit the general mold defined by Eqs. (7.68) and (7.69), so that a more general formulation is required to accommodate the system of Fig. 7.12.

Example 7.6

Derive the boundary-value problem for the beam in bending vibration considered in Sec. 7.2 under the assumption that the kinetic energy of rotation is not negligible.

The various terms are the same as in Sec. 7.2, except that the kinetic energy, Eq. (7.18), must be replaced by

$$T(t) = \frac{1}{2} \int_0^L \left\{ m(x) \left[\frac{\partial w(x, t)}{\partial t} \right]^2 + J(x) \left[\frac{\partial^2 w(x, t)}{\partial t \partial x} \right]^2 \right\} dx \quad (a)$$

where $J(x)$ is the mass moment of inertia per unit length of beam. Hence, combining Eqs. (a) and (7.19), we can write the Lagrangian density

$$\hat{L} = \hat{T} - \hat{V} = \frac{1}{2} \left[m \dot{w}^2 + J (\dot{w}')^2 - EI (w'')^2 - P (w')^2 \right] \quad (b)$$

and we note that there are no boundary Lagrangians.

Inserting Eq. (b) into Eq. (7.183), we obtain the explicit Lagrange equation

$$\frac{\partial}{\partial x} \left(P \frac{\partial w}{\partial x} \right) - \frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 w}{\partial x^2} \right) - m \frac{\partial^2 w}{\partial t^2} + \frac{\partial}{\partial x} \left(J \frac{\partial^3 w}{\partial x \partial t^2} \right) + f = 0, \quad 0 < x < L \quad (c)$$

Moreover, introducing Eq. (b) into Eqs. (7.186)–(7.189), we conclude that at $x = 0, L$ either

$$P \frac{\partial w}{\partial x} - \frac{\partial}{\partial x} \left(EI \frac{\partial^2 w}{\partial x^2} \right) + J \frac{\partial^3 w}{\partial x \partial t^2} = 0 \quad (d)$$

must be satisfied or

$$w = 0 \quad (e)$$

and either

$$EI \frac{\partial^2 w}{\partial x^2} = 0 \quad (f)$$

or

$$w' = 0 \quad (g)$$

for a total of two boundary conditions at each end.

From the differential equation, Eq. (c), we observe that there are two terms involving accelerations, and one of them involves spatial derivatives. Hence the mass density is no longer a simple function, but a differential expression. Moreover, from Eq. (d), we conclude that, if this boundary condition applies, then it depends on the angular acceleration. Note that the added terms involve J and are referred to as *rotatory inertia* terms. The differential eigenvalue problem for a beam in bending with the rotatory inertia included cannot be accommodated by the formulation of Sec. 7.5, and a generalization of the formulation is necessary.

7.9 GENERALIZATION OF THE DIFFERENTIAL EIGENVALUE PROBLEM FOR SELF-ADJOINT SYSTEMS

From Sec. 7.8, we conclude that the eigenvalue problem given by Eqs. (7.68) and (7.69) cannot accommodate a number of important problems. In view of this, we consider a more general eigenvalue problem and express the differential equation in the operator form

$$Lw = \lambda Mw, \quad x, y \text{ in } D \quad (7.190)$$

where L and M are linear homogeneous differential operators of order $2p$ and $2q$, and are referred to as *stiffness operator and mass operator*, respectively, λ is a parameter and D is the domain of definition of Eq. (7.190). The operators L and M are of the type (7.66) and their order is such that $p > q$. Associated with the differential equation (7.190) there are p boundary conditions to be satisfied by the solution w at every point of the boundary S of the domain D . The boundary conditions are of the type

$$B_i w = 0, \quad x, y \text{ on } S, \quad i = 1, 2, \dots, k \quad (7.191a)$$

$$B_i w = \lambda C_i w, \quad x, y \text{ on } S, \quad i = k + 1, k + 2, \dots, p \quad (7.191b)$$

where B_i and C_i are linear homogeneous differential *boundary operators* of maximum order $2p - 1$ and $2q - 1$, respectively.

Next, we consider two comparison functions u and v and state that *the stiffness operator L is self-adjoint* if

$$\int_D uLv dD + \sum_{i=k+1}^p \int_S uB_i v dS = \int_D vLu dD + \sum_{i=k+1}^p \int_S vB_i u dS \quad (7.192a)$$

Moreover, *the mass operator M is self-adjoint* if

$$\int_D uMv dD + \sum_{i=k+1}^p \int_S uC_i v dS = \int_D vMu dD + \sum_{i=k+1}^p \int_S vC_i u dS \quad (7.192b)$$

If L and M are self-adjoint, *the system, or the eigenvalue problem, is said to be self-adjoint*. As demonstrated in Sec. 7.5, self-adjointness can be ascertained through integration by parts with due consideration to the boundary conditions, and it implies certain mathematical symmetry. In fact, the concept of self-adjointness of the stiffness operator L and mass operator M in distributed systems is entirely analogous to the concept of symmetry of the stiffness matrix K and mass matrix M in discrete systems. This mathematical symmetry can be used to simplify the test for self-adjointness, as shown in Sec. 7.5. Indeed, if the left side of Eq. (7.192a), or of Eq. (7.192b), can be reduced to a symmetric form in u and v through integrations by parts, then the operator L , or operator M , is self-adjoint, and it is not really necessary to carry out the integrations on the right side of Eq. (7.192a), or Eq. (7.192b), as they are guaranteed to yield the same symmetric forms. For one-dimensional self-adjoint systems, we denote the symmetric result of the integrations by parts for the operator L by

$$\begin{aligned} [u, v]_P &= \int_D uLv dD + \sum_{i=k+1}^p \int_S uB_i v dS \\ &= \int_0^L \sum_{k=0}^p a_k \frac{d^k u}{dx^k} \frac{d^k v}{dx^k} dx + \sum_{\ell=0}^{p-1} b_\ell \frac{d^\ell u}{dx^\ell} \frac{d^\ell v}{dx^\ell} \Big|_0^L \end{aligned} \quad (7.193a)$$

and for the operator M by

$$\begin{aligned} [u, v]_K &= \int_D uMv dD + \sum_{i=k+1}^p \int_S uC_i v dS \\ &= \int_0^L \sum_{k=0}^q e_k \frac{d^k u}{dx^k} \frac{d^k v}{dx^k} dx + \sum_{\ell=0}^{q-1} f_\ell \frac{d^\ell u}{dx^\ell} \frac{d^\ell v}{dx^\ell} \Big|_0^L \end{aligned} \quad (7.193b)$$

where $[u, v]_P$ and $[u, v]_K$ will be referred to as *potential and kinetic energy inner products*, respectively, in which a_k , b_ℓ , e_k and f_ℓ are coefficients depending in general on x .

If for any comparison function u we have the inequality

$$\int_D uLu dD + \sum_{i=k+1}^p \int_S uB_i u dS \geq 0 \quad (7.194a)$$

and the equality sign holds if and only if $u \equiv 0$, then *the operator L is said to be positive definite*. If the expression can be zero without u being identically zero, then *the operator L is only positive semidefinite*. Similarly, if

$$\int_D u M u dD + \sum_{i=k+1}^p \int_S u C_i u dS \geq 0 \quad (7.194b)$$

and the equality holds if and only if $u \equiv 0$, *the operator M is positive definite*, and if the expression is zero without u being identically zero, *the operator M is only positive semidefinite*. If L and M are positive definite (semidefinite), then the system, or the eigenvalue problem, is positive definite (semidefinite). Unless otherwise stated, we will be concerned exclusively with systems for which M is positive definite. Hence, the sign properties of the system are governed by the sign properties of the stiffness operator L .

For $v = u$, Eqs. (7.193) reduce to

$$[u, u]_P = \int_0^L \sum_{k=0}^p a_k \left(\frac{d^k u}{dx^k} \right)^2 dx + \sum_{\ell=0}^{p-1} b_\ell \left(\frac{d^\ell u}{dx^\ell} \right)^2 \Big|_0^L \quad (7.195a)$$

$$[u, u]_K = \int_0^L \sum_{k=0}^q e_k \left(\frac{d^k u}{dx^k} \right)^2 dx + \sum_{\ell=0}^{q-1} f_\ell \left(\frac{d^\ell u}{dx^\ell} \right)^2 \Big|_0^L \quad (7.195b)$$

and we note that $[u, u]_P$ and $[u, u]_K$ are measures of the potential and kinetic energy, respectively, which explains the terms of potential and kinetic energy inner products for $[u, v]_P$ and $[u, v]_K$ introduced earlier. Equations (7.195) can be used to define the *potential and kinetic energy norms*

$$\|u\|_P = [u, u]_P^{1/2}, \quad \|u\|_K = [u, u]_K^{1/2} \quad (7.196a, b)$$

respectively.

Next, we introduce the sequence of approximations

$$u_n = \sum_{r=1}^n c_r \phi_r, \quad n = 1, 2, \dots \quad (7.197)$$

where ϕ_1, ϕ_2, \dots are given functions from an independent set. Then, if by choosing n sufficiently large,

$$\|u - u_n\|_P < \epsilon_P, \quad \|u - u_n\|_K < \epsilon_K \quad (7.198a, b)$$

where ϵ_P and ϵ_K are arbitrarily small positive numbers, the set of functions ϕ_1, ϕ_2, \dots is said to be *complete in energy*. Moreover, if

$$\lim_{n \rightarrow \infty} \|u - u_n\|_P = 0, \quad \lim_{n \rightarrow \infty} \|u - u_n\|_K = 0 \quad (7.199a, b)$$

the sequence of approximations u_1, u_2, \dots is said to *converge in energy* to u .

Next, we wish to examine the properties of the eigenvalues and eigenfunctions. Assuming that the problem is self-adjoint and inserting two distinct solutions λ_r, w_r

and λ_s , w_s of the eigenvalue problem, Eqs. (7.190) and (7.191), into Eq. (7.190), we can write

$$Lw_r = \lambda_r M w_r, \quad Lw_s = \lambda_s M w_s \quad (7.200a, b)$$

Multiplying Eq. (7.200a) by w_s and Eq. (7.200b) by w_r , subtracting the second from the first and integrating over domain D , we obtain

$$\int_D (w_s L w_r - w_r L w_s) dD = \int_D (\lambda_r w_s M w_r - \lambda_s w_r M w_s) dD \quad (7.201)$$

But, because the operators L and M are self-adjoint, we can use Eqs. (7.192) and (7.191b) to write

$$\begin{aligned} \int_D (w_s L w_r - w_r L w_s) dD &= \sum_{i=k+1}^p \int_S (w_r B_i w_s - w_s B_i w_r) dS \\ &= \sum_{i=k+1}^p \int_S (\lambda_s w_r C_i w_s - \lambda_r w_s C_i w_r) dS \end{aligned} \quad (7.202a)$$

and

$$\int_D w_s M w_r dD = \int_D w_r M w_s dD + \sum_{i=k+1}^p \int_S (w_r C_i w_s - w_s C_i w_r) dS \quad (7.202b)$$

so that, inserting Eqs. (7.202) into Eq. (7.201) and rearranging, we have

$$(\lambda_r - \lambda_s) \left(\int_D w_r M w_s dD + \sum_{i=k+1}^p \int_S w_r C_i w_s dS \right) = 0 \quad (7.203)$$

But, by assumption, the eigenvalues λ_r and λ_s are distinct. Hence, Eq. (7.203) can be satisfied if and only if

$$\int_D w_r M w_s dD + \sum_{i=k+1}^p \int_S w_r C_i w_s dS = 0, \quad \lambda_r \neq \lambda_s, \quad r, s = 1, 2, \dots \quad (7.204)$$

Equation (7.204) represents the *orthogonality relation* for the eigenfunctions of distributed-parameter systems described by the eigenvalue problem given by Eqs. (7.190) and (7.191). Multiplying Eq. (7.200b) by w_r , integrating over D and using Eqs. (7.191b) and (7.204), it can be shown that the eigenfunctions satisfy a *second orthogonality relation*, namely,

$$\int_D w_r L w_s dD + \sum_{i=k+1}^p \int_S w_r B_i w_s dS = 0, \quad \lambda_r \neq \lambda_s, \quad r, s = 1, 2, \dots \quad (7.205)$$

Clearly, the general orthogonality of the eigenfunctions solving the eigenvalue problem given by Eqs. (7.190) and (7.191) applies to self-adjoint systems alone. If an eigenvalue has multiplicity m ; then there are exactly m eigenfunctions belonging to

the repeated eigenvalue, and these eigenfunctions are generally not orthogonal to one another, although they are independent and orthogonal to the remaining eigenfunctions of the system. But, as pointed out in Sec. 7.5, independent functions can be orthogonalized by grouping them in proper linear combinations. Hence, *all the eigenfunctions of a self-adjoint system can be regarded as orthogonal, regardless of whether there are repeated eigenvalues or not.*

The eigenvalue problem, Eqs. (7.190) and (7.191), is homogeneous, so that only the shape of the eigenfunctions is unique, and the amplitude is arbitrary. This arbitrariness can be removed through normalization. A mathematically convenient normalization scheme is given by

$$\int_D w_r M w_r dD + \sum_{i=k+1}^p \int_S w_r C_i w_r dS = 1, \quad r = 1, 2, \dots \quad (7.206a)$$

which implies that

$$\int_D w_r L w_r dD + \sum_{i=k+1}^p \int_S w_r B_i w_r dS = \lambda_r, \quad r = 1, 2, \dots \quad (7.206b)$$

Then, Eqs. (7.204)–(7.206) can be combined into the *orthonormality relations*

$$\int_D w_r M w_s dD + \sum_{i=k+1}^p \int_S w_r C_i w_s dS = \delta_{rs}, \quad r, s = 1, 2, \dots \quad (7.207a)$$

$$\int_D w_r L w_s dD + \sum_{i=k+1}^p \int_S w_r B_i w_s dS = \lambda_r \delta_{rs}, \quad r, s = 1, 2, \dots \quad (7.207b)$$

In Sec. 7.5, we demonstrated that the eigenvalues and eigenfunctions of a self-adjoint system are real. We propose to prove here that the same is true for the more general eigenvalue problem. To this end, we consider a complex solution λ , w of the eigenvalue problem, Eqs. (7.190) and (7.191). Because all the operators are real, if λ , w are a complex solution of the eigenvalue problem, then the complex conjugates $\bar{\lambda}$, \bar{w} must also be a solution, so that Eq. (7.190) yields

$$Lw = \lambda Mw, \quad L\bar{w} = \bar{\lambda} M\bar{w} \quad (7.208a, b)$$

Multiplying Eq. (7.208a) by \bar{w} and Eq. (7.208b) by w , subtracting the second from the first and integrating over D , we obtain

$$\int_D (\bar{w} L w - w L \bar{w}) dD = \lambda \int_D \bar{w} M w dD - \bar{\lambda} \int_D w M \bar{w} dD \quad (7.209)$$

Letting $v = w$ and $u = \bar{w}$ in Eqs. (7.191b) and (7.192) and inserting the results into Eq. (7.209), we obtain after some manipulations

$$(\lambda - \bar{\lambda}) \left(\int_D \bar{w} M w dD + \sum_{i=k+1}^p \int_S \bar{w} C_i w dS \right) = 0 \quad (7.210)$$

Recalling Eqs. (7.99) and considering Eq. (7.192b), we conclude that the term in the second parentheses in Eq. (7.210) is real and positive, so that the only alternative is

$$\lambda - \bar{\lambda} = \alpha + i\beta - (\alpha - i\beta) = 2i\beta = 0 \quad (7.211)$$

Hence, as in Sec. 7.5, we conclude that *the eigenvalues of a self-adjoint system are real. As a corollary, the eigenfunctions of a self-adjoint system are real.* Moreover, considering inequality (7.194a), we conclude from Eq. (7.206b) that, *if the operator L is positive definite (semidefinite), all the eigenvalues are positive (nonnegative).*

Finally, we wish to extend the *expansion theorem for self-adjoint systems* of Sec. 7.5 as follows: *Every function w with continuous Lw and Mw and satisfying the boundary conditions of the system can be expanded in an absolutely and uniformly convergent series in the eigenfunctions in the form*

$$w = \sum_{r=1}^{\infty} c_r w_r \quad (7.212)$$

where the coefficients c_r are such that

$$c_r = \int_D w_r M w \, dD + \sum_{i=k+1}^P \int_S w_r C_i w \, dS \quad (7.213a)$$

$$\lambda_r c_r = \int_D w_r L w \, dD + \sum_{i=k+1}^P \int_S w_r B_i w \, dS \quad (7.213b)$$

We should note here that Eqs. (7.213) are based on the more general orthonormality relations, Eqs. (7.207). The expansion theorem just presented forms the basis for a modal analysis for self-adjoint systems with stiffness and mass operators L and M , respectively, and with boundary conditions depending on the eigenvalue λ . Although the expansion theorem, Eqs. (7.212) and (7.213), seems intimidating compared to the expansion theorem of Sec. 7.5, the process of using modal analysis to derive the modal equations remains essentially the same. Derivation of the system response by modal analysis is discussed later in this chapter.

Example 7.7

Derive the eigenvalue problem for the shaft in torsional vibration of Example 7.5 and show how it fits the formulation given by Eqs. (7.190) and (7.191).

From Example 7.5, the free vibration of the shaft, obtained by letting the distributed torque m be equal to zero, is described by the partial differential equation

$$\frac{\partial}{\partial x} \left[GJ(x) \frac{\partial \theta(x, t)}{\partial x} \right] = I(x) \frac{\partial^2 \theta(x, t)}{\partial t^2}, \quad 0 < x < L \quad (a)$$

and the boundary conditions

$$\theta(0, t) = 0; \quad I_D \frac{\partial^2 \theta(x, t)}{\partial t^2} + GJ(x) \frac{\partial \theta(x, t)}{\partial x} = 0, \quad x = L \quad (b)$$

To derive the eigenvalue problem, we assume a solution in the form

$$\theta(x, t) = \Theta(x)F(t) \quad (c)$$

Introducing Eq. (c) into Eq. (a) and separating variables, we can write

$$\frac{1}{I(x)\Theta(x)} \frac{d}{dx} \left[GJ(x) \frac{d\Theta(x)}{dx} \right] = \frac{1}{F(t)} \frac{d^2 F(t)}{dt^2}, \quad 0 < x < L \quad (d)$$

Following the developments of Sec. 7.4, it can be shown that the function $F(t)$ is harmonic, and it satisfies

$$\frac{d^2 F(t)}{dt^2} = -\lambda F(t), \quad \lambda = \omega^2 \quad (e)$$

where ω is the frequency of oscillation, so that the left side of Eq. (d) yields the differential equation

$$-\frac{d}{dx} \left[GJ(x) \frac{d\Theta(x)}{dx} \right] = \lambda I(x)\Theta(x), \quad 0 < x < L \quad (f)$$

Moreover, inserting Eqs. (c) and (e) into Eqs. (b), we obtain the boundary conditions

$$\Theta(0) = 0; \quad GJ(x) \frac{d\Theta(x)}{dx} = \lambda I_D \Theta(x), \quad x = L \quad (g)$$

Equations (f) and (g) constitute the desired differential eigenvalue problem.

Contrasting Eqs. (7.190) and (f) on the one hand and Eqs. (7.191) and (g) on the other hand, we conclude that the eigenvalue problem does fit the mold. The various operators can be identified as follows:

$$\begin{aligned} L &= -\frac{d}{dx} \left[GJ(x) \frac{d}{dx} \right], & p &= 1; \quad M = I(x), & q &= 0 \\ B_1 &= 1, \quad C_1 = 0 \quad \text{at } x = 0, & k &= 1 \\ B_1 &= GJ(x) \frac{d}{dx}, \quad C_1 = I_D \quad \text{at } x = L, & k &= 0 \end{aligned} \quad (h)$$

A solution of the eigenvalue problem given by Eqs. (f) and (g) for a uniform shaft is presented in Sec. 7.10.

Example 7.8

Derive the eigenvalue problem for the beam in bending of Example 7.6 and show the relation with the formulation given by Eqs. (7.190) and (7.191). Assume that the beam is clamped at $x = 0$ and free at $x = L$.

Letting $f = 0$, the free vibration problem can be obtained from Example 7.6 in the form of the partial differential equation

$$\begin{aligned} \frac{\partial}{\partial x} \left[P(x) \frac{\partial w(x, t)}{\partial x} \right] - \frac{\partial^2}{\partial x^2} \left[EI(x) \frac{\partial^2 w(x, t)}{\partial x^2} \right] - m(x) \frac{\partial^2 w(x, t)}{\partial t^2} \\ + \frac{\partial}{\partial x} \left[J(x) \frac{\partial^3 w(x, t)}{\partial x \partial t^2} \right] = 0, \quad 0 < x < L \quad (a) \end{aligned}$$

and the boundary conditions

$$\begin{aligned}
 w(x, t) = 0, \quad \frac{\partial w(x, t)}{\partial x} = 0, \quad x = 0 \\
 EI(x) \frac{\partial^2 w(x, t)}{\partial x^2} = 0, \quad P(x) \frac{\partial w(x, t)}{\partial x} - \frac{\partial}{\partial x} \left[EI(x) \frac{\partial^2 w(x, t)}{\partial x^2} \right] \\
 + J(x) \frac{\partial^3 w(x, t)}{\partial x \partial t^2} = 0, \quad x = L
 \end{aligned} \tag{b}$$

To derive the eigenvalue problem, we assume that the solution is separable in x and t , or

$$w(x, t) = W(x)F(t) \tag{c}$$

Inserting Eq. (c) into Eq. (a) and following the usual steps, we obtain

$$\frac{\frac{d^2}{dx^2} \left[EI(x) \frac{d^2 W(x)}{dx^2} \right] - \frac{d}{dx} \left[P(x) \frac{dW(x)}{dx} \right]}{m(x)W(x) - \frac{d}{dx} \left[J(x) \frac{dW(x)}{dx} \right]} = -\frac{1}{F(t)} \frac{d^2 F(t)}{dt^2} \tag{d}$$

Then, using the standard argument, we let both sides of Eq. (d) be equal to $\lambda = \omega^2$, so that $F(t)$ is harmonic with the frequency ω . Moreover, we obtain the differential equation

$$\begin{aligned}
 \frac{d^2}{dx^2} \left[EI(x) \frac{d^2 W(x)}{dx^2} \right] - \frac{d}{dx} \left[P(x) \frac{dW(x)}{dx} \right] \\
 = \lambda \left\{ m(x)W(x) - \frac{d}{dx} \left[J(x) \frac{dW(x)}{dx} \right] \right\}, \quad 0 < x < L
 \end{aligned} \tag{e}$$

Similarly, introducing Eq. (c) into Eqs. (b), in conjunction with $\ddot{F}(t) = -\lambda F(t)$, and dividing through by $F(t)$, we obtain the boundary conditions

$$\begin{aligned}
 W(x) = 0, \quad \frac{dW(x)}{dx} = 0, \quad x = 0 \\
 EI(x) \frac{d^2 W(x)}{dx^2} = 0, \\
 -\frac{d}{dx} \left[EI(x) \frac{d^2 W(x)}{dx^2} \right] + P(x) \frac{dW(x)}{dx} = \lambda J(x)W(x), \quad x = L
 \end{aligned} \tag{f}$$

Equations (e) and (f) represent the differential eigenvalue problem for the system at hand.

Comparing Eqs. (e) and (f) with Eqs. (7.190) and (7.191), respectively, we can identify the various operators as follows:

$$\begin{aligned}
 L &= \frac{d^2}{dx^2} \left[EI(x) \frac{d^2}{dx^2} \right] - \frac{d}{dx} \left[P(x) \frac{d}{dx} \right], & p &= 2 \\
 M &= m(x) - \frac{d}{dx} \left[J(x) \frac{d}{dx} \right], & q &= 1 \\
 B_1 &= 1, \quad B_2 = \frac{d}{dx}, \quad C_1 = C_2 = 0, \quad k = 2, \quad x = 0, & (g) \\
 B_1 &= EI(x) \frac{d^2}{dx^2}, \quad B_2 = -\frac{d}{dx} \left[EI(x) \frac{d^2}{dx^2} \right] + P(x) \frac{d}{dx}, \\
 C_1 &= 0, \quad C_2 = J(x), \quad k = 1, \quad x = L
 \end{aligned}$$

Clearly, the differential eigenvalue problem, Eqs. (e) and (f), does fit the general formulation given by Eqs. (7.190) and (7.191).

Closed-form solutions to the eigenvalue problem given by Eqs. (e) and (f) do not exist.

7.10 SYSTEMS WITH BOUNDARY CONDITIONS DEPENDING ON THE EIGENVALUE

Let us return to the system shown in Fig. 7.12 and recall that the boundary-value problem was derived in Example 7.5 and the eigenvalue problem in Example 7.7. In this section, we consider the solution of the eigenvalue problem.

In the case in which the shaft is uniform, $GJ(x) = GJ = \text{constant}$, $I(x) = I = \text{constant}$, the differential equation, Eq. (f) of Example 7.7, reduces to the familiar form

$$\frac{d^2\Theta(x)}{dx^2} + \beta^2\Theta(x) = 0, \quad \beta^2 = \frac{\lambda I}{GJ} = \frac{\omega^2 I}{GJ}, \quad 0 < x < L \quad (7.214)$$

Moreover, the boundary conditions, Eqs. (g) of Example 7.7, become

$$\Theta(0) = 0, \quad \left. \frac{d\Theta(x)}{dx} \right|_{x=L} = \frac{\lambda I_D}{GJ} \Theta(L) = \frac{\beta^2 I_D}{I} \Theta(L) \quad (7.215a, b)$$

so that boundary condition (7.215b) depends on the eigenvalue β . As in Sec. 7.6, the solution of Eq. (7.214) is

$$\Theta(x) = C_1 \sin \beta x + C_2 \cos \beta x \quad (7.216)$$

and boundary condition (7.215a) yields $C_2 = 0$. On the other hand, boundary condition (7.215b) leads to the characteristic equation

$$\tan \beta L = \frac{IL}{I_D} \frac{1}{\beta L} \quad (7.217)$$

which must be solved numerically for the eigenvalues $\beta_r L$ ($r = 1, 2, \dots$). If some accuracy can be sacrificed, then the solution can be obtained graphically, as shown in Fig. 7.13, in which the three lowest eigenvalues corresponding to $IL/I_D = 1$ were obtained. The natural modes are given by

$$\Theta_r(x) = A_r \sin \beta_r x, \quad r = 1, 2, \dots \quad (7.218)$$

and they are orthogonal. Using Eqs. (7.204) and (7.205) in conjunction with boundary conditions (7.215), the orthogonality relations can be shown to be

$$\int_0^L I \Theta_r(x) \Theta_s(x) dx + I_D \Theta_r(L) \Theta_s(L) = 0, \quad r, s = 1, 2, \dots; r \neq s \quad (7.219a)$$

$$\int_0^L GJ \Theta'_r(x) \Theta'_s(x) dx = 0, \quad r, s = 1, 2, \dots; r \neq s \quad (7.219b)$$

The natural frequencies are related to the eigenvalues by

$$\omega_r = \beta_r L \sqrt{\frac{GJ}{IL^2}}, \quad r = 1, 2, \dots \quad (7.220)$$

The first three natural modes and natural frequencies are displayed in Fig. 7.14.

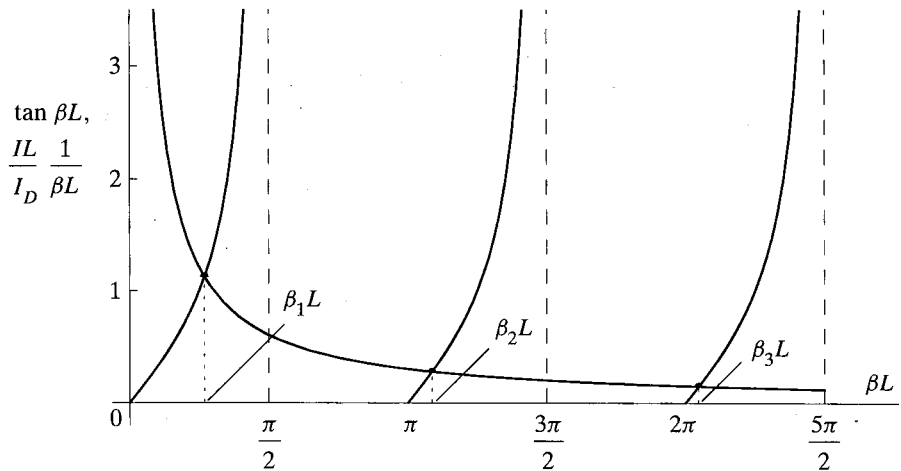


Figure 7.13 Graphical solution of the characteristic equation, Eq. (7.217)

From Eq. (7.217), as well as from Fig. 7.13, we observe that, as the eigenvalues $\beta_r L$ increase without bound, they tend to become integer multiples of π . Specifically,

$$\lim_{r \rightarrow \infty} \beta_r L = (r - 1)\pi \quad (7.221)$$

Inserting these values into Eq. (7.218), we conclude that the very high modes have nodes at $x = L$, which implies that the end disk is at rest for these modes, so that the end $x = L$ acts as if it were clamped.

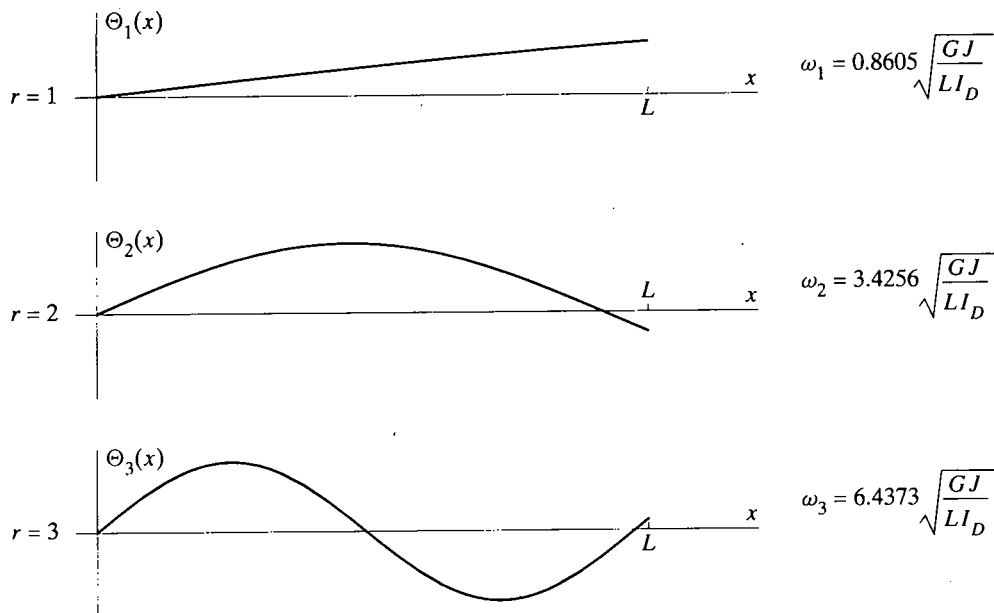


Figure 7.14 The first three modes of vibration of a shaft in torsion clamped at $x = 0$ and with a disk at $x = L$

7.11 TIMOSHENKO BEAM

In Sec. 7.2, we derived the boundary-value problem for the simplest model of a beam in bending vibration, namely, the Euler-Bernoulli model, which is based on the elementary beam theory. Then, in Sec. 7.6 we refined the model by including the rotatory inertia effects. The model of Sec. 7.6 can be further refined by considering the shear deformation effects. The inclusion of the shear deformation presents us with a problem not encountered before. Indeed, because in this case the slope of the deflection curve is not equal to the rotation of the beam cross section, we are faced with the problem of two dependent variables. The model of a beam including both rotatory inertia and shear deformation effects is commonly referred to as a *Timoshenko beam*.

Our objective is to derive the boundary-value problem for the nonuniform beam in bending shown in Fig. 7.15a. To this end, we consider the differential element of Fig. 7.15b and denote the mass per unit length at any point x by $m(x)$, the cross-sectional area by $A(x)$ and the area and mass moments of inertia about an axis normal to the plane of motion and passing through point C by $I(x)$ and $J(x)$, respectively, where C represents the mass center of the differential element. From Fig. 7.15b, the total deflection $w(x, t)$ of the beam consists of two parts, one caused by bending and one by shear, so that the slope of the deflection curve at point x can be written in the form

$$\frac{\partial w(x, t)}{\partial x} = \psi(x, t) + \beta(x, t) \tag{7.222}$$

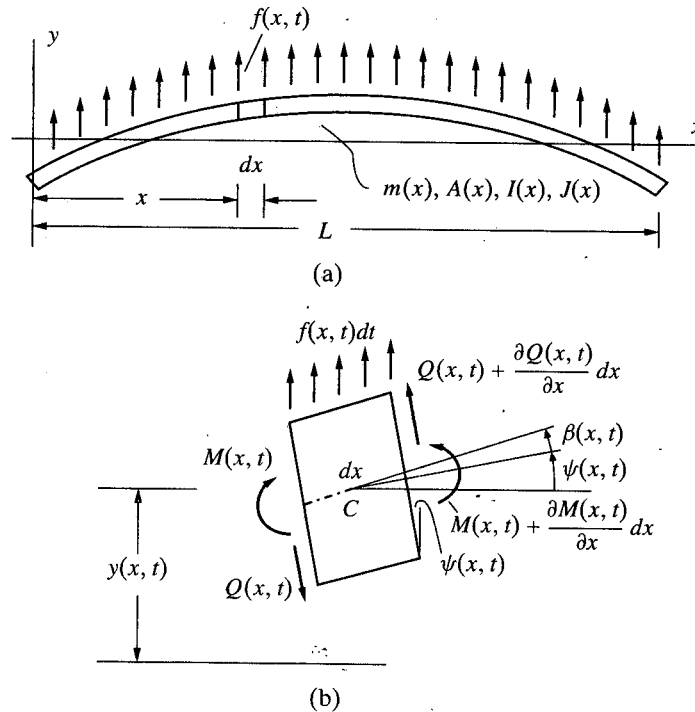


Figure 7.15 (a) Timoshenko beam (b) Timoshenko beam differential element

where $\psi(x, t)$ is the angle of rotation due to bending and $\beta(x, t)$ is the angle of distortion due to shear. As usual, the linear deflection and angular deflection are assumed small.

The relation between the bending moment and the bending deformation is

$$M(x, t) = EI(x) \frac{\partial \psi(x, t)}{\partial x} \quad (7.223)$$

and the relation between the shearing force and shearing deformation is given by

$$Q(x, t) = k'GA(x)\beta(x, t) \quad (7.224)$$

in which G is the shear modulus and k' is a numerical factor depending on the shape of the cross section. Because of shear alone, the element undergoes distortion but no rotation.

To formulate the boundary-value problem, we make use of the extended Hamilton's principle, Eq. (7.4), which requires the kinetic energy, potential energy and virtual work. The kinetic energy is due to translation and rotation and has the form

$$T(t) = \frac{1}{2} \int_0^L m(x) \left[\frac{\partial w(x, t)}{\partial t} \right]^2 dx + \frac{1}{2} \int_0^L J(x) \left[\frac{\partial \psi(x, t)}{\partial t} \right]^2 dx \quad (7.225)$$

where the mass moment of inertia density $J(x)$ is related to the area moment of inertia $I(x)$ by

$$J(x) = \rho I(x) = \frac{m(x)}{A(x)} I(x) = k^2(x)m(x) \quad (7.226)$$

in which ρ is the mass density and $k(x)$ is the radius of gyration about the neutral axis. The variation of the kinetic energy can be readily written as

$$\delta T = \int_0^L m \frac{\partial w}{\partial t} \delta \left(\frac{\partial w}{\partial t} \right) dx + \int_0^L k^2 m \frac{\partial \psi}{\partial t} \delta \left(\frac{\partial \psi}{\partial t} \right) dx \quad (7.227)$$

The potential energy has the expression

$$\begin{aligned} V(t) &= \frac{1}{2} \int_0^L M(x, t) \frac{\partial \psi(x, t)}{\partial x} dx + \frac{1}{2} \int_0^L Q(x, t) \beta(x, t) dx \\ &= \frac{1}{2} \int_0^L EI(x) \left[\frac{\partial \psi(x, t)}{\partial x} \right]^2 dx + \frac{1}{2} \int_0^L k'GA(x) \beta^2(x, t) dx \end{aligned} \quad (7.228)$$

so that the variation of the potential energy is simply

$$\begin{aligned} \delta V &= \int_0^L EI \frac{\partial \psi}{\partial x} \delta \left(\frac{\partial \psi}{\partial x} \right) dx + \int_0^L k'GA \beta \delta \beta dx \\ &= \int_0^L EI \frac{\partial \psi}{\partial x} \delta \left(\frac{\partial \psi}{\partial x} \right) dx + \int_0^L k'GA \left(\frac{\partial w}{\partial x} - \psi \right) \delta \left(\frac{\partial w}{\partial x} - \psi \right) dx \end{aligned} \quad (7.229)$$

The virtual work due to nonconservative forces is given by

$$\delta W_{nc}(t) = \int_0^L f(x, t) \delta w(x, t) dx \quad (7.230)$$

where f is the force density.

Introducing Eqs. (7.227), (7.229) and (7.230) into the extended Hamilton's principle, Eq. (7.4), we have

$$\begin{aligned} \int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{nc}) dt &= \int_{t_1}^{t_2} \left\{ \int_0^L \left[m \frac{\partial w}{\partial t} \delta \left(\frac{\partial w}{\partial t} \right) + k^2 m \frac{\partial \psi}{\partial t} \delta \left(\frac{\partial \psi}{\partial t} \right) \right. \right. \\ &\quad \left. \left. - EI \frac{\partial \psi}{\partial x} \delta \left(\frac{\partial \psi}{\partial x} \right) - k'GA \left(\frac{\partial w}{\partial x} - \psi \right) \delta \left(\frac{\partial w}{\partial x} - \psi \right) + f \delta w \right] dx \right\} dt = 0, \\ \delta w &= 0, \quad \delta \psi = 0, \quad t = t_1, t_2 \end{aligned} \quad (7.231)$$

and we note that we have two dependent variables, w and ψ . We carry out the operations involved in Eq. (7.231) term by term. Recalling that the order of the integrations with respect to x and t is interchangeable and that the variation and

differentiation operators are commutative, we can perform the following integration by parts with respect to time:

$$\begin{aligned} \int_{t_1}^{t_2} m \frac{\partial w}{\partial t} \delta \left(\frac{\partial w}{\partial t} \right) dt &= \int_{t_1}^{t_2} m \frac{\partial w}{\partial t} \frac{\partial}{\partial t} \delta w dt \\ &= m \frac{\partial w}{\partial t} \delta w \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(m \frac{\partial w}{\partial t} \right) \delta w dt \\ &= - \int_{t_1}^{t_2} m \frac{\partial^2 w}{\partial t^2} \delta w dt \end{aligned} \quad (7.232)$$

where we took into account that δw vanishes at $t = t_1$ and $t = t_2$. In a similar fashion, we obtain

$$\int_{t_1}^{t_2} k^2 m \frac{\partial \psi}{\partial t} \delta \left(\frac{\partial \psi}{\partial t} \right) dt = - \int_{t_1}^{t_2} k^2 m \frac{\partial^2 \psi}{\partial t^2} \delta \psi dt \quad (7.233)$$

On the other hand, integrations over the spatial variable yield

$$\begin{aligned} \int_0^L EI \frac{\partial \psi}{\partial x} \delta \left(\frac{\partial \psi}{\partial x} \right) dx &= \int_0^L EI \frac{\partial \psi}{\partial x} \frac{\partial}{\partial x} \delta \psi dx \\ &= \left(EI \frac{\partial \psi}{\partial x} \right) \delta \psi \Big|_0^L - \int_0^L \frac{\partial}{\partial x} \left(EI \frac{\partial \psi}{\partial x} \right) \delta \psi dx \end{aligned} \quad (7.234a)$$

$$\begin{aligned} &\int_0^L k' GA \left(\frac{\partial w}{\partial x} - \psi \right) \delta \left(\frac{\partial w}{\partial x} - \psi \right) dx \\ &= \int_0^L k' GA \left(\frac{\partial w}{\partial x} - \psi \right) \left(\frac{\partial}{\partial x} \delta w - \delta \psi \right) dx \\ &= \left[k' GA \left(\frac{\partial w}{\partial x} - \psi \right) \right] \delta w \Big|_0^L \\ &\quad - \int_0^L \left\{ \frac{\partial}{\partial x} \left[k' GA \left(\frac{\partial w}{\partial x} - \psi \right) \right] \delta w + k' GA \left(\frac{\partial w}{\partial x} - \psi \right) \delta \psi \right\} dx \end{aligned} \quad (7.234b)$$

Inserting Eqs. (7.232)–(7.234) into Eq. (7.231) and rearranging, we obtain

$$\begin{aligned} &\int_{t_1}^{t_2} (\delta T - \delta V + \delta W_{nc}) dt \\ &= \int_{t_1}^{t_2} \left[\int_0^L \left\{ \frac{\partial}{\partial x} \left[k' GA \left(\frac{\partial w}{\partial x} - \psi \right) \right] - m \frac{\partial^2 w}{\partial t^2} + f \right\} \delta w \right. \end{aligned}$$

$$\begin{aligned}
& + \left\{ \left[\frac{\partial}{\partial x} \left(EI \frac{\partial \psi}{\partial x} \right) + k'GA \left(\frac{\partial w}{\partial x} - \psi \right) \right] - k^2 m \frac{\partial^2 \psi}{\partial t^2} \right\} \delta \psi \Big|_0^L dx \\
& - \left(EI \frac{\partial \psi}{\partial x} \right) \delta \psi \Big|_0^L - \left[k'GA \left(\frac{\partial w}{\partial x} - \psi \right) \right] \delta w \Big|_0^L dt = 0 \quad (7.235)
\end{aligned}$$

The virtual displacements $\delta \psi$ and δw are arbitrary and independent, so that they can be taken equal to zero at $x = 0$ and $x = L$ and arbitrary for $0 < x < L$. Hence, we must have

$$\frac{\partial}{\partial x} \left[k'GA \left(\frac{\partial w}{\partial x} - \psi \right) \right] - m \frac{\partial^2 w}{\partial t^2} + f = 0, \quad 0 < x < L \quad (7.236a)$$

$$\frac{\partial}{\partial x} \left(EI \frac{\partial \psi}{\partial x} \right) + k'GA \left(\frac{\partial w}{\partial x} - \psi \right) - k^2 m \frac{\partial^2 \psi}{\partial t^2} = 0, \quad 0 < x < L \quad (7.236b)$$

In addition, if we write

$$\left(EI \frac{\partial \psi}{\partial x} \right) \delta \psi \Big|_0^L = 0 \quad (7.237a)$$

$$\left[k'GA \left(\frac{\partial w}{\partial x} - \psi \right) \right] \delta w \Big|_0^L = 0 \quad (7.237b)$$

we take into account the possibility that either $EI (\partial \psi / \partial x)$ or $\delta \psi$ on the one hand, and either $k'GA [(\partial w / \partial x) - \psi]$ or δw on the other hand vanishes at the ends $x = 0$ and $x = L$. Equations (7.236) are the differential equations of motion and Eqs. (7.237) represent the boundary conditions. The boundary-value problem consists of the differential equations, Eqs. (7.236) and two boundary conditions at each end to be chosen from Eqs. (7.237).

For a beam *clamped at both ends*, the deflection and rotation are zero, or

$$w(0, t) = 0, \quad \psi(0, t) = 0 \quad (7.238a, b)$$

$$w(L, t) = 0, \quad \psi(L, t) = 0 \quad (7.238c, d)$$

and note that it is the rotation that is zero and not the slope. All boundary conditions are geometric.

In the case of a *simply-supported* beam, i.e., a beam *pinned at both ends*, the boundary conditions are

$$w(0, t) = 0, \quad M(0, t) = EI \frac{\partial \psi}{\partial x} \Big|_{x=0} = 0 \quad (7.239a, b)$$

$$w(L, t) = 0, \quad M(L, t) = EI \frac{\partial \psi}{\partial x} \Big|_{x=L} = 0 \quad (7.239c, d)$$

so that there is one geometric and one natural boundary condition at each end.

For a beam cantilevered at $x = 0$, the boundary conditions at the clamped end are

$$w(0, t) = 0, \quad \psi(0, t) = 0 \quad (7.240a, b)$$

At the free end neither the deflection nor the rotation is zero, so that we must have

$$M(L, t) = EI \frac{\partial \psi}{\partial x} \Big|_{x=L} = 0, \quad Q(L, t) = \left[k'GA \left(\frac{\partial w}{\partial x} - \psi \right) \right]_{x=L} = 0 \quad (7.240c, d)$$

which reflects the fact that both the bending moment and the shearing force vanish at a free end. Hence, at the clamped end we have two geometric boundary conditions and at the free end we have two natural boundary conditions.

Finally, in this case of a *free-free* beam, the boundary conditions are

$$M(0, t) = EI \frac{\partial \psi}{\partial x} \Big|_{x=0} = 0, \quad Q(0, t) = \left[k'GA \left(\frac{\partial w}{\partial x} - \psi \right) \right]_{x=0} = 0 \quad (7.241a, b)$$

$$M(L, t) = EI \frac{\partial \psi}{\partial x} \Big|_{x=L} = 0, \quad Q(L, t) = \left[k'GA \left(\frac{\partial w}{\partial x} - \psi \right) \right]_{x=L} = 0 \quad (7.241c, d)$$

and they are all natural.

The interesting part about the new formulation is that the mass density is no longer an operator, as in Example 7.8, but a mere function. Moreover, boundary conditions (7.240d), (7.241b) and (7.241d) do not depend on the acceleration, in contrast with the case in which the shear deformation is absent, as can be concluded from Eq. (d) of Example 7.6. Both differences are due to the fact that the rotation is no longer equal to the spatial derivative of the displacement. The simplification gained in the mass density expression and the boundary condition involving the shearing force is balanced by the complication arising from the fact that now there are two dependent variables.

Next, we wish to derive the eigenvalue problem. In view of our past experience, we assume that $f = 0$ and that the solution of the boundary-value problem is separable in x and t , or

$$w(x, t) = W(x)F(t), \quad \psi(x, t) = \Psi(x)F(t) \quad (7.242)$$

where $F(t)$ is harmonic and it satisfies

$$\ddot{F}(t) = -\lambda F(t), \quad \lambda = \omega^2 \quad (7.243)$$

Introducing Eqs. (7.242) and (7.243) into Eqs. (7.236) with $f = 0$, we obtain the ordinary differential equations

$$-\frac{d}{dx} \left[k'GA \left(\frac{dW}{dx} - \Psi \right) \right] = \lambda mW, \quad 0 < x < L \quad (7.244a)$$

$$-\left\{ \frac{d}{dx} \left(EI \frac{d\Psi}{dx} \right) + k'GA \left(\frac{dW}{dx} - \Psi \right) \right\} = \lambda k^2 m \Psi, \quad 0 < x < L \tag{7.244b}$$

The boundary conditions transform accordingly.

The eigenvalue problem is defined by two differential equations in terms of two dependent variables, instead of one differential equation in one variable, so that the traditional ways of checking for self-adjointness and positive definiteness do not apply in the case of a Timoshenko beam. In the following, we define new criteria. To this end, we introduce the displacement vector

$$\mathbf{y}(x) = [W(x) \ \Psi(x)]^T \tag{7.245}$$

as well as the stiffness and mass operator matrices

$$\mathcal{L} = - \begin{bmatrix} \frac{d}{dx} \left(k'GA \frac{d}{dx} \right) & -\frac{d}{dx} (k'GA \cdot) \\ k'GA \frac{d}{dx} & \frac{d}{dx} \left(EI \frac{d}{dx} \right) - k'GA \end{bmatrix}, \quad \mathcal{M} = \begin{bmatrix} m & 0 \\ 0 & k^2 m \end{bmatrix} \tag{7.246a,b}$$

where the dot indicates the implied position of Ψ , and write Eqs. (7.244) in the operator matrix form

$$\mathcal{L}\mathbf{y} = \lambda \mathcal{M}\mathbf{y} \tag{7.247}$$

Then, by analogy with the scalar definitions, the problem is self-adjoint if for any two vectors \mathbf{u} and \mathbf{v} of comparison functions

$$\int_0^L \mathbf{u}^T \mathcal{L}\mathbf{v} dx = \int_0^L \mathbf{v}^T \mathcal{L}\mathbf{u} dx \tag{7.248a}$$

$$\int_0^L \mathbf{u}^T \mathcal{M}\mathbf{v} dx = \int_0^L \mathbf{v}^T \mathcal{M}\mathbf{u} dx \tag{7.248b}$$

Moreover, the problem is positive definite if

$$\int_0^L \mathbf{u}^T \mathcal{L}\mathbf{u} dx \geq 0, \quad \int_0^L \mathbf{u}^T \mathcal{M}\mathbf{u} dx \geq 0 \tag{7.249a, b}$$

and the equality sign holds true if and only if $\mathbf{u} \equiv \mathbf{0}$, and it is positive semidefinite if the equality sign holds true for some $\mathbf{u} \neq \mathbf{0}$. Because the mass operator matrix \mathcal{M} is self-adjoint and positive definite by definition, the system self-adjointness and positive definiteness depend on the stiffness operator \mathcal{L} .

To check for self-adjointness, we insert Eq. (7.246a) into the left side of Eq. (7.248a), carry out suitable integrations by parts and obtain

$$\int_0^L \mathbf{u}^T \mathcal{L}\mathbf{v} dx = - \int_0^L \mathbf{u}^T \begin{bmatrix} \frac{d}{dx} \left(k'GA \frac{d}{dx} \right) & -\frac{d}{dx} (k'GA \cdot) \\ k'GA \frac{d}{dx} & \frac{d}{dx} \left(EI \frac{d}{dx} \right) - k'GA \end{bmatrix} \mathbf{v} dx$$

$$\begin{aligned}
&= -\mathbf{u}^T \left[\begin{array}{cc} k'GA \frac{d}{dx} & -k'GA \\ 0 & EI \frac{d}{dx} \end{array} \right] \mathbf{v} \Big|_0^L \\
&\quad + \int_0^L \left(\frac{d\mathbf{u}^T}{dx} \left[\begin{array}{cc} k'GA & 0 \\ 0 & EI \end{array} \right] \frac{d\mathbf{v}}{dx} - \frac{d\mathbf{u}^T}{dx} \left[\begin{array}{cc} 0 & k'GA \\ 0 & 0 \end{array} \right] \mathbf{v} \right. \\
&\quad \left. - \mathbf{u}^T \left[\begin{array}{cc} 0 & 0 \\ k'GA & 0 \end{array} \right] \frac{d\mathbf{v}}{dx} + \mathbf{u}^T \left[\begin{array}{cc} 0 & 0 \\ 0 & k'GA \end{array} \right] \mathbf{v} \right) dx \quad (7.250)
\end{aligned}$$

We observe that the integral on the right side of Eq. (7.250) is symmetric in \mathbf{u} and \mathbf{v} . Hence, *all systems for which the boundary term is zero are self-adjoint*. This is certainly the case with the systems with the boundary conditions given by Eqs. (7.238)–(7.241).

Before we proceed with the check for positive definiteness, we should state that the concept applies only to self-adjoint systems. Hence, assuming that the boundary term is zero and letting $\mathbf{v} = \mathbf{u}$ in Eq. (7.250), we have

$$\begin{aligned}
\int_0^L \mathbf{u}^T \mathcal{L} \mathbf{u} dx &= \int_0^L \left(\frac{d\mathbf{u}^T}{dx} \left[\begin{array}{cc} k'GA & 0 \\ 0 & EI \end{array} \right] \frac{d\mathbf{u}}{dx} - \frac{d\mathbf{u}^T}{dx} \left[\begin{array}{cc} 0 & k'GA \\ 0 & 0 \end{array} \right] \mathbf{u} \right. \\
&\quad \left. - \mathbf{u}^T \left[\begin{array}{cc} 0 & 0 \\ k'GA & 0 \end{array} \right] \frac{d\mathbf{u}}{dx} + \mathbf{u}^T \left[\begin{array}{cc} 0 & 0 \\ 0 & k'GA \end{array} \right] \mathbf{u} \right) dx \\
&= \int_0^L \left[k'GA \left(\frac{du_1}{dx} - u_2 \right)^2 + EI \left(\frac{du_2}{dx} \right)^2 \right] dx \geq 0 \quad (7.251)
\end{aligned}$$

where u_1 and u_2 are the components of \mathbf{u} . It is easy to verify that the expression can be zero for the nontrivial case $u_1 = \text{constant}$, $u_2 = 0$. But, when one of the ends is clamped or pinned, such as in the case of the systems with the boundary conditions given by Eqs. (7.238)–(7.240), this constant must be zero. It follows that in the three cases covered by boundary conditions (7.238)–(7.240), the operator \mathcal{L} is positive definite, so that *the system is positive definite*. On the other hand, for a free-free beam with the boundary conditions given by Eqs. (7.241), $u_1 = \text{constant}$, $u_2 = 0$ is a possible solution of the eigenvalue problem so that the operator \mathcal{L} is only positive semidefinite, from which it follows that *the system is only positive semidefinite*. Consistent with this, the solution $u_1 = \text{constant}$, $u_2 = 0$ represents a rigid-body mode belonging to a zero eigenvalue.

7.12 VIBRATION OF MEMBRANES

All distributed systems considered until now were one-dimensional, which implies that the description of their motion requires a single spatial variable. At this point, we turn our attention to two-dimensional systems whose motion is described in terms of two spatial coordinates. We confine ourselves to the case in which the domain D is planar, with the motion being measured normal to the nominal plane, and the boundary S consists of one or two nonintersecting curves. Two-dimensional

problems introduce a new element into the boundary-value problem, namely, the shape of the boundary S . In two-dimensional problems, there are several choices of coordinates for describing the motion, such as rectangular, polar, elliptical, etc. The choice is generally not arbitrary but dictated by the shape of the boundary, because the boundary conditions for the most part involve derivatives along the normal direction n or along the tangent s to the boundary (Fig. 7.16). Hence, it is only natural to choose rectangular coordinates if the boundary S is a rectangle, polar coordinates if S is a circle, elliptical coordinates if S is an ellipse, etc. The question is not so clear when S has an irregular shape, in which case no closed-form solution can be expected. In this case, the choice of coordinates depends on the method used to produce an approximate solution.

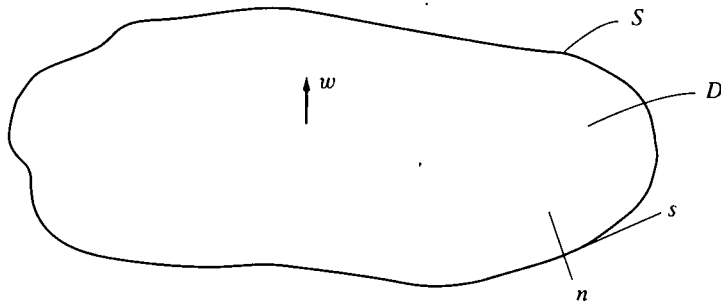


Figure 7.16 Two-dimensional distributed system

The simplest two-dimensional problem in vibrations is that of a membrane. Indeed, the membrane can be regarded as the two-dimensional counterpart of the string. The boundary-value problem can be obtained conveniently by means of the extended Hamilton's principle, Eq. (7.4). It is relatively easy to carry out the derivation in terms of rectangular coordinates. However, we opt for an approach valid for all types of coordinates. To this end, we consider a membrane fixed or free over a portion S_1 of the boundary S and supported by a distributed spring over the remaining portion S_2 and write the potential energy in the form

$$V = \frac{1}{2} \int_D T \nabla w \cdot \nabla w \, dD + \frac{1}{2} \int_{S_2} k w^2 \, dS \quad (7.252)$$

where w is the transverse displacement, ∇ a vector operator signifying the gradient, T the tension and k the distributed spring constant. For simplicity we assume that *the tension is constant*. The kinetic energy has the expression

$$T = \frac{1}{2} \int_D \rho \dot{w}^2 \, dD \quad (7.253)$$

in which ρ is the mass density, and the virtual work of the nonconservative forces is simply

$$\delta \bar{W}_{nc} = \int_D f \delta w \, dD \quad (7.254)$$

where f is the force density.

Next, we consider the variation in the potential energy, Eq. (7.252), in the form

$$\delta V = \int_D T \nabla w \cdot \delta \nabla w \, dD + \int_{S_2} k w \delta w \, dS = \int_D T \nabla w \cdot \nabla \delta w \, dD + \int_{S_2} k w \delta w \, dS \quad (7.255)$$

But, from Ref. 8, we can write the relation

$$u \nabla^2 v = u \nabla \cdot \nabla v = \nabla \cdot (u \nabla v) - \nabla u \cdot \nabla v \quad (7.256)$$

where $\nabla^2 = \nabla \cdot \nabla$ is the Laplacian, which is equal to the divergence of the gradient. Hence, letting $u = \delta w$, $v = w$, Eq. (7.255) becomes

$$\delta V = \int_D T [\nabla \cdot (\delta w \nabla w) - \delta w \nabla^2 w] \, dD + \int_{S_2} k w \delta w \, dS \quad (7.257)$$

At this point, we invoke the divergence theorem (Ref. 4)

$$\int_D \nabla \cdot \mathbf{A} \, dD = \int_S A_n \, dS \quad (7.258)$$

in which A_n is the component of the vector \mathbf{A} along the exterior normal to boundary S , so that, letting $\mathbf{A} = \delta w \nabla w$, $dA_n = \frac{\partial w}{\partial n} \delta w$, Eq. (7.257) can be rewritten as

$$\begin{aligned} \delta V &= - \int_D T \nabla^2 w \delta w \, dD + \int_S T \frac{\partial w}{\partial n} \delta w \, dS + \int_{S_2} k w \delta w \, dS \\ &= - \int_D T \nabla^2 w \delta w \, dD + \int_{S_1} T \frac{\partial w}{\partial n} \delta w \, dS + \int_{S_2} \left(T \frac{\partial w}{\partial n} + k w \right) \delta w \, dS \end{aligned} \quad (7.259)$$

Moreover, by analogy with the one-dimensional case, Eq. (7.12),

$$\int_{t_1}^{t_2} \delta T \, dt = - \int_{t_1}^{t_2} \int_D \rho \ddot{w} \delta w \, dD \, dt \quad (7.260)$$

Inserting Eqs. (7.254), (7.259) and (7.260) into the extended Hamilton's principle, Eq. (7.4), we obtain

$$\begin{aligned} \int_{t_1}^{t_2} \left[\int_D (T \nabla^2 w - \rho \ddot{w} + f) \delta w \, dD \right. \\ \left. + \int_{S_1} T \frac{\partial w}{\partial n} \delta w \, dS + \int_{S_2} \left(T \frac{\partial w}{\partial n} + k w \right) \delta w \, dS \right] dt = 0 \end{aligned} \quad (7.261)$$

Finally, using the usual argument, we conclude that Eq. (7.261) can be satisfied for arbitrary δw in D and on S if and only if w satisfies the partial differential equation

$$T \nabla^2 w + f = \rho \ddot{w} \text{ in } D \quad (7.262)$$

and, in addition, either

$$T \frac{\partial w}{\partial n} = 0 \text{ on } S_1 \quad (7.263a)$$

or

$$w = 0 \text{ on } S_1 \quad (7.263b)$$

and

$$T \frac{\partial w}{\partial n} + kw = 0 \text{ on } S_2 \quad (7.264)$$

The boundary-value problem consists of the partial differential equation (7.262) to be satisfied over D and appropriate boundary conditions. For a membrane fixed at every point of S_1 and supported by a spring at every point of S_2 , the boundary conditions consist of Eqs. (7.263b) and (7.264). On the other hand, if the membrane is free at every point of S_1 , instead of being fixed, the boundary conditions consist of Eqs. (7.263a) and (7.264).

To derive the differential eigenvalue problem, we follow the established procedure, i.e., we let $f = 0$, assume that $w = WF$, where W depends on the spatial position alone and F depends on time alone and satisfies $\dot{F} = -\lambda F$, eliminate the time dependence from the boundary-value problem, Eqs. (7.262)–(7.264), and obtain the partial differential equation

$$-T\nabla^2 W = \lambda\rho W, \quad \lambda = \omega^2 \text{ over } D \quad (7.265)$$

Moreover, for a membrane fixed at every point of S_1 and spring-supported at every point of S_2 , $S_1 + S_2 = S$, we obtain the boundary conditions

$$W = 0 \text{ on } S_1 \quad (7.266a)$$

$$T \frac{\partial W}{\partial n} + kW = 0 \text{ on } S_2 \quad (7.266b)$$

and if the membrane is free at every point of S_1 , the boundary conditions

$$T \frac{\partial W}{\partial n} = 0 \text{ on } S_1 \quad (7.267a)$$

$$T \frac{\partial W}{\partial n} + kW = 0 \text{ on } S_2 \quad (7.267b)$$

The just derived eigenvalue problem is of the special type given by Eqs. (7.68) and (7.69) in which we identify the stiffness operator and the mass density

$$L = -T\nabla^2, \quad m = \rho \text{ in } D \quad (7.268a, b)$$

Moreover, for a membrane fixed at all points of S_1 and spring-supported at all points of S_2 , the boundary operators are

$$B_1 = 1 \text{ on } S_1 \quad (7.269a)$$

$$B_1 = T \frac{\partial}{\partial n} + k \text{ on } S_2 \quad (7.269b)$$

and if the membrane is free at all points of S_1 , they are

$$B_1 = T \frac{\partial}{\partial n} \text{ on } S_1 \quad (7.270a)$$

$$B_1 = T \frac{\partial}{\partial n} + k \text{ on } S_2 \quad (7.270b)$$

The system is self-adjoint and positive definite for both types of boundary conditions. To verify self-adjointness, we consider two comparison functions u and v , use Eqs. (7.256) and (7.258), where in the latter $\mathbf{A} = u \nabla v$, and write

$$\begin{aligned} \int_D u L v \, dD &= - \int_D u T \nabla^2 v \, dD = - \int_D T [\nabla \cdot (u \nabla v) - \nabla u \cdot \nabla v] \, dD \\ &= \int_D T \nabla u \cdot \nabla v \, dD - \int_{S_2} T u \frac{\partial v}{\partial n} \, dS \\ &= \int_D T \nabla u \cdot \nabla v \, dD + \int_{S_2} k u v \, dS \end{aligned} \quad (7.271)$$

which is symmetric in u and v . Hence, as anticipated, the operator L , and hence the system, is self-adjoint. It follows that the eigenvalues are real and the eigenfunctions are real and orthogonal. The eigenfunctions can be normalized so as to satisfy the orthonormality relations

$$\int_D \rho W_r W_s \, dD = \delta_{rs}, \quad r, s = 1, 2, \dots \quad (7.272a)$$

$$\begin{aligned} \int_D W_r L W_s \, dD &= - \int_D T W_r \nabla^2 W_s \, dD \\ &= \int_D T \nabla W_r \cdot \nabla W_s \, dD + \int_{S_2} k W_r W_s \, dS = \lambda_r \delta_{rs}, \\ & \quad r, s = 1, 2, \dots \end{aligned} \quad (7.272b)$$

Moreover, to verify positive definiteness, we let $v = u$ in Eq. (7.271) and obtain

$$\begin{aligned} \int_D u L u \, dD &= \int_D T \nabla u \cdot \nabla u \, dD + \int_{S_2} k u^2 \, dS \\ &= \int_D T \|\nabla u\|^2 \, dD + \int_{S_2} k u^2 \, dS = \|u\|_E^2 > 0 \end{aligned} \quad (7.273)$$

where $\|u\|_E$ is the energy norm. Clearly, the energy norm is positive for nontrivial u , so that the operator L , and hence the system, is positive definite. It follows that all the eigenvalues are positive, a fact already taken into account when the time dependence was assumed to be harmonic.

In the above discussion, we carefully avoided reference to any particular set of coordinates. The deflection w can be given in terms of rectangular coordinates and time or curvilinear coordinates and time. Accordingly, the Laplacian ∇^2 can be

expressed in terms of rectangular or curvilinear coordinates. As pointed out earlier in this section, the shape of the boundary dictates the choice of coordinates, because the only way we can deal effectively with boundary conditions is by formulating the problem in terms of coordinates capable of matching the shape of the boundary. In fact, there are only a few boundary shapes permitting closed-form solutions. We confine our discussion to rectangular and circular membranes.

i. Rectangular membranes

Under consideration is a rectangular membrane extending over a domain D defined by $0 < x < a$ and $0 < y < b$. The boundaries of the domain are the straight lines $x = 0, a$ and $y = 0, b$. If we assume that the mass density is constant, then Eq. (7.265) can be written in the form

$$\nabla^2 W(x, y) + \beta^2 W(x, y) = 0, \quad \beta^2 = \frac{\rho\omega^2}{T}, \quad x, y \text{ in } D \quad (7.274)$$

where the Laplacian in rectangular coordinates has the expression

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (7.275)$$

For a membrane *fixed at all boundaries*, the boundary conditions are

$$W(0, y) = 0, \quad W(a, y) = 0 \quad (7.276a, b)$$

$$W(x, 0) = 0, \quad W(x, b) = 0 \quad (7.276c, d)$$

and we note that they are all geometric boundary conditions. The differential equation, Eq. (7.274) together with the boundary conditions, Eqs. (7.276), constitute the eigenvalue problem.

Equation (7.274) can be solved by the method of separation of variables. To this end, we let the solution have the form

$$W(x, y) = X(x)Y(y) \quad (7.277)$$

Upon substitution in Eq. (7.274), we obtain

$$\frac{d^2 X(x)}{dx^2} Y(y) + X(x) \frac{d^2 Y(y)}{dy^2} + \beta^2 X(x) Y(y) = 0 \quad (7.278)$$

which can be divided through by $X(x)Y(y)$ to yield

$$\frac{1}{X(x)} \frac{d^2 X(x)}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y(y)}{dy^2} + \beta^2 = 0 \quad (7.279)$$

This leads to the equations

$$\frac{d^2 X(x)}{dx^2} + \alpha^2 X(x) = 0 \quad (7.280a)$$

$$\frac{d^2 Y(y)}{dy^2} + \gamma^2 Y(y) = 0 \quad (7.280b)$$

where

$$\alpha^2 + \gamma^2 = \beta^2 \quad (7.281)$$

The solution of Eq. (7.280a) is

$$X(x) = C_1 \sin \alpha x + C_2 \cos \alpha x \quad (7.282a)$$

and the solution of Eq. (7.280b) is

$$Y(y) = C_3 \sin \gamma y + C_4 \cos \gamma y \quad (7.282b)$$

so that, introducing Eqs. (7.282) into Eq. (7.277), we obtain

$$\begin{aligned} W(x, y) = & A_1 \sin \alpha x \sin \gamma y + A_2 \sin \alpha x \cos \gamma y \\ & + A_3 \cos \alpha x \sin \gamma y + A_4 \cos \alpha x \cos \gamma y \end{aligned} \quad (7.283)$$

where A_1 , A_2 , A_3 and A_4 , as well as α and γ , must be determined by using the boundary conditions.

Boundary condition (7.276a) gives

$$W(0, y) = A_3 \sin \gamma y + A_4 \cos \gamma y = 0 \quad (7.284)$$

which can hold true for all y , assuming that $\gamma \neq 0$, if and only if A_3 and A_4 are zero. Assuming that A_3 and A_4 are zero, boundary condition (7.276b) yields

$$W(a, y) = A_1 \sin \alpha a \sin \gamma y + A_2 \sin \alpha a \cos \gamma y = 0 \quad (7.285)$$

which can be satisfied if A_1 and A_2 are zero. This would give the trivial solution $W(x, y) = 0$, however, so that we must consider the other possibility, namely,

$$\sin \alpha a = 0 \quad (7.286a)$$

Similarly, boundary condition (7.276c) leads us to the conclusion that $A_2 = A_4 = 0$, whereas boundary condition (7.276d) gives

$$\sin \gamma b = 0 \quad (7.286b)$$

Equations (7.286) play the role of characteristic, or frequency equations, because together they define the eigenvalues of the system. Indeed, Eq. (7.286a) yields the infinite set of discrete roots

$$\alpha_m a = m\pi, \quad m = 1, 2, \dots \quad (7.287a)$$

and Eq. (7.286b) gives another infinite set of roots, or

$$\gamma_n b = n\pi, \quad n = 1, 2, \dots \quad (7.287b)$$

It follows from Eqs. (7.281) and (7.287) that the solution of the eigenvalue problem consists of the eigenvalues

$$\beta_{mn} = \sqrt{\alpha_m^2 + \gamma_n^2} = \pi \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}, \quad m, n = 1, 2, \dots \quad (7.288)$$

and from Eq. (7.283) that the corresponding eigenfunctions are

$$W_{mn} = A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = \frac{2}{\sqrt{\rho ab}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad m, n = 1, 2, \dots \quad (7.289)$$

where the eigenfunctions have been normalized so as to satisfy $\int_0^a \int_0^b \rho W_{mn}^2 dx dy = 1$ ($m, n = 1, 2, \dots$). Moreover, from Eqs. (7.274) and (7.288), the natural frequencies are

$$\omega_{mn} = \beta_{mn} \sqrt{\frac{T}{\rho}} = \pi \sqrt{\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right] \frac{T}{\rho}}, \quad m, n = 1, 2, \dots \quad (7.290)$$

We have shown earlier that the problem is self-adjoint and positive definite. From Eqs. (7.272), the orthonormality relations are

$$\int_0^a \int_0^b \rho W_{mn}(x, y) W_{rs}(x, y) dx dy = \delta_{mr} \delta_{ns} \quad (7.291a)$$

$$\begin{aligned} & - \int_0^a \int_0^b W_{mn}(x, y) T \nabla^2 W_{rs}(x, y) dx dy \\ & = \int_0^a \int_0^b T \left(\frac{\partial W_{mn}}{\partial x} \frac{\partial W_{rs}}{\partial x} + \frac{\partial W_{mn}}{\partial y} \frac{\partial W_{rs}}{\partial y} \right) dx dy = \lambda_{mn} \delta_{mr} \delta_{ns} \end{aligned} \quad (7.291b)$$

where $\lambda_{mn} = \omega_{mn}^2$. The first four eigenfunctions are plotted in Fig. 7.17. The nodes are straight lines; the number of nodal lines parallel to the x axis is $n - 1$ and the number parallel to the y axis is $m - 1$.

We note that some but not all of the higher natural frequencies are integer multiples of the fundamental frequency, $\omega_{mm} = m\omega_{11}$. For example, ω_{12} is not an integer multiple of ω_{11} . Hence, the sounds produced by vibrating membranes are not as pleasant as the sounds produced by strings or any other system with harmonic overtones.

In the special case in which the ratio $R = a/b$ is a rational number, we have repeated natural frequencies $\omega_{mn} = \omega_{rs}$ if

$$m^2 + R^2 n^2 = r^2 + R^2 s^2 \quad (7.292)$$

For a ratio $R = 4/3$, we note that $\omega_{35} = \omega_{54}$, $\omega_{83} = \omega_{46}$, etc. For a square membrane, $a = b$, Eq. (7.292) reduces to

$$m^2 + n^2 = r^2 + s^2 \quad (7.293)$$

in which case we obtain repeated frequencies $\omega_{mn} = \omega_{nm}$. Hence, two distinct eigenfunctions W_{mn} and W_{nm} belong to the same eigenvalue, so that there are fewer eigenvalues than eigenfunctions. Such a case is said to be *degenerate*. As in the case of discrete systems, any linear combination of eigenfunctions belonging to repeated eigenvalues is also an eigenfunction. They are characterized by a large variety of

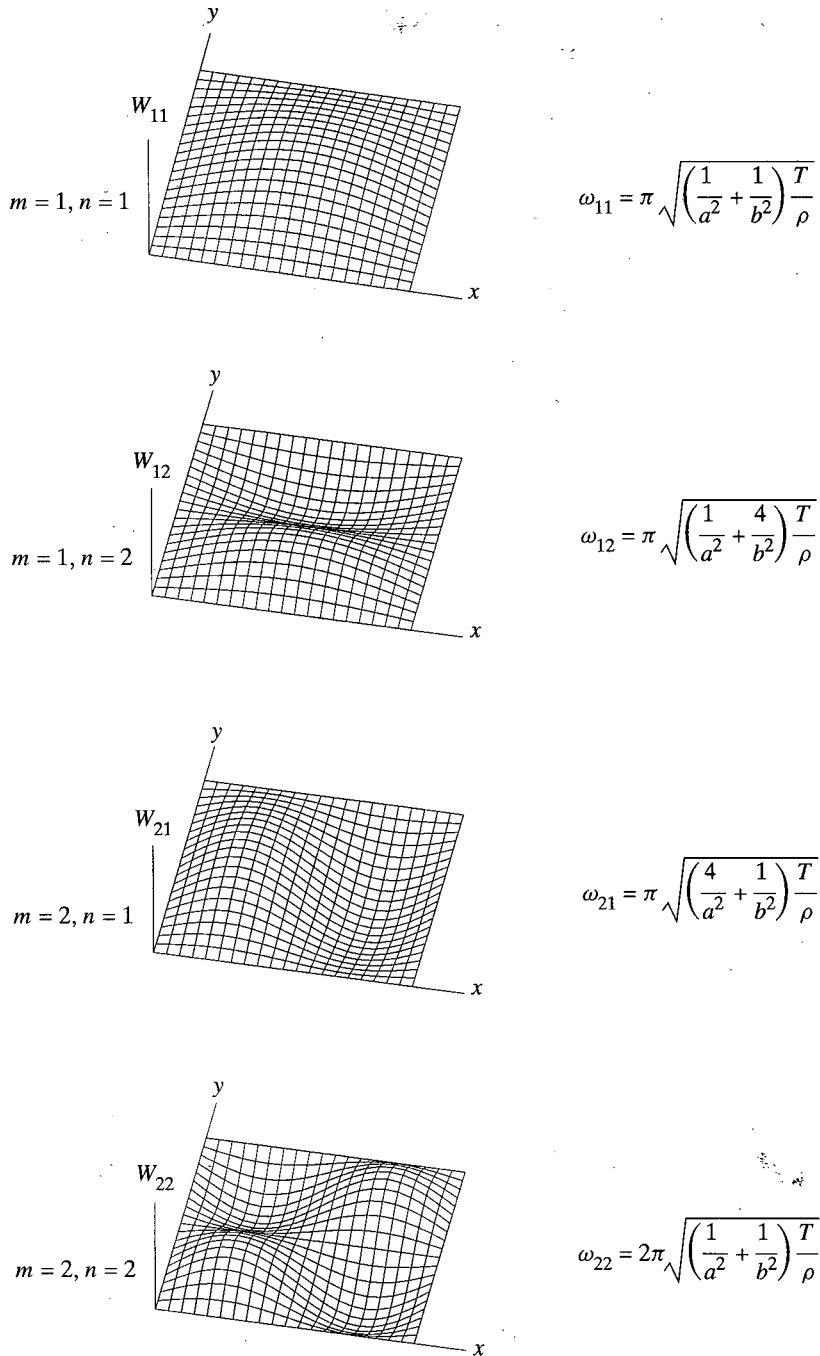


Figure 7.17 The first four modes of vibration of a uniform rectangular membrane fixed on all sides

nodal patterns. For the square membrane, the nodal lines are no longer straight lines except in special cases. The reader who wishes to pursue this subject further is referred to the text by Courant and Hilbert (Ref. 1, p. 302).

ii. Circular membranes

We consider a uniform circular membrane extending over a domain D defined by $0 < r < a$. The boundary of the domain is the circle S given by the equation $r = a$. Using the polar coordinates r and θ , the differential equation is

$$\nabla^2 W(r, \theta) + \beta^2 W(r, \theta) = 0, \quad \beta^2 = \frac{\rho \omega^2}{T}, \quad r, \theta \text{ in } D \quad (7.294)$$

where the Laplacian in polar coordinates is given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \quad (7.295)$$

Assuming a solution of the form

$$W(r, \theta) = R(r)\Theta(\theta) \quad (7.296)$$

Eq. (7.294) reduces to

$$\left(\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} \right) \Theta + \frac{R}{r^2} \frac{d^2 \Theta}{d\theta^2} + \beta^2 R \Theta = 0 \quad (7.297)$$

which can be separated into

$$\frac{d^2 \Theta}{d\theta^2} + m^2 \Theta = 0 \quad (7.298a)$$

$$\frac{d^2 R}{dr^2} + \frac{1}{r} \frac{dR}{dr} + \left(\beta^2 - \frac{m^2}{r^2} \right) R = 0 \quad (7.298b)$$

where the constant m^2 has been assumed to be positive so as to obtain a harmonic solution for Θ . Furthermore, because the solution must be continuous, implying that the solution at $\theta = \theta_0$ must be identical to the solution at $\theta = \theta_0 + j2\pi$ ($j = 1, 2, \dots$) for any value θ_0 , m must be an integer. Hence, the solution of Eq. (7.298a) is

$$\Theta_m(\theta) = C_{1m} \sin m\theta + C_{2m} \cos m\theta, \quad m = 0, 1, 2, \dots \quad (7.299)$$

Equation (7.298b), on the other hand, is a Bessel equation and its solution is

$$R_m(r) = C_{3m} J_m(\beta r) + C_{4m} Y_m(\beta r), \quad m = 0, 1, 2, \dots \quad (7.300)$$

where $J_m(\beta r)$ and $Y_m(\beta r)$ are Bessel functions of order m and of the first and second kind, respectively. The general solution can be written in the form

$$\begin{aligned} W_m(r, \theta) = & A_{1m} J_m(\beta r) \sin m\theta + A_{2m} J_m(\beta r) \cos m\theta \\ & + A_{3m} Y_m(\beta r) \sin m\theta + A_{4m} Y_m(\beta r) \cos m\theta, \\ & m = 0, 1, 2, \dots \end{aligned} \quad (7.301)$$

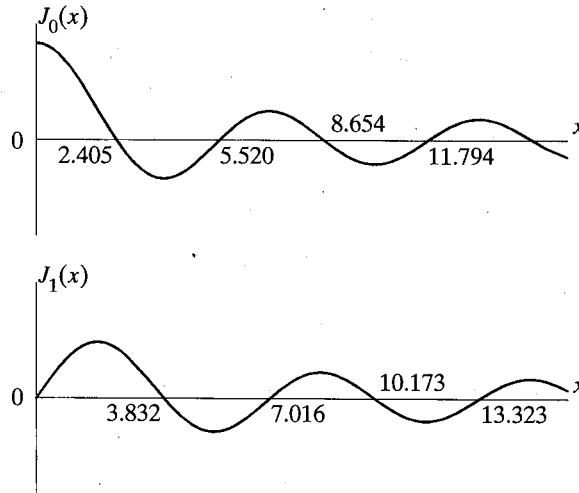


Figure 7.18 The zeros of the Bessel functions $J_0(x)$ and $J_1(x)$

Next, we consider a membrane *fixed* at the boundary $r = a$, so that the boundary condition at $r = a$ is

$$W_m(a, \theta) = 0, \quad m = 0, 1, 2, \dots \quad (7.302)$$

At every interior point of the membrane the displacement must be finite. But Bessel functions of the second kind tend to infinity as the argument approaches zero. It follows that $A_{3m} = A_{4m} = 0$, so that Eq. (7.301) reduces to

$$W_m(r, \theta) = A_{1m} J_m(\beta r) \sin m\theta + A_{2m} J_m(\beta r) \cos m\theta, \quad m = 0, 1, 2, \dots \quad (7.303)$$

At $r = a$, however, we have

$$W_m(a, \theta) = A_{1m} J_m(\beta a) \sin m\theta + A_{2m} J_m(\beta a) \cos m\theta = 0, \quad m = 0, 1, 2, \dots \quad (7.304)$$

regardless of the value of θ . This can be satisfied only if

$$J_m(\beta a) = 0, \quad m = 0, 1, 2, \dots \quad (7.305)$$

Equations (7.305) represent an infinite set of characteristic equations, or frequency equations, as for every m there is an infinite number of discrete solutions β_{mn} corresponding to the zeros of the Bessel functions J_m . As an illustration, the Bessel functions of zero and first order are plotted in Fig. 7.18. The intersections with the x -axis provide the roots $\beta_{mn}a$, from which we obtain the natural frequencies $\omega_{mn} = \beta_{mn} \sqrt{T/\rho}$. For each frequency ω_{mn} there are two modes, except when $m = 0$, for which we obtain only one mode. It follows that for $m \neq 0$ the natural modes are degenerate. The modes can be written as

$$W_{0n}(r, \theta) = A_{0n} J_0(\beta_{0n} r), \quad n = 1, 2, \dots \quad (7.306a)$$

$$\begin{aligned} W_{mnc}(r, \theta) &= A_{mnc} J_m(\beta_{mn} r) \cos m\theta, \\ W_{mns}(r, \theta) &= A_{mns} J_m(\beta_{mn} r) \sin m\theta, \end{aligned} \quad m, n = 1, 2, \dots \quad (7.306b)$$

The problem is self-adjoint and positive definite, so that the natural modes are orthogonal. From Eqs. (7.272), we can write the orthonormality relations

$$\int_D \rho W_{mn} W_{pq} dD = \int_0^{2\pi} \int_0^a \rho W_{mn} W_{pq} r dr d\theta = \delta_{mp} \delta_{nq} \quad (7.307a)$$

$$\begin{aligned} \int_D W_{mn} L W_{pq} dD &= - \int_0^{2\pi} \int_0^a W_{mn} T \nabla^2 W_{pq} r dr d\theta \\ &= \int_0^{2\pi} \int_0^a T \left(\frac{\partial W_{mn}}{\partial r} \frac{\partial W_{rs}}{\partial r} + \frac{1}{r^2} \frac{\partial W_{mn}}{\partial \theta} \frac{\partial W_{rs}}{\partial \theta} \right) r dr d\theta \\ &= \lambda_{mn} \delta_{mp} \delta_{nq} \end{aligned} \quad (7.307b)$$

where $\lambda_{mn} = \omega_{mn}^2$. To normalize the natural modes, we write (Ref. 7, p. 190)

$$\int_D \rho W_{0n}^2 dD = \int_0^{2\pi} \int_0^a \rho A_{0n}^2 J_0^2(\beta_{0n} r) r dr d\theta = \pi \rho a^2 A_{0n}^2 J_1^2(\beta_{0n} a) = 1 \quad (7.308)$$

so that

$$A_{0n}^2 = \frac{1}{\pi \rho a^2 J_1^2(\beta_{0n} a)} \quad (7.309)$$

Also

$$\begin{aligned} \int_D \rho W_{mnc}^2 dD &= \int_0^{2\pi} \int_0^a \rho A_{mnc}^2 J_m^2(\beta_{mnc} r) \cos^2 m\theta r dr d\theta \\ &= \frac{\pi}{2} \rho A_{mnc}^2 a^2 J_{m+1}^2(\beta_{mnc} a) = 1 \end{aligned} \quad (7.310)$$

or

$$A_{mnc}^2 = \frac{2}{\pi \rho a^2 J_{m+1}^2(\beta_{mnc} a)} \quad (7.311)$$

Similarly,

$$A_{mns}^2 = \frac{2}{\pi \rho a^2 J_{m+1}^2(\beta_{mns} a)} \quad (7.312)$$

The orthonormal modes take the form

$$W_{0n}(r, \theta) = \frac{1}{\sqrt{\pi \rho a} J_1(\beta_{0n} a)} J_0(\beta_{0n} r), \quad n = 1, 2, \dots \quad (7.313a)$$

$$\begin{aligned} W_{mnc}(r, \theta) &= \frac{\sqrt{2}}{\sqrt{\pi \rho a} J_{m+1}(\beta_{mnc} a)} J_m(\beta_{mnc} r) \cos m\theta, \\ W_{mns}(r, \theta) &= \frac{\sqrt{2}}{\sqrt{\pi \rho a} J_{m+1}(\beta_{mns} a)} J_m(\beta_{mns} r) \sin m\theta, \end{aligned} \quad m, n = 1, 2, \dots \quad (7.313b)$$

They are plotted in Figs. 7.19 and 7.20. The nodal lines are circles $r = \text{constant}$ and straight diametrical lines $\theta = \text{constant}$. For $m = 0$, there are no diametrical nodes and there are $n - 1$ circular nodes. The first three modes W_{0n} are plotted in

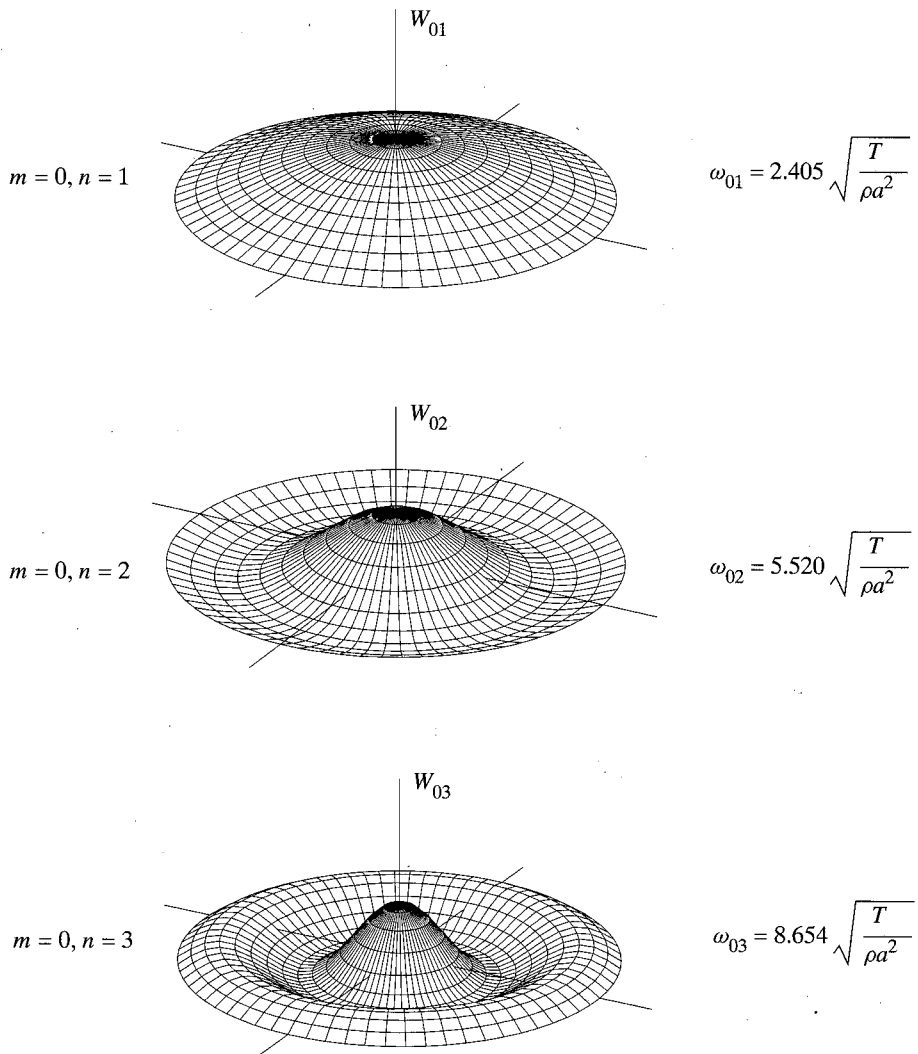


Figure 7.19 The three lowest symmetric modes of a uniform circular membrane fixed at $r = a$

Fig. 7.19. For $m = 1$ there is just one diametrical node and $n - 1$ circular nodes. The first two modes, W_{11c} and W_{12c} , are plotted in Fig. 7.20. In general, the mode W_{mn} has m equally spaced diametrical nodes and $n - 1$ circular nodes (the boundary is excluded) of radius $r_i = (\beta_{mi}/\beta_{mn}) a$ ($i = 1, 2, \dots, n - 1$).

Note that, for very large arguments, we have the relation

$$\lim_{z \rightarrow \infty} J_m(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{2m + 1}{4}\pi\right) \quad (7.314)$$

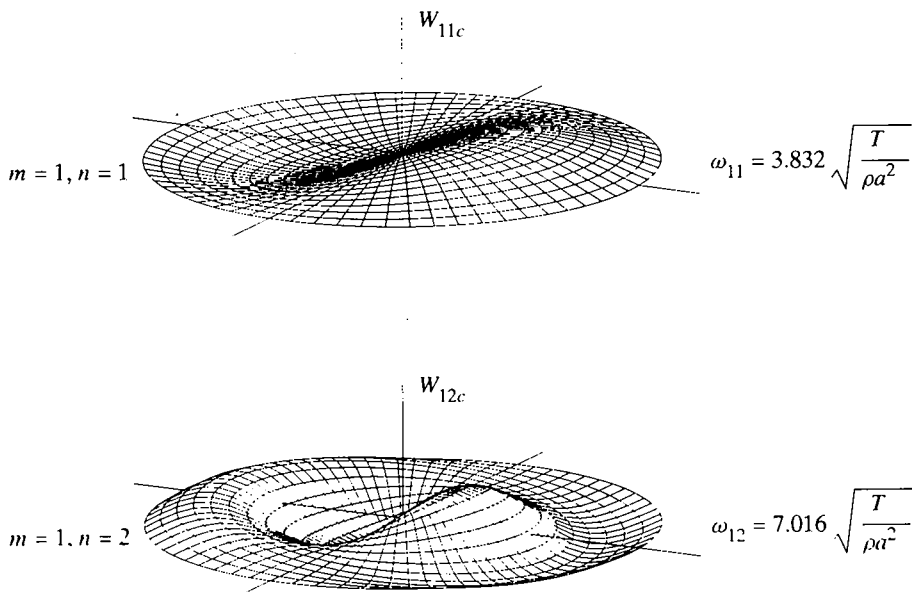


Figure 7.20 The two lowest antisymmetric modes of a uniform circular membrane fixed at $r = a$

so that the frequency equation, Eq. (7.305), leads us to the conclusion that, for very large n , the natural frequencies can be approximated by

$$\omega_{mn} = \left(\frac{m}{2} + n - \frac{1}{4} \right) \pi \sqrt{\frac{T}{\rho a^2}} \tag{7.315}$$

where both m and n are integers.

7.13 VIBRATION OF PLATES

In contrast to membranes, plates do have bending stiffness in a manner similar to beams in bending. There is one difference between beams and plates in bending, however. The beam is essentially a one-dimensional system. When a differential beam element bends, a portion of the material undergoes tension and the remaining portion undergoes compression, with the neutral axis acting as the dividing line between the two regions. The part in tension tends to contract laterally and the part in compression tends to expand. As long as the width of the beam is small, this lateral contraction and expansion is free to take place, and there are no lateral stresses. This is the essence of the ordinary beam theory. As the width of the beam increases, this effect tends to bend the cross section, so that a curvature is produced in the plane of the cross section, in addition to the curvature in the plane of bending. Let us now consider a plate and imagine for the moment that the plate is made up of individual parallel beams, obtained by dividing the plate by means of vertical planes, and focus our attention on material elements belonging to two such adjacent beams.

When undeformed, they share a lateral surface that is part of the dividing vertical plane. If allowed to behave like beams in bending, when the plate begins to bend these two adjacent beam elements would expand and contract laterally, so that in expanding each element would cross the dividing surface and occupy space belonging to the adjacent element, and in contracting each element would pull away from the dividing surface, resulting in a void in the material. In reality, this situation is not possible, so that internal lateral stresses must arise to prevent it from happening. Furthermore, in the case of plates one can think of two planes of bending, producing in general two distinct curvatures. In addition to bending, there is also twist present, because an element of plate area can be regarded as belonging to two orthogonal strips, so that bending of one strip can be looked upon as twisting of the orthogonal strip.

The *elementary theory of plates* is based on the following assumptions:

1. Deflections are small when compared with the plate thickness.
2. The normal stresses in the direction transverse to the plate can be ignored.
3. There is no force resultant on the cross-sectional area of a plate element. The middle plane of the plate does not undergo deformations during bending and can be regarded as a neutral plane.
4. Any straight line normal to the middle plane before deformation remains a straight line normal to the neutral plane during deformation.

These assumptions are reasonable for a relatively thin plate with no forces acting in the middle plane.

The boundary-value problem for a plate in bending vibration can be obtained by means of the extended Hamilton's principle, Eq. (7.4). To this end, we will find it convenient to begin with a description of the motion in terms of rectangular coordinates. The potential energy can be shown to have the expression (Ref. 15)

$$V = \frac{1}{2} \int_D D_E \left\{ (\nabla^2 w)^2 + 2(1 - \nu) \left[\left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right] \right\} dD \quad (7.316)$$

where

$$D_E = \frac{Eh^3}{12(1 - \nu^2)} \quad (7.317)$$

is the plate flexural rigidity, in which E is Young's modulus, h the plate thickness and ν Poisson's ratio. The kinetic energy is simply

$$T = \frac{1}{2} \int_D m \dot{w}^2 dD \quad (7.318)$$

and the virtual work of the nonconservative forces is given by

$$\overline{\delta W} = \int_D f \delta w dD \quad (7.319)$$

where f is the force density. Note that, for simplicity, we assumed that there are no lumped masses and springs at the boundaries.

The variation in the potential energy has the form

$$\delta V = \int_D D_E \left\{ \nabla^2 w \delta \nabla^2 w + (1 - \nu) \left[\frac{\partial^2 w}{\partial x \partial y} \delta \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 w}{\partial y \partial x} \delta \frac{\partial^2 w}{\partial y \partial x} - \frac{\partial^2 w}{\partial x^2} \delta \frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w}{\partial y^2} \delta \frac{\partial^2 w}{\partial y^2} \right] \right\} dD \quad (7.320)$$

To render δV in a form involving variations in the displacement and the first partial derivatives of the displacement with respect to x and y alone, we use the relation (Ref. 8)

$$\nabla^2 u \nabla^2 v = u \nabla^4 v - \nabla \cdot (u \nabla \nabla^2 v) + \nabla \cdot (\nabla^2 v \nabla u) \quad (7.321)$$

Then, assuming that the variation and differentiation processes are interchangeable, letting $u = \delta w$, $v = w$ and using the divergence theorem, Eq. (7.258), it can be shown that for *uniform flexural rigidity* Eq. (7.320) reduces to

$$\begin{aligned} \delta V = & \int_D D_E \nabla^4 w \delta w dD \\ & + \int_S D_E \left\{ \left[\frac{\partial}{\partial y} \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) + (1 - \nu) \frac{\partial^3 w}{\partial x^2 \partial y} \right] \delta w dx \right. \\ & - (1 - \nu) \frac{\partial^2 w}{\partial x \partial y} \frac{\partial \delta w}{\partial x} dx - \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \frac{\partial \delta w}{\partial y} dx \\ & - \left[\frac{\partial}{\partial x} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) + (1 - \nu) \frac{\partial^3 w}{\partial x \partial y^2} \right] \delta w dy \\ & \left. + (1 - \nu) \frac{\partial^2 w}{\partial x \partial y} \frac{\partial \delta w}{\partial y} dy + \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \frac{\partial \delta w}{\partial x} dy \right\} \quad (7.322) \end{aligned}$$

where $\nabla^4 = \nabla^2 \nabla^2$ is known as the *biharmonic operator*. Equation (7.322) can be expressed in terms of moments and forces by introducing the formulas (Ref. 13)

$$M_x = -D_E \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \quad (7.323a)$$

$$M_y = -D_E \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \quad (7.323b)$$

$$M_{xy} = -D_E (1 - \nu) \frac{\partial^2 w}{\partial x \partial y} \quad (7.323c)$$

$$Q_x = -D_E \left[\frac{\partial}{\partial x} \left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) + (1 - \nu) \frac{\partial^3 w}{\partial x \partial y^2} \right]$$

$$= \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \quad (7.323d)$$

$$\begin{aligned} Q_y &= -D_E \left[\frac{\partial}{\partial y} \left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) + (1 - \nu) \frac{\partial^2 w}{\partial x^2 \partial y} \right] \\ &= \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \end{aligned} \quad (7.323e)$$

where M_x and M_y are bending moments, M_{xy} is a twisting moment and Q_x and Q_y are shearing forces. Inserting Eqs. (7.323) into Eq. (7.322), we obtain

$$\begin{aligned} \delta V &= \int_D D_E \nabla^4 w \delta w dD \\ &+ \int_S \left\{ \left[- \left(\frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \right) \delta w + M_{xy} \frac{\partial \delta w}{\partial x} + M_y \frac{\partial \delta w}{\partial y} \right] dx \right. \\ &\left. + \left[\left(\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \right) \delta w - M_{xy} \frac{\partial \delta w}{\partial y} - M_x \frac{\partial \delta w}{\partial x} \right] dy \right\} \\ &= \int_D D_E \nabla^4 w \delta w dD + \int_S \left[\left(-Q_y \delta w + M_{xy} \frac{\partial \delta w}{\partial x} + M_y \frac{\partial \delta w}{\partial y} \right) dx \right. \\ &\left. + \left(Q_x \delta w - M_{xy} \frac{\partial \delta w}{\partial y} - M_x \frac{\partial \delta w}{\partial x} \right) dy \right] \end{aligned} \quad (7.324)$$

At this point, we wish to express the boundary integral in terms of components normal and tangent to the boundary, n and s , respectively. To this end, we refer to Fig. 7.21 and write

$$\begin{aligned} dx &= -ds \sin \phi, & dy &= ds \cos \phi \\ \frac{\partial}{\partial x} &= \frac{\partial}{\partial n} \frac{\partial n}{\partial x} + \frac{\partial}{\partial s} \frac{\partial s}{\partial x} = \cos \phi \frac{\partial}{\partial n} - \sin \phi \frac{\partial}{\partial s} \\ \frac{\partial}{\partial y} &= \frac{\partial}{\partial n} \frac{\partial n}{\partial y} + \frac{\partial}{\partial s} \frac{\partial s}{\partial y} = \sin \phi \frac{\partial}{\partial n} + \cos \phi \frac{\partial}{\partial s} \end{aligned} \quad (7.325)$$

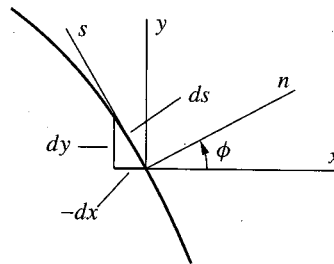


Figure 7.21 Tangential and normal directions at a plate boundary

Moreover, the moments and forces transform as follows (Ref. 13):

$$\begin{aligned} M_x \cos^2 \phi + 2M_{xy} \sin \phi \cos \phi + M_y \sin^2 \phi &= M_n \\ (M_y - M_x) \sin \phi \cos \phi + M_{xy} (\cos^2 \phi - \sin^2 \phi) &= M_{ns} \\ Q_x \cos \phi + Q_y \sin \phi &= Q_n \end{aligned} \quad (7.326)$$

Introducing Eqs. (7.325) and (7.326) into Eq. (7.324), we have

$$\delta V = \int_D D_E \nabla^4 w \delta w dD + \int_S \left(-M_n \frac{\partial \delta w}{\partial n} - M_{ns} \frac{\partial \delta w}{\partial s} + Q_n \delta w \right) ds \quad (7.327)$$

Equation (7.327) is still not in a form suitable for the derivation of the boundary-value problem. Indeed, the boundary integral would lead to three boundary conditions to be satisfied at every point of S , when in fact only two are called for. To resolve this apparent paradox, we carry out the following integration by parts

$$\int_S M_{ns} \frac{\partial \delta w}{\partial s} ds = M_{ns} \delta w \Big|_S - \int_S \frac{\partial M_{ns}}{\partial s} \delta w ds \quad (7.328)$$

If S is a closed smooth curve, then $M_{ns} \delta w \Big|_S = 0$, and Eq. (7.328) reduces to

$$\int_S M_{ns} \frac{\partial \delta w}{\partial s} ds = - \int_S \frac{\partial M_{ns}}{\partial s} \delta w ds \quad (7.329)$$

Inserting (7.329) into Eq. (7.327), we obtain the variation in the potential energy in the desired form

$$\delta V = \int_D D_E \nabla^4 w \delta w dD + \int_S \left[-M_n \delta \frac{\partial w}{\partial n} + \left(Q_n + \frac{\partial M_{ns}}{\partial s} \right) \delta w \right] ds \quad (7.330)$$

If S is not a smooth curve, as in the case in which the boundary is in the form of a polygon, the term $M_{ns} \delta w \Big|_S$ gives rise to a so-called *corner condition* (Ref. 13). For a clamped corner, or a simply-supported corner, $\delta w = 0$ and for a free corner $M_{ns} = 0$. With the proviso that the term $M_{ns} \delta w \Big|_S$ is either zero or is handled separately, we accept δV as given by Eq. (7.330).

Using Eq. (7.318) and recalling that δw vanishes at $t = t_1, t_2$, we can carry out an integration by parts with respect to t and write

$$\begin{aligned} \int_{t_1}^{t_2} \delta T dt &= \int_{t_1}^{t_2} \int_D m \dot{w} \delta \dot{w} dD dt = \int_D \int_{t_1}^{t_2} m \frac{\partial w}{\partial t} \frac{\partial}{\partial t} \delta w dt dD \\ &= \int_D \left[m \frac{\partial w}{\partial t} \delta w \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial}{\partial t} \left(m \frac{\partial w}{\partial t} \right) \delta w dt \right] dD \\ &= - \int_{t_1}^{t_2} \int_D m \ddot{w} \delta w dD dt \end{aligned} \quad (7.331)$$

Inserting Eqs. (7.319), (7.330) and (7.331) into the extended Hamilton's principle, Eq. (7.4), we obtain

$$\int_{t_1}^{t_2} \left\{ - \int_D (D_E \nabla^4 w + m \ddot{w} - f) \delta w dD + \int_S \left[M_n \delta \frac{\partial w}{\partial n} - \left(Q_n + \frac{\partial M_{ns}}{\partial s} \right) \delta w \right] ds \right\} dt = 0 \quad (7.332)$$

Then, using the customary arguments concerning the arbitrariness of the virtual displacements, we conclude that Eq. (7.332) is satisfied if and only if

$$-D_E \nabla^4 w + f = m \ddot{w} \text{ in } D \quad (7.333)$$

and either

$$M_n = 0 \text{ on } S \quad (7.334a)$$

or

$$\frac{\partial w}{\partial n} = 0 \text{ on } S \quad (7.334b)$$

and either

$$Q_{\text{eff}} = Q_n + \frac{\partial M_{ns}}{\partial s} = 0 \text{ on } S \quad (7.335a)$$

where Q_{eff} denotes an "effective" shearing force, or

$$w = 0 \text{ on } S \quad (7.335b)$$

The boundary-value problem consists of the partial differential equation Eq. (7.333) and two boundary conditions, one from Eqs. (7.334) and one from Eqs. (7.335).

The relations between the moments and shearing forces and deformations, in terms of normal and tangential coordinates (see Problem 7.43), are

$$M_n = -D_E \nabla^2 w + (1 - \nu) D_E \left(\frac{1}{R} \frac{\partial w}{\partial n} + \frac{\partial^2 w}{\partial s^2} \right) \quad (7.336a)$$

$$M_{ns} = -(1 - \nu) D_E \left(\frac{\partial^2 w}{\partial n \partial s} - \frac{1}{R} \frac{\partial w}{\partial s} \right) \quad (7.336b)$$

$$Q_n = -D_E \frac{\partial}{\partial n} \nabla^2 w \quad (7.336c)$$

where the Laplacian has the form

$$\nabla^2 = \frac{\partial^2}{\partial n^2} + \frac{1}{R} \frac{\partial}{\partial n} + \frac{\partial^2}{\partial s^2} \quad (7.337)$$

in which R is the radius of curvature of the boundary.

Boundary condition (7.335a) is associated with the name of Kirchhoff and is of some historical interest. Poisson believed that M_n , Q_n and M_{ns} must be independently zero at a free boundary, yielding a total of three boundary conditions, one too many for a fourth-order differential equation. Later, however, Kirchhoff

cleared up the problem by pointing out that Q_n and M_{ns} are related as indicated by Eq. (7.335a). If variational principles are used to formulate the boundary-value problem, the boundary conditions are obtained both in the right number and in the correct form.

To derive the eigenvalue problem, we let $f = 0$ and assume a solution in the form

$$w = WF \quad (7.338)$$

where W depends on the spatial coordinates only and F is a time-dependent harmonic function of frequency ω . Then, following the usual steps involved in the separation of variables, the differential equation, Eq. (7.333), reduces to

$$D_E \nabla^4 W = \lambda m W, \quad \lambda = \omega^2, \quad \text{in } D \quad (7.339)$$

As mentioned earlier in this section, the boundary conditions to be satisfied at every point of S must be chosen from Eqs. (7.334) and (7.335) on the basis of physical considerations. For example, at a clamped edge, the displacement and slope must be zero. Hence, using Eq. (7.338) and dividing through by F , the boundary conditions for a *clamped edge* are simply

$$W = 0, \quad \frac{\partial W}{\partial n} = 0 \quad (7.340a, b)$$

Moreover, from Eqs. (7.335b) and (7.334a), with due consideration to Eq. (7.336a), the boundary conditions for a *simply supported edge* become

$$W = 0, \quad \nabla^2 W - (1 - \nu) \left(\frac{1}{R} \frac{\partial W}{\partial n} + \frac{\partial^2 W}{\partial s^2} \right) = 0 \quad (7.341a, b)$$

and, because W does not vary along the edge, Eqs. (7.341) assume the simplified form

$$W = 0, \quad \frac{\partial^2 W}{\partial n^2} + \frac{\nu}{R} \frac{\partial W}{\partial n} = 0 \quad (7.342a, b)$$

Similarly using Eqs. (7.334a) and (7.335a), in conjunction with Eqs. (7.336), the boundary conditions along a *free edge* are

$$\nabla^2 W - (1 - \nu) \left(\frac{1}{R} \frac{\partial W}{\partial n} + \frac{\partial^2 W}{\partial s^2} \right) = 0 \quad (7.343a)$$

$$\frac{\partial}{\partial n} \nabla^2 W + (1 - \nu) \frac{\partial}{\partial s} \left(\frac{\partial^2 W}{\partial n \partial s} - \frac{1}{R} \frac{\partial W}{\partial s} \right) = 0 \quad (7.343b)$$

Again, we must recognize that the eigenvalue problem for the transverse vibration of uniform plates fits the pattern of Sec. 7.5. In this case, the stiffness operator and mass density are

$$L = D_E \nabla^4, \quad M = m \quad (7.344)$$

and it follows that the eigenvalue problem is of the special type, in the sense that M is a mere function and the boundary conditions do not depend on λ .

Before proceeding to the solution of the eigenvalue problem for some cases of interest, we propose to derive a criterion for the self-adjointness of L , and hence of the system. To this end, we let u and v be two comparison functions, use Eq. (7.321), consider the divergence theorem, Eq. (7.258), and write

$$\begin{aligned} \int_D uLv \, dD &= \int_D D_E u \nabla^4 v \, dD \\ &= \int_D D_E [\nabla \cdot (u \nabla \nabla^2 v) - \nabla \cdot (\nabla^2 v \nabla u) + \nabla^2 u \nabla^2 v] \, dD \\ &= \int_S D_E \left(u \frac{\partial}{\partial n} \nabla^2 v - \nabla^2 v \frac{\partial u}{\partial n} \right) ds + \int_D D_E \nabla^2 u \nabla^2 v \, dD \quad (7.345) \end{aligned}$$

If the integral over the boundary S vanishes, the right side of Eq. (7.345) is symmetric in u and v and the eigenvalue problem is self-adjoint. This is the case when the boundary points are simply supported, clamped, or free. Note that when some or all boundary points are supported by springs, the potential energy, Eq. (7.316), must be modified so as to include a boundary term of the type $\frac{1}{2} \int_S k w^2 \, dS$. In this case, the integral over the boundary S in Eq. (7.345) does not vanish but is symmetric in u and v , so that the eigenvalue problem is once again self-adjoint.

For self-adjoint systems the eigenvalues are real and the eigenfunctions are real and orthogonal. We assume that the eigenfunctions have been normalized so as to satisfy the orthonormality conditions

$$\int_D m W_r W_s \, dD = \delta_{rs}, \quad r, s = 1, 2, \dots \quad (7.346a)$$

$$\int_D D_e W_r \nabla^4 W_s \, dD = \lambda_r \delta_{rs}, \quad r, s = 1, 2, \dots \quad (7.346b)$$

As in the case of vibration of thin membranes, the shape of the boundary dictates the type of coordinates to be used. For plates, however, the satisfaction of the boundary conditions turns out to be a much more formidable task than for membranes. Only rectangular and circular plates will be discussed here.

The boundary-value problem defined by the differential equation (7.333) and boundary conditions from Eqs. (7.334) and (7.335) and the eigenvalue problem defined by the differential equation (7.339) and boundary conditions from Eqs. (7.340)–(7.343) are for plates of constant flexural rigidity alone. If the flexural rigidity is not constant, such as when the plate thickness varies, additional terms must be included (Ref. 15). No closed-form solutions can be expected for variable-thickness plates.

The plate theory presented here ignores shear deformation and rotatory inertia effects and is known as the classical plate theory. An extension of the theory so as to include shear deformation in the static deflection of plates was carried out by Reissner (Ref. 12) and to include both shear deformation and rotatory inertia in the vibration of plates by Mindlin (Ref. 10). For a discussion of Mindlin's higher-order plate theory, see the monograph by Leissa (Ref. 6).

i. Rectangular plates

We consider a *uniform rectangular plate* extending over a domain D defined by $0 < x < a$ and $0 < y < b$. The boundaries of the domain are the straight lines $x = 0, a$ and $y = 0, b$. Equation (7.339), in rectangular coordinates, takes the form

$$\nabla^4 W(x, y) - \beta^4 W(x, y) = 0, \quad \beta^4 = \frac{\omega^2 m}{D_E}, \quad x, y \text{ in } D \quad (7.347)$$

where the biharmonic operator is given by

$$\nabla^4 = \nabla^2 \nabla^2 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \quad (7.348)$$

Equation (7.347) can be expressed in the operator form

$$(\nabla^4 - \beta^4) W(x, y) = (\nabla^2 + \beta^2)(\nabla^2 - \beta^2) W(x, y) = 0 \quad (7.349)$$

which permits us to write

$$(\nabla^2 - \beta^2) W = W_1, \quad (\nabla^2 + \beta^2) W_1 = 0 \quad (7.350a, b)$$

Because β^2 is constant, the solution of Eq. (7.350a), and hence the solution of Eq. (7.347), is

$$W = W_1 + W_2 \quad (7.351)$$

where W_2 is the solution of the homogeneous equation

$$(\nabla^2 - \beta^2) W_2 = [\nabla^2 + (i\beta)^2] W_2 = 0 \quad (7.352)$$

We note here that the proportionality factor $-1/2\beta^2$ multiplying W_1 in Eq. (7.351) was omitted as irrelevant, because W_1 is obtained by solving a homogeneous equation, Eq. (7.350b). Equation (7.350b) resembles the equation for the vibration of a thin uniform membrane, whose general solution was obtained in Sec. 7.12 in the form of Eq. (7.283). Moreover, Eq. (7.352) has the same form as Eq. (7.350b), except that β is replaced by $i\beta$. Hence, the solution of Eq. (7.352) can be obtained from Eq. (7.283) by replacing the trigonometric functions by hyperbolic functions. It follows that the general solution of Eq. (7.347) is

$$\begin{aligned} W(x, y) = & A_1 \sin \alpha x \sin \gamma y + A_2 \sin \alpha x \cos \gamma y + A_3 \cos \alpha x \sin \gamma y \\ & + A_4 \cos \alpha x \cos \gamma y + A_5 \sinh \alpha_1 x \sinh \gamma_1 y \\ & + A_6 \sinh \alpha_1 x \cosh \gamma_1 y + A_7 \cosh \alpha_1 x \sinh \gamma_1 y \\ & + A_8 \cosh \alpha_1 x \cosh \gamma_1 y, \quad \alpha^2 + \gamma^2 = \alpha_1^2 + \gamma_1^2 = \beta^2 \end{aligned} \quad (7.353)$$

We consider a *simply supported* plate. Because for a straight boundary the radius of curvature R is infinite the boundary conditions, Eqs. (7.342), reduce to

$$W = 0, \quad \frac{\partial^2 W}{\partial x^2} = 0, \quad x = 0, a \quad (7.354a, b)$$

$$W = 0, \quad \frac{\partial^2 W}{\partial y^2} = 0, \quad y = 0, b \quad (7.354c, d)$$

Upon using boundary conditions (7.354), we conclude that all the coefficients A_i , with the exception of A_1 , vanish and, in addition, we obtain the two characteristic equations

$$\sin \alpha a = 0, \quad \sin \gamma b = 0 \quad (7.355a, b)$$

Their solutions are

$$\alpha_m a = m\pi, \quad m = 1, 2, \dots \quad (7.356a)$$

$$\gamma_n b = n\pi, \quad n = 1, 2, \dots \quad (7.356b)$$

so that the natural frequencies of the system become

$$\omega_{mn} = \beta_{mn}^2 \sqrt{\frac{D_E}{m}} = \pi^2 \left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 \right] \sqrt{\frac{D_E}{m}}, \quad m, n = 1, 2, \dots \quad (7.357)$$

and the corresponding natural modes, normalized so that $\int_0^a \int_0^b m W_{mn}^2 dx dy = 1$, are

$$W_{mn}(x, y) = \frac{2}{\sqrt{mab}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad m, n = 1, 2, \dots \quad (7.358)$$

which are identical to the modes of the clamped rectangular membrane. However, the natural frequencies are different from those of the membrane.

It is easy to see that boundary conditions (7.354) render the boundary integral in Eq. (7.345) zero, so that the eigenvalue problem is self-adjoint. It follows immediately that the eigenfunctions are orthonormal, satisfying Eqs. (7.346).

A special class of eigenvalue problems for rectangular plates admitting closed-form solution is characterized by the fact that two opposing sides are simply supported. The interesting part is that attempts to obtain closed-form solutions by means of Eq. (7.353) do not bear fruit. An approach yielding results uses experience gained from the simply supported plate to assume a solution separable in x and y in which the part associated with the simply supported sides is given. To illustrate the approach, we consider a plate *simply supported at* $x = 0, a$ and *clamped at* $y = 0, b$. Then, consistent with results obtained for the plate simply supported on all sides, we assume a solution of the form (Ref. 14)

$$W_m(x, y) = Y_m(y) \sin \alpha_m x \quad (7.359)$$

in which, according to Eq. (7.356a), $\alpha_m = m\pi/a$ ($m = 1, 2, \dots$). Inserting Eq. (7.359) into Eq. (7.347), recalling Eq. (7.348) and dividing through by $\sin \alpha_m x$, we obtain

$$\frac{d^4 Y_m(y)}{dy^4} - 2\alpha_m^2 \frac{d^2 Y_m(y)}{dy^2} + (\alpha_m^4 - \beta_m^4) Y_m(y) = 0, \quad 0 < y < b \quad (7.360)$$

where, in view of Eqs. (7.340), Y_m must satisfy the boundary conditions

$$Y_m = 0, \quad \frac{dY_m}{dy} = 0, \quad y = 0, b \quad (7.361a, b)$$

The solution of Eq. (7.360) has the exponential form

$$Y_m(y) = Ae^{s_m y} \tag{7.362}$$

Inserting Eq. (7.362) into Eq. (7.360) and dividing through by $e^{s_m y}$, we obtain the characteristic equation

$$s_m^4 - 2\alpha_m^2 s_m^2 + \alpha_m^4 - \beta_m^4 = 0 \tag{7.363}$$

which represents a quadratic equation in s_m^2 . The solutions of Eq. (7.363) can be shown to be

$$s_{1m} = -s_{2m} = \gamma_{1m} = \sqrt{\beta_m^2 + \alpha_m^2}, \quad s_{3m} = -s_{4m} = i\gamma_{2m} = i\sqrt{\beta_m^2 - \alpha_m^2} \tag{7.364}$$

so that solution (7.362) can be rewritten as

$$\begin{aligned} Y_m(y) &= A_1 e^{s_{1m} y} + A_2 e^{s_{2m} y} + A_3 e^{s_{3m} y} + A_4 e^{s_{4m} y} \\ &= C_1 \cosh \gamma_{1m} y + C_2 \sinh \gamma_{1m} y + C_3 \cos \gamma_{2m} y + C_4 \sin \gamma_{2m} y \end{aligned} \tag{7.365}$$

Introducing Eq. (7.365) into Eqs. (7.361), we obtain

$$\begin{aligned} Y_m(0) &= C_1 + C_3 = 0 \\ Y'_m(0) &= C_2 \gamma_{1m} + C_4 \gamma_{2m} = 0 \\ Y_m(b) &= C_1 \cosh \gamma_{1m} b + C_2 \sinh \gamma_{1m} b + C_3 \cos \gamma_{2m} b + C_4 \sin \gamma_{2m} b = 0 \\ Y'_m(b) &= C_1 \gamma_{1m} \sinh \gamma_{1m} b + C_2 \gamma_{1m} \cosh \gamma_{1m} b \\ &\quad - C_3 \gamma_{2m} \sin \gamma_{2m} b + C_4 \gamma_{2m} \cos \gamma_{2m} b = 0 \end{aligned} \tag{7.366}$$

where primes denote derivatives with respect to y . Equations (7.366) have a solution provided the determinant of the coefficients is zero, or

$$\begin{aligned} \Delta(\beta_m) &= \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & \gamma_{1m} & 0 & \gamma_{2m} \\ \cosh \gamma_{1m} b & \sinh \gamma_{1m} b & \cos \gamma_{2m} b & \sin \gamma_{2m} b \\ \gamma_{1m} \sinh \gamma_{1m} b & \gamma_{1m} \cosh \gamma_{1m} b & -\gamma_{2m} \sin \gamma_{2m} b & \gamma_{2m} \cos \gamma_{2m} b \end{vmatrix} \\ &= 2\gamma_{1m} \gamma_{2m} (1 - \cosh \gamma_{1m} b \cos \gamma_{2m} b) \\ &\quad + (\gamma_{1m}^2 - \gamma_{2m}^2) \sinh \gamma_{1m} b \sin \gamma_{2m} b = 0, \quad m = 1, 2, \dots \end{aligned} \tag{7.367}$$

Equations (7.367) with γ_{1m} and γ_{2m} given by Eqs. (7.364) represent an infinity of characteristic equations, one for every m , and each equation has an infinity of roots β_m . We identify these roots by $n = 1, 2, \dots$ and denote the double infinity of roots by β_{mn} ($m, n = 1, 2, \dots$). Then, inserting these values into Eqs. (7.364), we obtain

$$\gamma_{1mn} = \sqrt{\beta_{mn}^2 + \alpha_m^2}, \quad \gamma_{2mn} = \sqrt{\beta_{mn}^2 - \alpha_m^2}, \quad m, n = 1, 2, \dots \tag{7.368}$$

Moreover, solving Eqs. (7.366) for C_2 , C_3 and C_4 in terms of C_1 and using Eqs. (7.368), we can write

$$Y_{mn}(y) = C_{mn} \left[\cosh \gamma_{1mn} y - \cos \gamma_{2mn} y - \frac{\cosh \gamma_{1mn} b - \cos \gamma_{2mn} b}{\sinh \gamma_{1mn} b - (\gamma_{1mn}/\gamma_{2mn}) \sin \gamma_{2mn} b} (\sinh \gamma_{1mn} y - \frac{\gamma_{1mn}}{\gamma_{2mn}} \sin \gamma_{2mn} y) \right], \quad m, n = 1, 2, \dots \quad (7.369)$$

Finally, inserting Eq. (7.369) into Eq. (7.359), we obtain the desired eigenfunctions in the general form

$$W_{mn}(x, y) = Y_{mn}(y) \sin \alpha_m x, \quad m, n = 1, 2, \dots \quad (7.370)$$

Moreover, from Eq. (7.347), we conclude that the natural frequencies are

$$\omega_{mn} = \beta_{mn}^2 \sqrt{D_E/m}, \quad m, n = 1, 2, \dots \quad (7.371)$$

No confusion should arise from the fact that the symbol m denotes both the mass density and the first subscript in the natural frequencies and modes. We observe from Eq. (7.368) that the quantities γ_{1mn} and γ_{2mn} , defining the dependence of the eigenfunctions on y , are functions of the quantities α_m , defining the dependence of the eigenfunctions on x . By contrast, in the case of a plate simply supported on all sides, α_m and γ_n are independent of one another.

The eigenvalue problem has been solved numerically for a plate of sides ratio $a/b = 1.5$. Table 7.1 shows normalized natural frequencies for $m, n = 1, 2, 3$.

TABLE 7.1 Normalized Natural Frequencies

$$\bar{\omega}_{mn} = \omega_{mn} b^2 \sqrt{m/D_E}$$

$m \backslash n$	1	2	3
1	25.043584	65.007865	124.51603
2	35.103815	75.604983	135.61235
3	54.743071	94.585278	154.77570

ii. Circular plates

Now we consider a uniform circular plate extending over a domain D given by $0 < r < a$, where the boundary S of the domain is the circle $r = a$. Because the boundary is circular, we use the polar coordinates r and θ , so that the differential equation is

$$\nabla^4 W(r, \theta) - \beta^4 W(r, \theta) = 0, \quad \beta^4 = \frac{\omega^2 m}{D_E}, \quad r, \theta \text{ in } D \quad (7.372)$$

where the biharmonic operator, in polar coordinates, has the form

$$\nabla^4 = \nabla^2 \nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \quad (7.373)$$

Following the same pattern as for rectangular plates, Eq. (7.372) can be written in the operator form

$$(\nabla^2 + \beta^2) W_1(r, \theta) = 0, \quad [\nabla^2 + (i\beta)^2] W_2(r, \theta) = 0 \quad (7.374a, b)$$

which must be satisfied over the domain D . Equation (7.374a) has precisely the same form as the equation for the vibration of circular membranes, Eq. (7.294), so that its solution is given by Eq. (7.301), although the definition of β is not the same as for membranes. Moreover, the solution of Eq. (7.374b) is obtained from Eq. (7.301) by replacing β by $i\beta$. The Bessel functions of imaginary argument, $J_m(ix)$ and $Y_m(ix)$, are called *modified or hyperbolic Bessel functions* and denoted by $I_m(x)$ and $K_m(x)$, respectively. The hyperbolic Bessel functions are not equal to the ordinary Bessel functions of imaginary argument but are proportional to them (Ref. 7, p. 113). It follows that the solution of Eq. (7.372), which is the sum of the solutions of Eqs. (7.374), has the form

$$\begin{aligned} W_m(r, \theta) = & [A_{1m} J_m(\beta r) + A_{3m} Y_m(\beta r) + B_{1m} I_m(\beta r) + B_{3m} K_m(\beta r)] \sin m\theta \\ & + [A_{2m} J_m(\beta r) + A_{4m} Y_m(\beta r) + B_{2m} I_m(\beta r) + B_{4m} K_m(\beta r)] \cos m\theta, \\ & m = 0, 1, 2, \dots \quad (7.375) \end{aligned}$$

where $W_m(r, \theta)$ is subject to given boundary conditions.

As an example, we consider the case of a *clamped plate*, for which the boundary conditions are

$$W(a, \theta) = 0, \quad \left. \frac{\partial W(r, \theta)}{\partial r} \right|_{r=a} = 0 \quad (7.376a, b)$$

In addition, *the solution must be finite at every interior point*. This immediately eliminates Bessel functions of the second kind, Y_m and K_m , which become infinite at $r = 0$. Hence, solution (7.375) reduces to

$$\begin{aligned} W_m(r, \theta) = & [A_{1m} J_m(\beta r) + B_{1m} I_m(\beta r)] \sin m\theta \\ & + [A_{2m} J_m(\beta r) + B_{2m} I_m(\beta r)] \cos m\theta, \quad m = 0, 1, 2, \dots \quad (7.377) \end{aligned}$$

Boundary condition (7.376a) yields

$$B_{1m} = -\frac{J_m(\beta a)}{I_m(\beta a)} A_{1m}, \quad B_{2m} = -\frac{J_m(\beta a)}{I_m(\beta a)} A_{2m}, \quad m = 0, 1, 2, \dots \quad (7.378)$$

so that

$$\begin{aligned} W_m(r, \theta) = & \left[J_m(\beta r) - \frac{J_m(\beta a)}{I_m(\beta a)} I_m(\beta r) \right] (A_{1m} \sin m\theta + A_{2m} \cos m\theta), \\ & m = 0, 1, 2, \dots \quad (7.379) \end{aligned}$$

On the other hand, boundary condition (7.376b) leads to the set of characteristic equations

$$\left[\frac{d}{dr} J_m(\beta r) - \frac{J_m(\beta a)}{I_m(\beta a)} \frac{d}{dr} I_m(\beta r) \right]_{r=a} = 0, \quad m = 0, 1, 2, \dots \quad (7.380)$$

But

$$\frac{d}{dr} J_m(\beta r) = \beta \left[J_{m-1}(\beta r) - \frac{m}{\beta r} J_m(\beta r) \right] \quad (7.381a)$$

$$\frac{d}{dr} I_m(\beta r) = \beta \left[I_{m-1}(\beta r) - \frac{m}{\beta r} I_m(\beta r) \right] \quad (7.381b)$$

so that the set of characteristic equations reduces to

$$I_m(\beta a) J_{m-1}(\beta a) - J_m(\beta a) I_{m-1}(\beta a) = 0, \quad m = 0, 1, 2, \dots \quad (7.382)$$

For a given m , we must solve Eq. (7.382) numerically for the eigenvalues β_{mn} . The natural frequencies are related to the eigenvalues by

$$\omega_{mn} = \beta_{mn}^2 \sqrt{\frac{DE}{m}} \quad (7.383)$$

For each frequency ω_{mn} there are two corresponding natural modes, except for $m = 0$, for which there is just one mode. Hence, as for membranes, all modes for which $m \neq 0$ are *degenerate*. The natural modes can be written in the form

$$W_{0n}(r, \theta) = A_{0n} [I_0(\beta_{0n}a) J_0(\beta_{0n}r) - J_0(\beta_{0n}a) I_0(\beta_{0n}r)], \quad n = 1, 2, \dots \quad (7.384a)$$

$$\begin{aligned} W_{mnc}(r, \theta) &= A_{mnc} [I_m(\beta_{mn}a) J_m(\beta_{mn}r) - J_m(\beta_{mn}a) I_m(\beta_{mn}r)] \cos m\theta, \\ W_{mns}(r, \theta) &= A_{mns} [I_m(\beta_{mn}a) J_m(\beta_{mn}r) - J_m(\beta_{mn}a) I_m(\beta_{mn}r)] \sin m\theta, \end{aligned} \quad m, n = 1, 2, \dots \quad (7.384b)$$

For $m = 0$, there are no diametrical nodes and there are $n - 1$ circular nodes. The modes W_{01} and W_{02} are plotted in Fig. 7.22. For $m = 1$, there is one diametrical node and $n - 1$ circular nodes. The mode W_{11c} is plotted in Fig. 7.23. Note that the overtones are not harmonic.

It is easy to see that boundary conditions (7.376) render the boundary integral in Eq. (7.345) equal to zero, so that the problem is self-adjoint. Consequently, the natural modes are orthogonal.

Defining the modified Bessel functions of the first kind by the relation $I_m(x) = i^{-m} J_m(ix)$ and using Eq. (7.314), it can be shown that, for large argument, the solutions of the characteristic equation, Eq. (7.382), tend to

$$a\beta_{mn} = \left(\frac{m}{2} + n \right) \pi \quad (7.385)$$

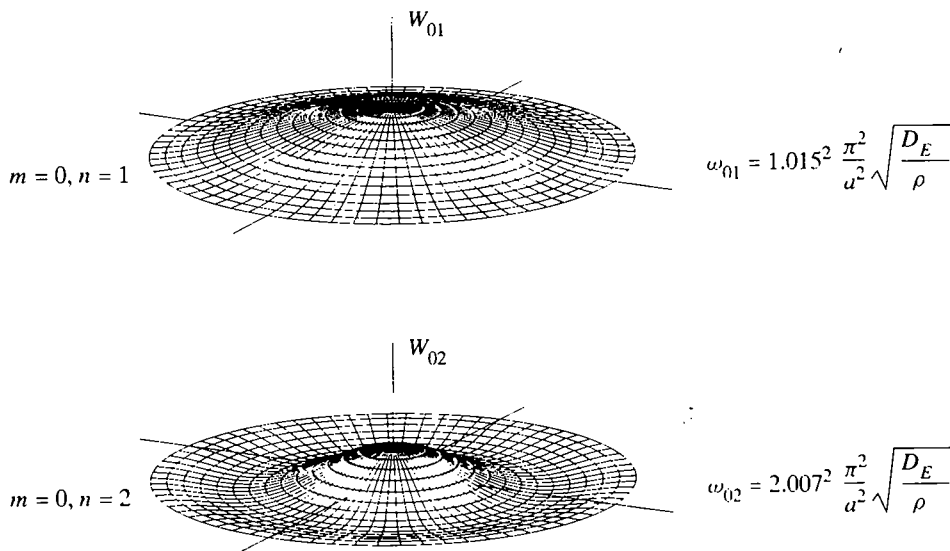


Figure 7.22 The two lowest symmetric modes of a uniform circular plate clamped at $r = a$

and, consequently, for large n , the natural frequencies tend to

$$\omega_{mn} = \left(\frac{m}{2} + n\right)^2 \frac{\pi^2}{a^2} \sqrt{\frac{D_E}{\rho}} \tag{7.386}$$

Because of the limitations of the elementary plate theory, however, this result is only of academic value.

In view of the fact that for circular plates the boundary $r = a$ is a closed smooth curve, the boundary integral in Eq. (7.345) simply vanishes. It follows that eigenvalue problems for circular plates are self-adjoint, so that the natural modes are orthogonal. Related to this is the fact that the boundary conditions for circular plates can be satisfied by working with the radial variable r alone, with the angle θ playing no role. This explains why eigenvalue problems for circular plates admit many more closed-form solutions than for rectangular plates (Ref. 6).

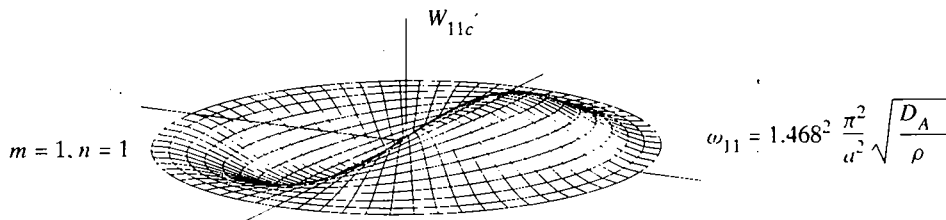


Figure 7.23 The lowest antisymmetric mode of a uniform circular plate clamped at $r = a$

7.14 VARIATIONAL FORMULATION OF THE DIFFERENTIAL EIGENVALUE PROBLEM

In Chapter 4, we demonstrated that the algebraic eigenvalue problem can be formulated as a variational problem consisting of rendering Rayleigh's quotient stationary. The extension of the approach to distributed systems can be advantageous at times, particularly when a closed-form solution to the differential eigenvalue problem proves elusive, and one must be content with an approximate solution.

Under consideration is a self-adjoint eigenvalue problem defined by the differential equation

$$Lw(x, y) = \lambda m(x, y)w(x, y), \quad \lambda = \omega^2, \quad x, y \text{ in } D \quad (7.387)$$

where L is a differential operator of order $2p$, and the boundary conditions

$$B_i w(x, y) = 0, \quad i = 1, 2, \dots, p, \quad x, y \text{ on } S \quad (7.388)$$

in which B_i are boundary differential operators of maximum order $2p - 1$. Multiplication of Eq. (7.387) by w and integration over D yields

$$\int_D wLw \, dD = \lambda \int_D mw^2 \, dD \quad (7.389)$$

which can be rewritten as

$$R(w) = \lambda = \omega^2 = \frac{\int_D wLw \, dD}{\int_D mw^2 \, dD} \quad (7.390)$$

Equation (7.390) represents *Rayleigh's quotient* for a distributed system. Clearly, if w is an eigenfunction, say w_r , then Rayleigh's quotient is the associated eigenvalue λ_r . In general, w can be regarded as a trial function, and the question arises as to the behavior of Rayleigh's quotient as w varies over the κ_B^{2p} space (Sec. 7.5). To answer this question, we invoke the expansion theorem, Eq. (7.101), and write

$$w = \sum_{r=1}^{\infty} c_r w_r \quad (7.391)$$

where w_r ($r = 1, 2, \dots$) are orthonormal eigenfunctions satisfying Eqs. (7.96). Introducing Eq. (7.391) into Eq. (7.390) and using the orthonormality relations (7.96), we obtain

$$R(c_1, c_2, \dots) = \frac{\sum_{r=1}^{\infty} \sum_{s=1}^{\infty} c_r c_s \int_D w_r L w_s \, dD}{\sum_{r=1}^{\infty} \sum_{s=1}^{\infty} c_r c_s \int_D m w_r w_s \, dD} = \frac{\sum_{r=1}^{\infty} c_r^2 \lambda_r}{\sum_{r=1}^{\infty} c_r^2} \quad (7.392)$$

and we note that Rayleigh's quotient is now a function of the coefficients c_1, c_2, \dots . Hence, the first variation in Rayleigh's quotient is simply

$$\delta R = \sum_{i=1}^{\infty} \frac{\partial R}{\partial c_i} \delta c_i \quad (7.393)$$

If the first variation δR vanishes, then Rayleigh's quotient has a stationary value. Because the coefficients c_i are all independent, the stationarity conditions are

$$\frac{\partial R}{\partial c_i} = 0, \quad i = 1, 2, \dots \quad (7.394)$$

Introducing Eq. (7.392) into Eqs. (7.394), we obtain

$$\begin{aligned} \frac{\partial R}{\partial c_i} &= \frac{\left(\sum_{r=1}^{\infty} 2c_r \frac{\partial c_r}{\partial c_i} \lambda_r \right) \sum_{r=1}^{\infty} c_r^2 - \left(\sum_{r=1}^{\infty} 2c_r \frac{\partial c_r}{\partial c_i} \right) \sum_{r=1}^{\infty} c_r^2 \lambda_r}{\left(\sum_{r=1}^{\infty} c_r^2 \right)^2} \\ &= \frac{2c_i \lambda_i \sum_{r=1}^{\infty} c_r^2 - 2c_i \sum_{r=1}^{\infty} c_r^2 \lambda_r}{\left(\sum_{r=1}^{\infty} c_r^2 \right)^2} = \frac{2c_i \sum_{r=1}^{\infty} (\lambda_i - \lambda_r) c_r^2}{\left(\sum_{r=1}^{\infty} c_r^2 \right)^2} = 0, \\ & \qquad \qquad \qquad i = 1, 2, \dots \quad (7.395) \end{aligned}$$

If w coincides with one of the eigenfunctions, say $w = w_i$, then $c_r = c_i \delta_{ir}$ ($r = 1, 2, \dots$), where δ_{ir} is the Kronecker delta. It follows that every term in the series at the numerator of (7.395) is zero, except for the term corresponding to $r = i$, and this latter term vanishes because $\lambda_i - \lambda_r = 0$ for $r = i$. Hence, conditions (7.394) are satisfied when the trial function coincides with an eigenfunction, so that *Rayleigh's quotient has stationary points at the system eigenfunctions*. These are the only stationary points of Rayleigh's quotient. Letting $w = c_j w_j$ in Eq. (7.392), we conclude that

$$R(w_j) = \lambda_j, \quad j = 1, 2, \dots \quad (7.396)$$

so that *the stationary values of Rayleigh's quotient are precisely the system eigenvalues*. These results are to be expected, as they constitute *Rayleigh's principle* for self-adjoint distributed systems and they represent the counterpart of Rayleigh's principle demonstrated in Sec. 5.2 for symmetric discrete systems.

The variational characterization of the eigenvalue problem just presented is equivalent to a certain form of the differential eigenvalue problem, Eqs. (7.387) and (7.388). To show this, we multiply Eq. (7.387) by an admissible function v , integrate over the domain D and write

$$\int_D v L w \, dD = \lambda \int_D m v w \, dD \quad (7.397)$$

Then, integrating the left side of Eq. (7.397) by parts with due consideration to the boundary conditions, Eqs. (7.388), we can write the result in the form

$$[v, w] = \lambda (\sqrt{m}v, \sqrt{m}w) \quad (7.398)$$

where $[v, w]$ is an energy inner product, Eq. (7.84), and $(\sqrt{m}v, \sqrt{m}w)$ is a weighted inner product. Equation (7.398) represents the *weak form* of the eigenvalue problem and can be stated as follows: *Determine a scalar λ and a function w in the admissible space κ_G^p such that Eq. (7.398) is satisfied for all v in κ_G^p .*

Next, we integrate the numerator of Rayleigh's quotient, Eq. (7.390), by parts and rewrite the quotient as

$$R(w) = \lambda = \frac{[w, w]}{(\sqrt{m}w, \sqrt{m}w)} \quad (7.399)$$

Then, we consider an admissible function in the neighborhood of w and write it in the form $w + \epsilon v$, where v is a function from κ_G^p and ϵ is a small parameter. Replacing w in Eq. (7.399) by the varied function $w + \epsilon v$ and carrying out a binomial expansion of the denominator, we can write

$$\begin{aligned} R(w + \epsilon v) &= \frac{[w + \epsilon v, w + \epsilon v]}{(\sqrt{m}(w + \epsilon v), \sqrt{m}(w + \epsilon v))} \\ &= \frac{[w, w] + 2\epsilon [v, w] + \epsilon^2 [v, v]}{(\sqrt{m}w, \sqrt{m}w) + 2\epsilon (\sqrt{m}v, \sqrt{m}w) + \epsilon^2 (\sqrt{m}v, \sqrt{m}v)} \\ &= R(w) + 2\epsilon \frac{[v, w] (\sqrt{m}w, \sqrt{m}w) - [w, w] (\sqrt{m}v, \sqrt{m}v)}{(\sqrt{m}w, \sqrt{m}w)^2} + O(\epsilon^2) \\ &= R(w) + 2\epsilon \frac{[v, w] - \lambda (\sqrt{m}v, \sqrt{m}w)}{(\sqrt{m}w, \sqrt{m}w)} + O(\epsilon^2). \end{aligned} \quad (7.400)$$

For a given function v , $R(w + \epsilon v)$ depends only on ϵ . If the linear term in ϵ in Eq. (7.400) is zero, i.e., *if the first variation of R vanishes, then R has a stationary value at w .* For this to happen, the coefficient of ϵ must be zero, which is the same as satisfying Eq. (7.398). Hence, *rendering Rayleigh's quotient stationary is equivalent to solving the weak form of the eigenvalue problem.*

In our discussions of the differential eigenvalue problem we assumed implicitly that the eigenvalues are ordered so as to satisfy $\lambda_1 \leq \lambda_2 \leq \dots$. Then, from Eq. (7.392), it is easy to see that *Rayleigh's quotient is an upper bound for the lowest eigenvalue, or*

$$R(w) \geq \lambda_1 \quad (7.401)$$

which also implies that the minimum value Rayleigh's quotient can take is λ_1 , or

$$\lambda_1 = \min R(w) \quad (7.402)$$

Equation (7.402) is very important for two reasons. In the first place, because λ is proportional to ω^2 , it characterizes the lowest natural frequency ω_1 , which is the most important one. Then, the fact that λ_1 is a minimum value, as opposed to a mere stationary value, Eq. (7.402) forms the basis for certain methods for the computation of approximate solutions to the differential eigenvalue problem. In view of this, Eq. (7.402) alone is often referred to as *Rayleigh's principle*.

If the lower s eigenfunctions w_i are known, then the lower $s + 1$ eigenvalues can be characterized by *constraining the trial function w to be orthogonal to w_i ($i = 1, 2, \dots, s$)*, in which case *Rayleigh's quotient is an upper bound for λ_{s+1}* , or

$$R(w) \geq \lambda_{s+1}, \quad (w, w_i) = 0, \quad i = 1, 2, \dots, s \quad (7.403)$$

This characterization is primarily of academic interest, as the eigenfunctions w_i ($i = 1, 2, \dots, s$) are not available.

By analogy with the approach used in Sec. 5.3, a characterization of λ_{s+1} independent of the eigenfunctions w_i ($i = 1, 2, \dots, s$) can be obtained by constraining the trial function w to be orthogonal to s independent, but otherwise arbitrary, functions v_i ($i = 1, 2, \dots, s$) and writing

$$\lambda_{s+1} = \max_{v_i} \min_w R(w), \quad (w, v_i) = 0, \quad i = 1, 2, \dots, s \quad (7.404)$$

The *maximum-minimum characterization of the eigenvalues*, Eq. (7.404), represents the *Courant and Fischer maximin theorem for distributed systems* and can be stated as follows: *The eigenvalue λ_{s+1} of the system described by Eqs. (7.387) and (7.388) is the maximum value that can be given to $\min R(w)$ by the imposition of the s constraints $(w, v_i) = 0$ ($i = 1, 2, \dots, s$), where the maximum is with respect to all sets containing v_1, v_2, \dots, v_s and the minimum is with respect to all functions in κ_G^P satisfying the imposed constraints.*

The geometric interpretation of the stationarity of Rayleigh's quotient and of the Courant and Fischer maximin theorem for distributed systems is similar to that for discrete systems given in Secs. 5.2 and 5.3, respectively.

The variational approach presented in this section forms the basis for the classical Rayleigh-Ritz method and the finite element method for generating approximate solutions to the differential eigenvalue problem.

7.15 INTEGRAL FORMULATION OF THE EIGENVALUE PROBLEM

Up to this point, we have formulated the eigenvalue problem for distributed systems as a differential problem, consisting of one (or two) differential equation(s) and an appropriate number of boundary conditions. The differential formulation emerges naturally from the boundary-value problem, which is how the motion of distributed systems is described almost exclusively. The eigenvalue problem for distributed systems, however, can also be described in integral form. Whereas the differential form remains the preferred choice, a discussion of the integral form should prove rewarding.

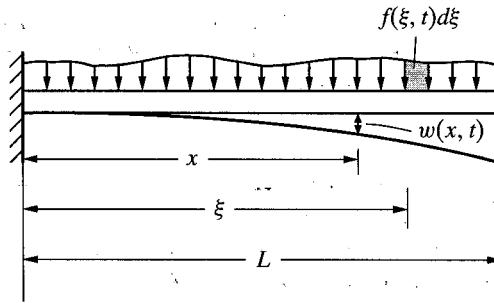


Figure 7.24 Cantilever beam in bending

The integral form of the eigenvalue problem is based on the concept of *flexibility influence function*. To introduce the concept, we consider a self-adjoint system, such as the cantilever beam of Fig. 7.24, and define the flexibility influence function $a(x, \xi)$ as the displacement at point x due to a unit force at point ξ . Then, the total displacement at point x due to the entire distributed force is simply

$$w(x, t) = \int_0^L a(x, \xi) f(\xi, t) d\xi \quad (7.405)$$

But, because displacements of elastic members increase linearly with forces, the system potential energy can be written as

$$V(t) = \frac{1}{2} \int_0^L w(x, t) f(x, t) dx = \frac{1}{2} \int_0^L \int_0^L a(x, \xi) f(\xi, t) f(x, t) dx d\xi \quad (7.406)$$

The potential energy expression can also be derived beginning with the displacement $w(\xi, t)$ instead of $w(x, t)$. Because the potential energy must be the same, irrespective of how it is derived, we conclude that the flexibility influence function is symmetric in x and ξ , or

$$a(x, \xi) = a(\xi, x) \quad (7.407)$$

Equation (7.407) represents *Maxwell's reciprocity theorem* and it states: *The displacement at point x due to a unit force at point ξ is equal to the displacement at point ξ due to a unit force at point x .* The symmetry of the flexibility influence function is consistent with the mathematical symmetry implied by the self-adjointness of the stiffness operator L . The flexibility influence function $a(x, \xi)$ is commonly known as a *Green's function*. Its expression differs from one elastic member to another, which is consistent with the fact that the expression for the differential operator L does. However, it should be observed that, whereas the self-adjointness of L depends not only on the expression for L but also on expressions for the boundary conditions, the boundary conditions are already built into the influence function $a(x, \xi)$. Moreover, whereas L is defined for positive definite as well as positive semidefinite systems, $a(x, \xi)$ is defined for positive definite systems alone.

In free vibration there are no external forces, so that the force density $f(\xi, t)$ in Eq. (7.405) is due entirely to inertial forces. Hence, denoting the mass density by $m(\xi)$, we have

$$f(\xi, t) = -m(\xi) \frac{\partial^2 w(\xi, t)}{\partial t^2} \quad (7.408)$$

But, as established in Sec. 7.4, free vibration of conservative systems is harmonic, so that

$$w(\xi, t) = w(\xi) \cos(\omega t - \phi) \quad (7.409)$$

where $w(\xi)$ is the vibration amplitude, ω the vibration frequency and ϕ an inconsequential phase angle. Hence, inserting Eqs. (7.408) and (7.409) into Eq. (7.405) and dividing through by $\cos(\omega t - \phi)$, we obtain the desired *integral form of the eigenvalue problem*

$$w(x) = \lambda \int_0^L a(x, \xi) m(\xi) w(\xi) d\xi, \quad \lambda = \omega^2 \quad (7.410)$$

which represents a homogeneous linear integral equation, often referred to as a Fredholm homogeneous linear integral equation of the second kind (Ref. 16). It also represents a linear transformation in which the product $a(x, \xi)m(\xi)$ plays the role of the *kernel* of the transformation.

Equation (7.410) can be generalized by writing

$$w(P) = \lambda \int_D G(P, Q) m(Q) w(Q) dD(Q), \quad \lambda = \omega^2 \quad (7.411)$$

where the position P is defined by one or two spatial coordinates, according to the nature of the problem. The function $G(P, Q)$ is a more general type of influence function, or Green's function. For a self-adjoint system, Green's function is symmetric in P and Q , $G(P, Q) = G(Q, P)$. The kernel $G(P, Q)m(Q)$ of the integral transformation (7.411) is not symmetric, unless $m(Q)$ is constant. It can be symmetrized, however, by introducing the function

$$v(P) = m^{1/2}(P)w(P) \quad (7.412)$$

and multiplying both sides of (7.411) by $m^{1/2}(P)$ to obtain

$$v(P) = \lambda \int_D K(P, Q) v(Q) dD(Q) \quad (7.413)$$

where the kernel $K(P, Q)$ is symmetric,

$$K(P, Q) = G(P, Q)m^{1/2}(P)m^{1/2}(Q) = K(Q, P) \quad (7.414)$$

For certain values λ_i , Eq. (7.413) has nontrivial solutions $v_i(P)$, which are related to the solutions $w_i(P)$ of Eq. (7.411) by Eq. (7.412). The values λ_i are the eigenvalues of the system and $w_i(P)$ are the associated eigenfunctions. Whereas the functions $v_i(P)$ are orthogonal in an ordinary sense, the functions $w_i(P)$ are

orthogonal with respect to the function $m(P)$. To show this, we consider two distinct solutions of Eq. (7.413), or

$$v_i(P) = \lambda_i \int_D K(P, Q)v_i(Q) dD(Q) \quad (7.415a)$$

$$v_j(P) = \lambda_j \int_D K(P, Q)v_j(Q) dD(Q) \quad (7.415b)$$

Multiplying Eq. (7.415a) by $v_j(P)$, integrating over domain D and using Eq. (7.415b), we obtain

$$\begin{aligned} \int_D v_i(P)v_j(P) dD(P) &= \lambda_i \int_D v_j(P) \left[\int_D K(P, Q)v_i(Q) dD(Q) \right] dD(P) \\ &= \lambda_i \int_D v_i(Q) \left[\int_D K(Q, P)v_j(P) dD(P) \right] dD(Q) \\ &= \frac{\lambda_i}{\lambda_j} \int_D v_i(Q)v_j(Q) dD(Q) \end{aligned} \quad (7.416)$$

from which we obtain

$$(\lambda_i - \lambda_j) \int_D v_i(P)v_j(P) dD(P) = 0 \quad (7.417)$$

For two distinct eigenvalues, we obtain the orthogonality relation

$$\int_D v_i(P)v_j(P) dD(P) = 0, \quad \lambda_i \neq \lambda_j \quad (7.418)$$

Introducing Eq. (7.412) in Eq. (7.418) and normalizing the eigenfunctions, we can write the orthonormality relations

$$\int_D m(P)w_i(P)w_j(P) dD(P) = \delta_{ij}, \quad i, j = 1, 2, \dots \quad (7.419)$$

where δ_{ij} is the Kronecker delta.

As for the differential eigenvalue problem, there is an expansion theorem concerning the eigenfunctions $w_i(P)$ according to which we represent a function satisfying the boundary conditions and possessing a continuous Lw by the infinite series

$$w(P) = \sum_{i=1}^{\infty} c_i w_i(P) \quad (7.420)$$

where the coefficients c_i are given by

$$c_i = \int_D m(P)w(P)w_i(P) dD(P), \quad i = 1, 2, \dots \quad (7.421)$$

There are several methods for solving Eq. (7.411). The description of these methods is beyond the scope of this text. We discuss here the iteration method,

which is similar in principle to the matrix iteration method using the power method. The iteration process for the first eigenfunction is defined by

$$w_1^{(k+1)}(P) = \int_D G(P, Q)m(Q)w_1^{(k)}(Q)dD(Q), \quad k = 1, 2, \dots \quad (7.422)$$

To demonstrate convergence of the algorithm, we choose an initial trial function $w_1^{(1)}(P)$, which can be assumed to have the form of the series given by Eq. (7.420), insert it into Eq. (7.422) with $k = 1$, carry out the integration and write

$$\begin{aligned} w_1^{(2)}(P) &= \int_D G(P, Q)m(Q)w_1^{(1)}(Q)dD(Q) \\ &= \sum_{i=1}^{\infty} c_i \int_D G(P, Q)m(Q)w_i(Q)dD(Q) = \sum_{i=1}^{\infty} c_i \frac{w_i(P)}{\lambda_i} \end{aligned} \quad (7.423)$$

Using $w_1^{(2)}(P)$ as an improved trial function, we obtain

$$w_1^{(3)}(P) = \int_D G(P, Q)m(Q)w_1^{(2)}(Q)dD(Q) = \sum_{i=1}^{\infty} c_i \frac{w_i(P)}{\lambda_i^2} \quad (7.424)$$

In general, we have

$$w_1^{(p)}(P) = \sum_{i=1}^{\infty} c_i \frac{w_i(P)}{\lambda_i^{p-1}} \quad (7.425)$$

If the eigenvalues are such that $\lambda_1 < \lambda_2 < \lambda_3 < \dots$, the first term in the series in Eq. (7.425) becomes increasingly large in comparison with the remaining ones and, as $p \rightarrow \infty$, $w_1^{(p)}(P)$ becomes proportional to the first eigenfunction, or

$$\lim_{p \rightarrow \infty} w_1^{(p)}(P) = w_1(P) \quad (7.426)$$

where the proportionality constant has been ignored as immaterial. After convergence has been reached, λ_1 is obtained as the ratio of two subsequent trial functions,

$$\lambda_1 = \lim_{p \rightarrow \infty} \frac{w_1^{(p)}(P)}{w_1^{(p+1)}(P)} \quad (7.427)$$

In practice, if the iterated functions are normalized by prescribing the value of the function at a given point, and this value is kept the same, then the normalization constant approaches λ_1 as p increases.

To obtain the second mode, we must insist that the trial function $w_2^{(1)}(P)$ used for iteration to the second mode be entirely free of the first mode. To this end, we use the iteration process

$$\varphi_2^{(k+1)} = \int_D G(P, Q)m(Q)w_2^{(k)}(Q)dD(Q), \quad k = 1, 2, \dots \quad (7.428)$$

and begin the iteration with the first trial function in the form

$$w_2^{(1)}(P) = \varphi_2^{(1)}(P) - a_1 w_1(P) \quad (7.429)$$

where $\varphi_2^{(1)}(P)$ is an arbitrarily chosen function and a_1 is a coefficient determined from the orthogonality requirement by writing

$$\begin{aligned} & \int_D m(P) w_2^{(1)}(P) w_1(P) dD(P) \\ &= \int_D m(P) \varphi_2^{(1)}(P) w_1(P) dD(P) - a_1 \int_D m(P) [w_1(P)]^2 dD(P) = 0 \end{aligned} \quad (7.430)$$

which yields

$$a_1 = \frac{\int_D m(P) \varphi_2^{(1)}(P) w_1(P) dD(P)}{\int_D m(P) w_1^2(P) dD(P)} \quad (7.431)$$

and if $w_1(P)$ is normalized so that $\int_D m w_1^2 dD = 1$, then

$$a_1 = \int_D m(P) \varphi_2^{(1)}(P) w_1(P) dD(P) \quad (7.432)$$

Introducing $w_2^{(1)}(P)$ in Eq. (7.428) with $k = 1$ and performing the integration, we have

$$\varphi_2^{(2)}(P) = \int_D G(P, Q) m(Q) w_2^{(1)}(Q) dD(Q) \quad (7.433)$$

For the next iteration step, we use

$$w_2^{(2)}(P) = \varphi_2^{(2)}(P) - a_2 w_1(P) \quad (7.434)$$

where, for normalized $w_1(P)$, we have

$$a_2 = \int_D m(P) \varphi_2^{(2)}(P) w_1(P) dD(P) \quad (7.435)$$

In general, for the p th iteration step, we use

$$w_2^{(p)}(P) = \varphi_2^{(p)}(P) - a_p w_1(P) \quad (7.436)$$

Convergence is achieved when, as $p \rightarrow \infty$, $a_p \rightarrow 0$ and

$$\lim_{p \rightarrow \infty} w_2^{(p)}(P) = w_2(P) \quad (7.437)$$

$$\lambda_2 = \lim_{p \rightarrow \infty} \frac{w_2^{(p)}(P)}{w_2^{(p+1)}(P)} \quad (7.438)$$

Similarly, for the third mode, we use the trial function

$$w_3^{(1)}(P) = \varphi_3^{(1)}(P) - a_1 w_1(P) - b_1 w_2(P) \quad (7.439)$$

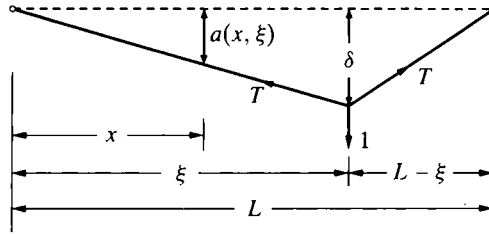


Figure 7.25 Displacement of a uniform string fixed at both ends due to a unit force

where $\varphi_3^{(1)}(P)$ is an arbitrary function and a_1 and b_1 are obtained by insisting that $w_3^{(1)}(P)$ be orthogonal to both $w_1(P)$ and $w_2(P)$. The same procedure is used to iterate to the third mode and, subsequently, to higher modes. In practice, a finite number p of iterations is needed for each mode.

As an illustration, we consider the free vibration of a string of uniformly distributed mass ρ , clamped at both ends and subjected to a constant tension T . The flexibility influence function $a(x, \xi)$ is obtained by applying a unit force at point ξ and calculating the deflection at point x (Fig. 7.25). For small angles α_1 and α_2 , the equilibrium condition at the point of application of the load is

$$T \frac{\delta}{\xi} + T \frac{\delta}{L - \xi} = 1 \quad (7.440)$$

from which we obtain the influence function

$$a(x, \xi) = \delta \frac{x}{\xi} = \frac{x(L - \xi)}{TL}, \quad \xi > x \quad (7.441a)$$

and it can be readily shown that

$$a(x, \xi) = \frac{\xi(L - x)}{TL}, \quad \xi < x \quad (7.441b)$$

As expected, Green's function $G(x, \xi) = a(x, \xi)$ is symmetric in x and ξ , because the system is self-adjoint.

We use the iteration method to solve the eigenvalue problem. To this end, we assume

$$w_1^{(1)} = \frac{x}{L} \quad (7.442)$$

and obtain in sequence

$$\begin{aligned} w_1^{(2)}(x) &= \int_0^L G(x, \xi) \rho(\xi) w_1^{(1)}(\xi) d\xi \\ &= \frac{\rho}{TL^2} \left[\int_0^x \xi(L - x) \xi d\xi + \int_x^L x(L - \xi) d\xi \right] \\ &= \frac{\rho L^2}{6} \left(\frac{x}{L} - \frac{x^3}{L^3} \right) \end{aligned} \quad (7.443a)$$

$$\begin{aligned}
 w_1^{(3)}(x) &= \int_0^L G(x, \xi) \rho(\xi) w_1^{(2)}(\xi) d\xi \\
 &= \frac{\rho L^2}{6T} \frac{7\rho L^2}{60T} \left(\frac{x}{L} - \frac{10x^3}{7L^3} + \frac{3x^5}{7L^5} \right)
 \end{aligned} \tag{7.443b}$$

$$\begin{aligned}
 w_1^{(4)}(x) &= \int_0^L G(x, \xi) \rho(\xi) w_1^{(3)}(\xi) d\xi \\
 &= \frac{\rho L^2}{6T} \frac{7\rho L^2}{60T} \frac{31\rho L^2}{294T} \left(\frac{x}{L} - \frac{49x^3}{31L^3} + \frac{21x^5}{31L^5} - \frac{3x^7}{31L^7} \right)
 \end{aligned} \tag{7.443c}$$

$$\begin{aligned}
 w_1^{(5)}(x) &= \int_0^L G(x, \xi) \rho(\xi) w_1^{(4)}(\xi) d\xi \\
 &= \frac{\rho L^2}{6T} \frac{7\rho L^2}{60T} \frac{31\rho L^2}{294T} \frac{2667\rho L^2}{26040T} \left(\frac{x}{L} - \frac{4340x^3}{2667L^3} + \frac{2058x^5}{2667L^5} \right. \\
 &\quad \left. - \frac{420x^7}{2667L^7} + \frac{35x^9}{2667L^9} \right)
 \end{aligned} \tag{7.443d}$$

At this point we pause to check convergence. It can be easily verified that $w_1^{(5)}(x)$, as given by Eq. (7.443d), is almost proportional to $\sin \pi x/L$, which is the first natural mode. Moreover, letting $p = 4$ in Eq. (7.427) (and ignoring the limit), we have

$$\lambda_1 = \omega_1^2 \cong \frac{w_1^{(4)}(L/2)}{w_1^{(5)}(L/2)} = \frac{26040T}{2667\rho L^2} \tag{7.444}$$

so that

$$\omega_1 \cong 3.12 \sqrt{\frac{T}{\rho L^2}} \tag{7.445}$$

The approximation is quite good, because the exact value of the first natural frequency is $\omega_1 = \pi \sqrt{T/\rho L^2}$. Hence, Eqs. (7.443d) and (7.445) can be accepted as representing the first natural mode and the first natural frequency, respectively. An interesting aspect of this iteration process is that, although $w_1^{(1)}$ violates the boundary condition at $x = L$, all iterates do satisfy both boundary conditions. It follows that multiplication by Green's function, in conjunction with integration over the domain, imposes the system boundary conditions on the iterates.

To obtain the second mode, we must use a trial function orthogonal to $w_1(x)$. This is left as an exercise to the reader.

It should be pointed out that Green's functions can be determined only for simple systems. Hence, the approach based on Green's functions has limited appeal for distributed-parameter systems. However, the approach proves useful in Sec. 8.1, where we use it for an approximate technique.

7.16 RESPONSE OF UNDAMPED DISTRIBUTED SYSTEMS

Using developments from Secs. 7.1, 7.2 and 7.5, we can write a typical boundary-value problem describing the behavior of vibrating undamped systems in the operator form

$$Lw(P, t) + m(P)\ddot{w}(P, t) = f(P, t), \quad P \text{ in } D \quad (7.446)$$

where $w(P, t)$ is the displacement of a point P in the domain D , L a linear homogeneous self-adjoint stiffness differential operator of order $2p$, $m(P)$ the mass density and $f(P, t)$ the force density, and we note that any concentrated forces acting at $P = P_j$ can be treated as distributed by means of spatial Dirac delta functions defined by

$$\begin{aligned} \delta(P - P_j) &= 0, & P \neq P_j \\ \int_D \delta(P - P_j) dD(P) &= 1 \end{aligned} \quad (7.447)$$

The solution $w(P, t)$ of Eq. (7.446) must satisfy the boundary conditions

$$B_i w(P, t) = 0, \quad i = 1, 2, \dots, p, \quad P \text{ on } S \quad (7.448)$$

where B_i are linear homogeneous boundary differential operators ranging in order from zero to $2p - 1$ and S is the boundary of D . In addition, the solution is subject to the initial conditions

$$w(P, 0) = w_0(P), \quad \dot{w}(P, 0) = v_0(P) \quad (7.449a, b)$$

In a manner analogous to that for discrete systems, the solution to the combined boundary-value problem and initial-value problem can be obtained conveniently by modal analysis. To this end, we must first solve the eigenvalue problem defined by the differential equation

$$Lw(P) = \lambda m(P)w(P), \quad \lambda = \omega^2, \quad P \text{ in } D \quad (7.450a)$$

and the boundary conditions

$$B_i w(P) = 0, \quad i = 1, 2, \dots, p, \quad P \text{ on } S \quad (7.450b)$$

The solution of Eqs. (7.450) consists of a denumerably infinite set of eigenvalues $\lambda_r = \omega_r^2$, where ω_r are the natural frequencies, and associated eigenfunctions $w_r(P)$ ($r = 1, 2, \dots$). Because L is self-adjoint, the eigenvalues are real and the eigenfunctions are real and orthogonal. On the assumption that L is positive definite and that the eigenfunctions have been normalized, the orthonormality relations are

$$\int_D m(P)w_r(P)w_s(P) dD(P) = \delta_{rs}, \quad r, s = 1, 2, \dots \quad (7.451a)$$

$$\int_D w_r(P)Lw_s(P) dD(P) = \omega_r^2 \delta_{rs}, \quad r, s = 1, 2, \dots \quad (7.451b)$$

where δ_{rs} is the Kronecker delta.

Using the expansion theorem, Eqs. (7.101) and (7.102), we can express the solution of Eq. (7.446) as a linear combination of the system eigenfunctions of the form

$$w(P, t) = \sum_{s=1}^{\infty} w_s(P) \eta_s(t) \quad (7.452)$$

where $\eta_s(t)$ are time-dependent generalized coordinates, referred to as *normal coordinates*, or *modal coordinates*, and playing the role of the expansion coefficients c_r . Strictly speaking, the expansion theorem is in terms of constant coefficients. To resolve this issue, we can conceive of expansion (7.452) being applied at the discrete times t_1, t_2, \dots , resulting in constant coefficients $\eta_s(t_1), \eta_s(t_2), \dots$. Then, if the times t_1, t_2, \dots are brought closer and closer together, the coefficients $\eta_s(t_1), \eta_s(t_2), \dots$ change in a continuous manner with time, thus justifying Eq. (7.452). Inserting Eq. (7.452) into Eq. (7.446), multiplying through by $w_r(P)$, integrating over the domain D and using the orthonormality relations, Eqs. (7.451), we obtain the infinite set of independent equations

$$\ddot{\eta}_r(t) + \omega_r^2 \eta_r(t) = N_r(t), \quad r = 1, 2, \dots \quad (7.453)$$

known as *normal equations*, or *modal equations*, in which

$$N_r(t) = \int_D w_r(P) f(P, t) dD(P), \quad r = 1, 2, \dots \quad (7.454)$$

are *generalized forces*, referred to as *modal forces*. Equations (7.453) are subject to the *initial generalized displacements and velocities*, or *initial modal displacements and velocities* $\eta_r(0)$ and $\dot{\eta}_r(0)$, respectively. They can be obtained by letting $t = 0$ in Eq. (7.452) multiplying through by $m(P)w_r(P)$, integrating over D and considering Eqs. (7.449a) and (7.451a). The result is

$$\eta_r(0) = \int_D m(P)w_r(P)w_0(P) dD(P), \quad r = 1, 2, \dots \quad (7.455a)$$

In a similar fashion, we obtain

$$\dot{\eta}_r(0) = \int_D m(P)w_r(P)v_0(P) dD(P), \quad r = 1, 2, \dots \quad (7.455b)$$

Equations (7.453) are identical in form to the modal equations for discrete systems, Eqs. (4.220), so that the solution is simply

$$\eta_r(t) = \int_0^t \left[\int_0^\tau N_r(\sigma) d\sigma \right] d\tau + \eta_r(0) + \dot{\eta}_r(0)t \quad (7.456a)$$

for rigid-body modes and

$$\eta_r(t) = \frac{1}{\omega_r} \int_0^t N_r(\tau) \sin \omega_r(t - \tau) d\tau + \eta_r(0) \cos \omega_r t + \frac{\dot{\eta}_r(0)}{\omega_r} \sin \omega_r t \quad (7.456b)$$

for elastic modes. The formal solution is completed by inserting Eqs. (7.456) into Eq. (7.452).

The boundary-value problem described by Eqs. (7.446) and (7.448) can be generalized so as to accommodate systems of the type discussed in Secs. 7.8 and 7.9. In particular, the differential equation can be generalized to

$$Lw(P, t) + M\ddot{w}(P, t) = f(P, t), \quad P \text{ in } D \quad (7.457)$$

where the various quantities are as defined earlier in this section. The one exception is M , which in the case at hand is a linear homogenous self-adjoint mass differential operator of order $2q$, $q < p$, as opposed to a mere function in Eq. (7.446). Moreover, the boundary conditions are

$$B_i w(P, t) = 0, \quad P \text{ on } S, \quad i = 1, 2, \dots, k \quad (7.458a)$$

$$B_i w(P, t) + C_i \dot{w}(P, t) = 0, \quad P \text{ on } S, \quad i = k + 1, k + 2, \dots, p \quad (7.458b)$$

The differential eigenvalue problem corresponding to Eqs. (7.457) and (7.458) is given by the differential equation

$$Lw(P) = \lambda Mw(P), \quad \lambda = \omega^2, \quad P \text{ in } D \quad (7.459)$$

and the boundary conditions

$$B_i w(P) = 0, \quad P \text{ on } S, \quad i = 1, 2, \dots, k \quad (7.460a)$$

$$B_i w(P) = \omega^2 C_i w(P), \quad P \text{ on } S, \quad i = k + 1, k + 2, \dots, p \quad (7.460b)$$

Moreover, we recall from Sec. 7.9 that the orthonormality relations for the system eigenfunctions are given by

$$\int_D w_r(P) M w_s(P) dD + \sum_{i=k+1}^p \int_S w_r(P) C_i w_s(P) dS = \delta_{rs}, \quad r, s = 1, 2, \dots \quad (7.461a)$$

$$\int_D w_r(P) L w_s(P) dD + \sum_{i=k+1}^p \int_S w_r(P) B_i w_s(P) dS = \omega_r^2 \delta_{rs}, \quad r, s = 1, 2, \dots \quad (7.461b)$$

Introducing solution (7.452) in Eq. (7.457), multiplying through by $w_r(P)$, integrating over D and using the orthonormality relations, we obtain

$$\begin{aligned} & \sum_{s=1}^{\infty} \left(\delta_{rs} - \sum_{i=k+1}^p \int_S w_r C_i w_s dS \right) \ddot{\eta}_s(t) \\ & + \sum_{s=1}^{\infty} \left(\omega_r^2 \delta_{rs} - \sum_{i=k+1}^p \int_S w_r B_i w_s dS \right) \eta_s(t) = \int_D w_r f dD, \end{aligned} \quad r = 1, 2, \dots \quad (7.462)$$

But, in view of Eq. (7.454) and boundary conditions (7.458b), Eqs. (7.462) reduce to the set of independent modal equations

$$\ddot{\eta}_r(t) + \omega_r^2 \eta_r(t) = N_r(t), \quad r = 1, 2, \dots \quad (7.463)$$

Equations (7.463) are the same as Eqs. (7.453), so that the solution is once again given by Eqs. (7.456). The conclusion is that, in spite of the intimidating appearance of the orthonormality relations, Eqs. (7.461), modal analysis for the solution of the general boundary-value problem, Eqs. (7.457) and (7.458), retains the same simplicity as for the common one, Eqs. (7.446) and (7.448).

Example 7.9

A uniform beam of mass density m , bending stiffness EI and length L is simply supported at both ends. Derive the response to the initial displacement

$$w(x, 0) = w_0(x) = A \left(\frac{x}{L} - 2 \frac{x^3}{L^3} + \frac{x^4}{L^4} \right) \quad (a)$$

and note that $w_0(x)$ is symmetric with respect to $x = L/2$. The initial velocity is zero and there are no external forces. The normal modes and natural frequencies of a uniform simply supported beam are

$$w_r(x) = \sqrt{\frac{2}{mL}} \sin \frac{r\pi x}{L}, \quad \omega_r = (r\pi)^2 \sqrt{\frac{EI}{mL^4}}, \quad r = 1, 2, \dots \quad (b)$$

The response to the initial displacement, Eq. (a), is given by

$$w(x, t) = \sum_{r=1}^{\infty} w_r(x) \eta_r(t) \quad (c)$$

where, in the absence of initial velocities and external forces, the modal coordinates $\eta_r(t)$ are obtained from Eqs. (7.456) in the form

$$\eta_r(t) = \eta_r(0) \cos \omega_r t, \quad r = 1, 2, \dots \quad (d)$$

in which, using Eqs. (a) and (b), we have

$$\begin{aligned} \eta_r(0) &= \int_0^L m(x) w_r(x) w_0(x) dx = A \sqrt{\frac{2m}{L}} \int_0^L \sin \frac{r\pi x}{L} \left(\frac{x}{L} - 2 \frac{x^3}{L^3} + \frac{x^4}{L^4} \right) dx \\ &= A \sqrt{2mL} \frac{24}{r^5 \pi^5} [1 - (-1)^r], \quad r = 1, 2, \dots \end{aligned} \quad (e)$$

When r is even,

$$\eta_r(0) = 0 \quad (f)$$

and when r is odd,

$$\eta_r(0) = \frac{48A}{r^5 \pi^5} \sqrt{2mL} \quad (g)$$

Combining the above results, the response to the initial displacement, Eq. (a), is

$$w(x, t) = \frac{96A}{\pi^5} \sum_{r=1}^{\infty} \frac{1}{(2r-1)^5} \sin \frac{(2r-1)\pi x}{L} \cos \omega_r t \quad (h)$$

where

$$\omega_n = [(2r-1)\pi]^2 \sqrt{\frac{EI}{mL^4}}, \quad r = 1, 2, \dots \quad (i)$$

Examining Eq. (h), we note that the terms in the series are symmetric with respect to the middle of the beam. This should come as no surprise, because the initial displacement, Eq. (a), is symmetric. In fact, it represents the static deflection caused by a uniformly distributed load. We must also note that the amplitude of the second harmonic is only 0.41% of the amplitude of the first harmonic, so that motion resembles the first mode very closely. This is to be expected, because the initial displacement resembles the first mode.

Example 7.10

An unrestrained uniform rod lies at rest on a smooth horizontal surface (Fig. 7.26). Derive the response to an axial force in the form of a step function of magnitude F_0 applied at $x = 0$.

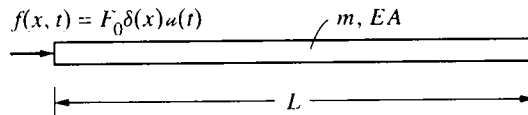


Figure 7.26 Unrestrained uniform rod in axial motion due to a force at $x = 0$

The longitudinal displacement of the rod can be written in the form

$$u(x, t) = \sum_{r=0}^{\infty} U_r(x) \eta_r(t) \quad (a)$$

where, from Eqs. (7.119), (7.137) and (7.138), the normal modes and natural frequencies are

$$U_0(x) = \frac{1}{\sqrt{mL}}, \quad \omega_0 = 0 \quad (b)$$

$$U_r(x) = \sqrt{\frac{2}{mL}} \cos \frac{r\pi x}{L}, \quad \omega_r = r\pi \sqrt{\frac{EA}{mL^2}}, \quad r = 1, 2, \dots$$

and we note the presence of one rigid-body mode.

The applied force can be written in the form of a distributed force as follows:

$$f(x, t) = F_0 \delta(x) u(t) \quad (c)$$

where $\delta(x)$ is a spatial Dirac delta function applied at $x = 0$ and $u(t)$ is a unit step function applied at $t = 0$. The modal coordinates can be obtained from Eqs. (7.456) in the form

$$\eta_0(t) = \int_0^t \left[\int_0^\tau N_0(\sigma) d\sigma \right] d\tau \quad (d)$$

$$\eta_r(t) = \frac{1}{\omega_r} \int_0^t N_r(\tau) \sin \omega_r(t - \tau) d\tau, \quad r = 1, 2, \dots,$$

where from Eqs. (7.454), the modal forces are given by

$$N_r(t) = \int_0^L U_r(x) f(x, t) dx = U_r(0) F_0 u(t), \quad r = 0, 1, 2, \dots \quad (e)$$

Hence, using Eqs. (a), (b), (d) and (e), we obtain

$$\begin{aligned}
 u(x, t) &= U_0(x)U_0(0)F_0 \int_0^t \left[\int_0^\tau u(\sigma) d\sigma \right] d\tau \\
 &\quad + \sum_{r=1}^{\infty} \frac{U_r(x)U_r(0)}{\omega_r} F_0 \int_0^t u(\tau) \sin \omega_r(t - \tau) d\tau \\
 &= \frac{F_0}{2mL} t^2 + F_0 \sum_{r=1}^{\infty} \frac{U_r(x)U_r(0)}{\omega_r^2} (1 - \cos \omega_r t) \\
 &= \frac{F_0}{2mL} t^2 + \frac{2F_0L}{\pi^2 EA} \sum_{r=1}^{\infty} \frac{1}{r^2} \cos \frac{r\pi x}{L} (1 - \cos \omega_r t) \quad (f)
 \end{aligned}$$

Furthermore,¹

$$\sum_{r=1}^{\infty} \frac{1}{r^2} \cos \frac{r\pi x}{L} = \frac{\pi^2}{2L^2} \left[\frac{(L-x)^2}{2} - \frac{1}{6}L^2 \right] \quad (g)$$

so that the general response is

$$u(x, t) = \frac{1}{2} \frac{F_0}{mL} t^2 + \frac{F_0}{EAL} \left[\frac{(L-x)^2}{2} - \frac{1}{6}L^2 \right] - \frac{2F_0L}{\pi^2 EA} \sum_{r=1}^{\infty} \frac{1}{r^2} \cos \frac{r\pi x}{L} \cos \omega_r t \quad (h)$$

The first term in Eq. (h) represents the rigid-body motion, and it is the only one to survive if the stiffness becomes infinitely large. The second term in Eq. (h) can be looked upon as the static deformation and the third term represents vibration. The first two terms can be interpreted as an average position about which the vibration takes place.

The same system can be looked upon as force-free with a nonhomogeneous boundary condition at the end $x = 0$. This approach is discussed in Sec. 7.19.

Example 7.11

Determine the response of a circular membrane of uniform thickness clamped at $r = a$ and subjected to the distributed force

$$f(r, \theta, t) = \begin{cases} f(t), & 0 \leq r \leq b \\ 0, & b < r \leq a \end{cases} \quad (a)$$

as shown in Fig. 7.27. The membrane is at rest initially.

The response of the membrane can be written in the form of the series

$$\begin{aligned}
 w(r, \theta, t) &= \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} W_{mn}(r, \theta) \eta_{mn}(t) \\
 &= \sum_{n=1}^{\infty} W_{0n}(r) \eta_{0n}(t) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mnc}(r, \theta) \eta_{mnc}(t) \\
 &\quad + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mns}(r, \theta) \eta_{mns}(t) \quad (b)
 \end{aligned}$$

¹ See Peirce, B. O. and Foster, R. M. *A Short Table of Integrals*, 4th ed., Ginn, Boston, 1957, Formulas 889 and 891.

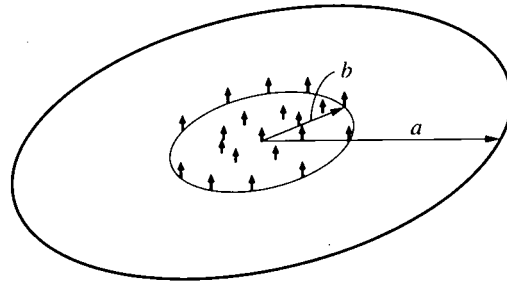


Figure 7.27 Uniform circular membrane fixed at $r = a$ with force distributed over the region $0 \leq r \leq b < a$

where W_{0n} , W_{mnc} and W_{mns} are the normal modes of a uniform membrane clamped at $r = a$, Eqs. (7.306). Moreover, the modal coordinates are given by

$$\begin{aligned} \eta_{0n}(t) &= \frac{1}{\omega_{0n}} \int_0^t N_{0n}(\tau) \sin \omega_{0n}(t - \tau) d\tau \\ \eta_{mnc}(t) &= \frac{1}{\omega_{mn}} \int_0^t N_{mnc}(\tau) \sin \omega_{mn}(t - \tau) d\tau \\ \eta_{mns}(t) &= \frac{1}{\omega_{mn}} \int_0^t N_{mns}(\tau) \sin \omega_{mn}(t - \tau) d\tau \end{aligned} \tag{c}$$

in which N_{0n} , N_{mnc} and N_{mns} are the modal forces. Inserting Eq. (a) into Eqs. (7.454) and considering Eqs. (7.306), the modal forces take the form

$$\begin{aligned} N_{0n}(t) &= \int_0^{2\pi} \int_0^a W_{0n}(r) f(r, \theta, t) r dr d\theta = \frac{2\pi f(t)}{\sqrt{\pi \rho a} J_1(\beta_{0n} a)} \frac{b}{\beta_{0n}} J_1(\beta_{0n} b) \\ N_{mnc}(t) &= \int_0^{2\pi} \int_0^a W_{mnc}(r, \theta) f(r, \theta, t) r dr d\theta = 0 \\ N_{mns}(t) &= \int_0^{2\pi} \int_0^a W_{mns}(r, \theta) f(r, \theta, t) r dr d\theta = 0 \end{aligned} \tag{d}$$

Hence,

$$\begin{aligned} \eta_{0n}(t) &= \frac{2\sqrt{\pi} b J_1(\beta_{0n} b)}{\sqrt{T} a \beta_{0n}^2 J_1(\beta_{0n} a)} \int_0^t f(\tau) \sin \omega_{0n}(t - \tau) d\tau \\ \eta_{mnc}(t) &= \eta_{mns}(t) = 0 \end{aligned} \tag{e}$$

where $\omega_{0n} = \beta_{0n} \sqrt{T/\rho}$. Introducing Eqs. (7.306) and (e) in the series (b), we obtain the transverse displacement of the membrane in the form

$$w(r, \theta, t) = \frac{2b}{a^2} \sqrt{\frac{1}{T\rho}} \sum_{n=1}^{\infty} \frac{J_1(\beta_{0n} b) J_0(\beta_{0n} r)}{\beta_{0n}^2 J_1^2(\beta_{0n} a)} \int_0^t f(\tau) \sin \omega_{0n}(t - \tau) d\tau \tag{g}$$

It appears that only Bessel functions of zero order, $m = 0$, participate in the motion. This is to be expected, owing to the nature of the load. The load is distributed uniformly over $0 \leq r \leq b$, so that there cannot be any trigonometric functions present,

as modes with diametrical nodes cannot take part in the motion. Furthermore, the Bessel functions of order higher than zero have a zero at $r = 0$, so that, for continuity, they must be antisymmetric with respect to the vertical through $r = 0$, and hence ruled out.

Example 7.12

Obtain the response of a uniform rectangular plate extending over the domain $0 < x < a$, $0 < y < b$ and simply supported along the boundaries $x = 0, a$ and $y = 0, b$ to a concentrated force at the point $x = 3/4a$, $y = 1/2b$, as shown in Fig. 7.28. The plate is at rest initially.

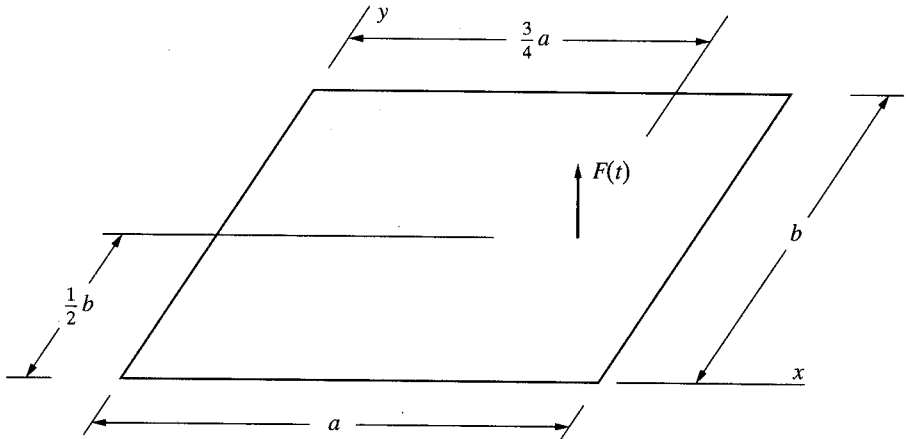


Figure 7.28 Uniform rectangular plate simply supported on all sides and subjected to a concentrated force

The force can be described mathematically as a distributed force given by

$$f(x, y, t) = F(t) \delta \left(x - \frac{3}{4}a, y - \frac{1}{2}b \right) \quad (a)$$

where $F(t)$ is the time-dependent amplitude of the force and $\delta(x - 3a/4, y - b/2)$ is a two-dimensional spatial Dirac delta function defined by

$$\delta \left(x - \frac{3}{4}a, y - \frac{1}{2}b \right) = 0, \quad x \neq \frac{3}{4}a \text{ and/or } y \neq \frac{1}{2}b \quad (b)$$

$$\int_0^a \int_0^b \delta \left(x - \frac{3}{4}a, y - \frac{1}{2}b \right) dx dy = 1$$

The normal modes of the simply supported uniform plate are

$$W_{mn}(x, y) = \frac{2}{\sqrt{\rho ab}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}, \quad m, n = 1, 2, \dots \quad (c)$$

and the corresponding natural frequencies are

$$\omega_{mn} = \pi^2 \sqrt{\frac{D_E}{\rho}} \left[\left(\frac{m}{a} \right)^2 + \left(\frac{n}{b} \right)^2 \right], \quad m, n = 1, 2, \dots \quad (d)$$

where D_E is the plate flexural rigidity and ρ is the mass per unit area of plate.

Using the expansion theorem, the transverse displacement of the plate is

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn}(x, y) \eta_{mn}(t) \quad (e)$$

where $\eta_{mn}(t)$ are the modal coordinates having the expressions

$$\eta_{mn}(t) = \frac{1}{\omega_{mn}} \int_0^t N_{mn}(\tau) \sin \omega_{mn}(t - \tau) d\tau \quad (f)$$

in which $N_{mn}(t)$ are the modal forces given by

$$\begin{aligned} N_{mn}(t) &= \int_0^a \int_0^b W_{mn}(x, y) f(x, y, t) dx dy \\ &= \frac{2F(t)}{\sqrt{\rho ab}} \int_0^a \int_0^b \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \delta\left(x - \frac{3}{4}a, y - \frac{1}{2}b\right) dx dy \\ &= \frac{2F(t)}{\sqrt{\rho ab}} \sin \frac{3m\pi}{4} \sin \frac{n\pi}{2} \end{aligned} \quad (g)$$

Introducing Eq. (g) into Eq. (f), we obtain

$$\eta_{mn}(t) = \frac{2}{\omega_{mn} \sqrt{\rho ab}} \sin \frac{3m\pi}{4} \sin \frac{n\pi}{2} \int_0^t F(\tau) \sin \omega_{mn}(t - \tau) d\tau \quad (h)$$

so that, using Eq. (e), the response can be written in the form

$$w(x, y, t) = \frac{4}{\rho ab} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin(3m\pi/4) \sin(n\pi/2)}{\omega_{mn}} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \times \int_0^t F(\tau) \sin \omega_{mn}(t - \tau) d\tau \quad (i)$$

where the frequencies ω_{mn} are given by Eq. (d).

It can be easily verified that, if m is an integer multiple of 4, the corresponding term in the series in Eq. (i) vanishes. This is consistent with the fact that a concentrated force applied at $x = 3a/4$ cannot excite the modes $\sin 4\pi x/a$, $\sin 8\pi x/a$, etc., which have nodes at that point. The same argument explains why all the terms for which n is an even number vanish.

7.17 DISTRIBUTED GYROSCOPIC SYSTEMS

In many cases of interest, the vibrating structure rotates about a given axis. If the structure has velocity components normal to the axis of rotation, then gyroscopic effects arise. We encountered gyroscopic effects for the first time in connection with discrete systems in Sec. 2.12 and then in Sec. 4.1. In this section, we consider such effects in connection with distributed systems.

We begin the study of distributed gyroscopic systems with the derivation of the boundary-value problem for a rotating elastic shaft simply supported at both ends, as shown in Fig. 7.29. The shaft rotates about axis X with the constant angular velocity Ω relative to the inertial axes XYZ . For convenience, we use a set of body axes xyz ,

with x coinciding with the rotation axis X , and playing the role of the spatial variable, and with axes y and z rotating together with the body with the same angular velocity Ω about axis $X = x$. Moreover, we let \mathbf{i} , \mathbf{j} and \mathbf{k} be unit vectors along the rotating axes x , y and z , respectively. The shaft undergoes the bending displacements w_y and w_z in the y and z directions, respectively, so that the angular velocity vector and displacement vector can be written in the vector form

$$\boldsymbol{\omega} = \Omega \mathbf{i} \quad (7.464a)$$

$$\mathbf{w}(x, t) = w_y(x, t)\mathbf{j} + w_z(x, t)\mathbf{k} \quad (7.464b)$$

Then, using the analogy with Eq. (d) of Example 4.1, the velocity vector of a typical point on the shaft can be shown to be

$$\mathbf{v}(x, t) = (\dot{w}_y - \Omega w_z)\mathbf{j} + (\dot{w}_z + \Omega w_y)\mathbf{k} \quad (7.465)$$

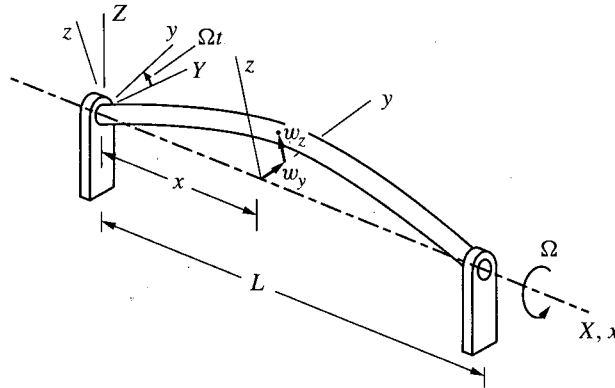


Figure 7.29 Rotating elastic shaft in bending simply supported at both ends

We propose to derive the boundary-value problem by means of Lagrange's equations for distributed systems. This requires an extension of the approach of Sec. 7.3 from systems defined by a single dependent variable to systems defined by two. To this end, we rewrite the extended Hamilton's principle, Eq. (7.31), in the form

$$\int_{t_1}^{t_2} (\delta L + \overline{\delta W}_{nc}) dt = 0, \quad \delta w_y = \delta w_z = 0, \quad t = t_1, t_2 \quad (7.466)$$

where $L = T - V$ is the Lagrangian, in which T is the kinetic energy and V the potential energy, and $\overline{\delta W}_{nc}$ is the virtual work performed by the nonconservative forces. Letting $m = m(x)$ be the mass density and using Eq. (7.465), the kinetic energy has the expression

$$\begin{aligned} T &= \frac{1}{2} \int_0^L m \mathbf{v}^T \mathbf{v} dx = \frac{1}{2} \int_0^L m \left[(\dot{w}_y - \Omega w_z)^2 + (\dot{w}_z + \Omega w_y)^2 \right] dx \\ &= \int_0^L \hat{T}(w_y, w_z, \dot{w}_y, \dot{w}_z) dx \end{aligned} \quad (7.467)$$

in which \hat{T} is the kinetic energy density. The potential energy is due to bending alone and has the form

$$V = \frac{1}{2} \int_0^L \left[EI_y (w_y'')^2 + EI_z (w_z'')^2 \right] dx = \int_0^L \hat{V} (w_y'', w_z'') dx \quad (7.468)$$

where \hat{V} is the potential energy density, in which EI_y and EI_z are bending stiffnesses. Finally, the virtual work performed by the nonconservative distributed forces is simply

$$\overline{\delta W}_{nc} = \int_0^L (f_y \delta w_y + f_z \delta w_z) dx \quad (7.469)$$

The boundary-value problem can be derived by inserting Eqs. (7.467)–(7.469) into Eq. (7.466) and carrying out the customary integrations by parts. All these operations have been performed in Sec. 7.3, however, and need not be repeated. Hence, using the analogy with Eq. (7.41), it is not difficult to show that Lagrange's differential equations of motion are

$$\frac{\partial \hat{L}}{\partial w_y} + \frac{\partial^2}{\partial x^2} \left(\frac{\partial \hat{L}}{\partial w_y''} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}_y} \right) + f_y = 0, \quad 0 < x < L \quad (7.470a)$$

$$\frac{\partial \hat{L}}{\partial w_z} + \frac{\partial^2}{\partial x^2} \left(\frac{\partial \hat{L}}{\partial w_z''} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \hat{L}}{\partial \dot{w}_z} \right) + f_z = 0, \quad 0 < x < L \quad (7.470b)$$

where $\hat{L} = \hat{T} - \hat{V}$ is the Lagrangian density. Moreover, one boundary condition for w_y and one for w_z must be selected at each end from

$$\frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w_y''} \right) \delta w_y \Big|_0^L = 0, \quad \frac{\partial}{\partial x} \left(\frac{\partial \hat{L}}{\partial w_z''} \right) \delta w_z \Big|_0^L = 0 \quad (7.471a, b)$$

and one from

$$\frac{\partial \hat{L}}{\partial w_y''} \delta w_y' \Big|_0^L = 0, \quad \frac{\partial \hat{L}}{\partial w_z''} \delta w_z' \Big|_0^L = 0 \quad (7.471c, d)$$

In view of Eqs. (7.467) and (7.468), the partial differential equations of motion have the explicit form

$$m \ddot{w}_y - 2m\Omega \dot{w}_z - m\Omega^2 w_y + \frac{\partial^2}{\partial x^2} \left(EI_y \frac{\partial^2 w_y}{\partial x^2} \right) = f_y, \quad 0 < x < L \quad (7.472a)$$

$$m \ddot{w}_z + 2m\Omega \dot{w}_y - m\Omega^2 w_z + \frac{\partial^2}{\partial x^2} \left(EI_z \frac{\partial^2 w_z}{\partial x^2} \right) = f_z, \quad 0 < x < L \quad (7.472b)$$

and, because the shaft is simply supported at both ends, the boundary conditions are

$$w_y = 0, \quad EI_y \frac{\partial^2 w_y}{\partial x^2} = 0, \quad x = 0, L \quad (7.473a, b)$$

$$w_z = 0, \quad EI_z \frac{\partial^2 w_z}{\partial x^2} = 0, \quad x = 0, L \quad (7.473c, d)$$

We observe that the boundary-value problem, Eqs. (7.472) and (7.473), is coupled by the gyroscopic terms alone.

To derive the eigenvalue problem, we let $f_y = f_z = 0$ in Eqs. (7.472) and assume a solution of the resulting homogeneous problem in the form

$$w_y(x, t) = w_y(x)e^{\lambda t}, \quad w_z(x, t) = w_z(x)e^{\lambda t} \quad (7.474)$$

Inserting Eqs. (7.474) into Eqs. (7.472) and (7.473) and dividing through by $e^{\lambda t}$, we obtain the ordinary differential equations

$$\lambda^2 m w_y - 2\lambda m \Omega w_z + \frac{d^2}{dx^2} \left(EI_y \frac{d^2 w_y}{dx^2} \right) - m \Omega^2 w_y = 0, \quad 0 < x < L \quad (7.475a)$$

$$\lambda^2 m w_z + 2\lambda m \Omega w_y + \frac{d^2}{dx^2} \left(EI_z \frac{d^2 w_z}{dx^2} \right) - m \Omega^2 w_z = 0, \quad 0 < x < L \quad (7.475b)$$

as well as the boundary conditions

$$w_y = 0, \quad EI_y \frac{d^2 w_y}{dx^2} = 0, \quad x = 0, L \quad (7.476a, b)$$

$$w_z = 0, \quad EI_z \frac{d^2 w_z}{dx^2} = 0, \quad x = 0, L \quad (7.476c, d)$$

Closed-form solutions of the eigenvalue problem are not possible in general.

Next, we assume that the shaft is *uniform*, $m(x) = m = \text{constant}$, $EI_y(x) = EI_y = \text{constant}$ and $EI_z(x) = EI_z = \text{constant}$. Moreover, for simplicity, we assume that $I_y = I_z = I$. Then, dividing through by m , the differential equations can be rewritten as

$$\lambda^2 w_y - 2\lambda \Omega w_z + \frac{EI}{m} \frac{d^4 w_y}{dx^4} - \Omega^2 w_y = 0, \quad 0 < x < L \quad (7.477a)$$

$$\lambda^2 w_z + 2\lambda \Omega w_y + \frac{EI}{m} \frac{d^4 w_z}{dx^4} - \Omega^2 w_z = 0, \quad 0 < x < L \quad (7.477b)$$

and the boundary conditions reduce to

$$w_y = 0, \quad \frac{d^2 w_y}{dx^2} = 0, \quad x = 0, L \quad (7.478a, b)$$

$$w_z = 0, \quad \frac{d^2 w_z}{dx^2} = 0, \quad x = 0, L \quad (7.478c, d)$$

It is easy to verify that the eigenfunctions of the system are

$$w_{yj} = a_j \sin \frac{j\pi x}{L}, \quad w_{zj} = b_j \sin \frac{j\pi x}{L}, \quad j = 1, 2, \dots \quad (7.479)$$

Introducing the solutions

$$w_y = \sum_{j=1}^{\infty} a_j \sin \frac{j\pi x}{L}, \quad w_z = \sum_{j=1}^{\infty} b_j \sin \frac{j\pi x}{L} \quad (7.480)$$

in Eqs. (7.477), multiplying the resulting equations by $\sin k\pi x/L$, integrating over the domain $0 < x < L$ and considering the orthogonality of the eigenfunctions, we obtain the infinite set of pairs of homogeneous algebraic equations

$$\begin{aligned} (\lambda^2 + \omega_j^2 - \Omega^2) a_j - 2\lambda\Omega b_j &= 0, \\ 2\lambda\Omega a_j + (\lambda^2 + \omega_j^2 - \Omega^2) b_j &= 0, \end{aligned} \quad j = 1, 2, \dots \quad (7.481)$$

where

$$\omega_j' = (j\pi)^2 \sqrt{\frac{EI}{mL^4}}, \quad j = 1, 2, \dots \quad (7.482)$$

are recognized as the natural frequencies of the nonrotating shaft. The solution of Eqs. (7.481) can be verified to be

$$\lambda_{2j-1} = i\omega_{2j-1}, \quad \bar{\lambda}_{2j-1} = -i\omega_{2j-1}, \quad \lambda_{2j} = i\omega_{2j}, \quad \bar{\lambda}_{2j} = -i\omega_{2j}, \quad j = 1, 2, \dots \quad (7.483a)$$

$$\omega_{2j-1} = \Omega + \omega_j', \quad \omega_{2j} = \Omega - \omega_j', \quad j = 1, 2, \dots \quad (7.483b)$$

$$b_{2j-1} = ia_{2j-1}, \quad \bar{b}_{2j-1} = -i\bar{a}_{2j-1}, \quad b_{2j} = ia_{2j}, \quad \bar{b}_{2j} = -i\bar{a}_{2j}, \quad j = 1, 2, \dots \quad (7.483c)$$

where overbars denote complex conjugates.

The free vibration solution can be obtained by inserting Eqs. (7.480) and (7.483) into Eqs. (7.474), with the result

$$w_y(x, t) = \sum_{j=1}^{\infty} \left(a_{2j-1} e^{\lambda_{2j-1}t} + \bar{a}_{2j-1} e^{\bar{\lambda}_{2j-1}t} + a_{2j} e^{\lambda_{2j}t} + \bar{a}_{2j} e^{\bar{\lambda}_{2j}t} \right) \sin \frac{j\pi x}{L} \quad (7.484a)$$

$$w_z(x, t) = \sum_{j=1}^{\infty} \left(b_{2j-1} e^{\lambda_{2j-1}t} + \bar{b}_{2j-1} e^{\bar{\lambda}_{2j-1}t} + b_{2j} e^{\lambda_{2j}t} + \bar{b}_{2j} e^{\bar{\lambda}_{2j}t} \right) \sin \frac{j\pi x}{L} \quad (7.484b)$$

Then, using Eqs. (7.483) and introducing the notation

$$a_{2j-1} = \frac{1}{2} A_{2j-1} e^{-i\phi_{2j-1}}, \quad a_{2j} = \frac{1}{2} A_{2j} e^{-i\phi_{2j}}, \quad j = 1, 2, \dots \quad (7.485)$$

where A_{2j-1} and A_{2j} are real amplitudes and ϕ_{2j-1} and ϕ_{2j} are corresponding phase angles, quantities depending on the initial conditions, the response becomes

$$w_y(x, t) = \sum_{j=1}^{\infty} [A_{2j-1} \cos(\omega_{2j-1}t - \phi_{2j-1}) + A_{2j} \cos(\omega_{2j}t - \phi_{2j})] \sin \frac{j\pi x}{L} \quad (7.486a)$$

$$w_z(x, t) = - \sum_{j=1}^{\infty} [A_{2j-1} \sin(\omega_{2j-1}t - \phi_{2j-1}) + A_{2j} \sin(\omega_{2j}t - \phi_{2j})] \sin \frac{j\pi x}{L} \quad (7.486b)$$

Note that the results obtained here are consistent with those for discrete gyroscopic systems obtained in Sec. 4.7.

7.18 DISTRIBUTED DAMPED SYSTEMS

The systems considered in this chapter until now share one characteristic, namely, in the absence of external forces they are conservative. This implies that, if they are excited initially and then allowed to vibrate freely, the vibration will continue ad infinitum. But, conservative systems represent mathematical idealizations, and in practice all systems possess some degree of damping, so that free vibration dies out eventually. Nevertheless, the idealization is useful when damping is very small and the interest lies in time intervals too short for damping effects to become measurable. At this point, however, we consider the case in which damping is not negligible.

As established for discrete systems, and implied above, damping forces are nonconservative. Perhaps the simplest way to account for viscous damping forces is to treat them as a special type of nonconservative forces in the boundary-value problem for undamped systems of Sec. 7.16, Eqs. (7.457) and (7.458). We assume that the damping force at any point P is proportional to the velocity and opposite in direction to the velocity, or

$$f_d(P, t) = -C\dot{w}(P, t) \quad (7.487)$$

where C is a linear homogeneous differential operator of order $2p$. In fact, C is an operator similar to the operator L defined in Sec. 7.5. Inserting Eq. (7.487) into Eq. (7.457) and assuming that the force density $f(P, t)$ includes all nonconservative forces other than damping forces of the type defined by Eq. (7.487), we define the boundary-value problem for viscously damped systems as consisting of the differential equation

$$Lw(P, t) + C\dot{w}(P, t) + M\ddot{w}(P, t) = f(P, t), \quad P \text{ in } D \quad (7.488)$$

and the boundary conditions

$$B_i w(P, t) = 0, \quad P \text{ on } S, \quad i = 1, 2, \dots, k \quad (7.489a)$$

$$B_i \dot{w}(P, t) + C_i \ddot{w}(P, t) = 0, \quad P \text{ on } S, \quad i = k + 1, k + 2, \dots, p \quad (7.489b)$$

Closed-form solutions of the boundary-value problem for damped systems, Eqs. (7.488) and (7.489), are not possible in general due to difficulties in solving the eigenvalue problem. Under certain circumstances, however, the eigenfunctions of the undamped system can be used to decouple the modal equations, in a manner similar to that for discrete systems. To this end, we assume a solution of Eq. (7.488) in the form

$$w(P, t) = \sum_{s=1}^{\infty} w_s(P) \eta_s(t) \quad (7.490)$$

where $w_s(P)$ are the eigenfunctions of the undamped system, obtained by letting $C = 0$ in Eq. (7.488). The corresponding eigenvalue problem is given by Eqs. (7.459) and (7.460). Following the approach of Sec. 7.16, we introduce Eq. (7.490) in Eq. (7.488), multiply through by $w_r(P)$, consider the orthonormality relations, Eqs. (7.461), and obtain the modal equations

$$\ddot{\eta}_r(t) + \sum_{s=1}^{\infty} c_{rs} \dot{\eta}_s(t) + \omega_r^2 \eta_r(t) = N_r(t), \quad r = 1, 2, \dots \quad (7.491)$$

where

$$c_{rs} = \int_D w_r(P) C w_s(P) dD(P), \quad r, s = 1, 2, \dots \quad (7.492)$$

are damping coefficients and $N_r(t)$ are modal forces having the form given by Eqs. (7.454).

Equations (7.491) represent an infinite set of coupled ordinary differential equations, so that in general damping produces coupling of the modal equations. In the special case in which the damping operator C can be expressed as a linear combination of the stiffness operator L and mass operator M of the form

$$C = \alpha L + \beta M \quad (7.493)$$

where α and β are constant scalars, the damping coefficients, Eqs. (7.492), can be rewritten as

$$c_{rs} = c_r \delta_{rs} = 2\zeta_r \omega_r \delta_{rs}, \quad r, s = 1, 2, \dots \quad (7.494)$$

in which case Eqs. (7.491) reduce to the independent set

$$\ddot{\eta}_r(t) + 2\zeta_r \omega_r \dot{\eta}_r(t) + \omega_r^2 \eta_r(t) = N_r(t), \quad r = 1, 2, \dots \quad (7.495)$$

and we note that the notation in Eq. (7.494) was chosen so as to render Eqs. (7.495) similar in structure to the modal equations of viscously damped systems. The solution of Eqs. (7.495) was obtained in Sec. 4.10 in the form of Eqs. (4.229). Damping of the type represented by Eq. (7.493) is known as *proportional damping*.

Several damping models in common use are merely special cases of proportional damping. We distinguish between external and internal damping. External damping generally carries the implication that the mass operator is a mere function, $M = m$, where m is the mass density. In this case, $\alpha = 0$ and the damping operator is assumed to be proportional to the mass density, $C = \beta m = c$, where c represents a viscous damping density function. This is the case of *distributed viscous damping*, a

concept that raises as many questions as it answers. Internal damping is based on the assumption that the *material behaves viscoelastically*. A commonly used viscoelastic model is the *Kelvin-Voigt* model, whereby the normal stress is related to the strain and strain rate by

$$\sigma = E(\epsilon + c\dot{\epsilon}) = E\left(\frac{\partial u}{\partial x} + c\frac{\partial^2 u}{\partial t \partial x}\right) \quad (7.496)$$

where E is Young's modulus and c is a given constant. It is easy to verify that the Kelvin-Voigt model represents the distributed counterpart of the spring and dashpot in parallel model used repeatedly in Chapter 3.

In the case of a thin rod in axial vibration, the normal stresses are assumed to be distributed uniformly over the cross-sectional area, so that the axial force is related to the axial displacement by

$$F(x, t) = EA(x)\left[\frac{\partial u(x, t)}{\partial x} + c\frac{\partial^2 u(x, t)}{\partial t \partial x}\right] \quad (7.497)$$

It is not difficult to show that in this case

$$C = cL = -c\frac{\partial}{\partial x}\left(EA\frac{\partial}{\partial x}\right) \quad (7.498)$$

Comparing Eqs. (7.493) and (7.498), we conclude that the Kelvin-Voigt viscoelastic model is indeed a special case of proportional damping, in which $\alpha = c$, $\beta = 0$. Note that, in view of Eq. (7.497), any existing natural boundary conditions must be modified to include the contribution of viscosity to the force.

In the case of an Euler-Bernoulli beam in bending vibration, the assumption that cross-sectional areas remain planar during deformations, when used in conjunction with the Kelvin-Voigt model, implies that the bending moment is related to the bending displacement by

$$M(x, t) = EI(x)\left[\frac{\partial^2 w(x, t)}{\partial x^2} + c\frac{\partial^3 w(x, t)}{\partial t \partial x^2}\right] \quad (7.499)$$

so that once again we encounter a special case of proportional damping. This time

$$C = cL = c\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2}{\partial x^2}\right) \quad (7.500)$$

so that $\alpha = c$, $\beta = 0$. Here too, we must recognize that any existing natural boundary conditions must be modified to include the viscosity effect. In this regard, we should recognize that the shearing force Q is also involved, as it is equal to $-\partial M/\partial x$.

Finally, we consider *structural damping*. As indicated in Sec. 4.10, structural damping can be treated as viscous damping, provided the excitation is harmonic. To emphasize this point, we rewrite Eq. (7.488) in the form

$$Lw(P, t) + C\dot{w}(P, t) + M\ddot{w}(P, t) = f(P)e^{i\Omega t} \quad (7.501)$$

where $f(P)$ is a force density amplitude, generally a complex quantity, and Ω is the driving frequency. But, steady-state response is harmonic, so that

$$\dot{w}(P, t) = i\Omega w(P, t) \quad (7.502)$$

Inserting Eq. (7.502) into Eq. (7.501), we have

$$Lw(P, t) + i\Omega Cw(P, t) + M(P)\ddot{w}(P, t) = f(P)e^{i\Omega t} \quad (7.503)$$

By analogy with the assumption made in Sec. 4.10 in connection with discrete systems, according to which the structural damping matrix is proportional to the stiffness matrix, it is customary to assume that the damping operator C is proportional to the stiffness operator L , or

$$C = \frac{\gamma}{\Omega} L \quad (7.504)$$

where γ is a *structural damping factor*. Introducing Eq. (7.504) in Eq. (7.503), we obtain

$$(1 + i\gamma)Lw(P, t) + M\ddot{w}(P, t) = f(P)e^{i\Omega t} \quad (7.505)$$

Then, using Eq. (7.490) and following the same procedure as earlier in this section, we obtain the infinite set of decoupled modal equations

$$\ddot{\eta}_r(t) + (1 + i\gamma)\omega_r^2\eta_r(t) = N_r e^{i\Omega t}, \quad r = 1, 2, \dots \quad (7.506)$$

where ω_r ($r = 1, 2, \dots$) are the natural frequencies of undamped oscillation and

$$N_r = \int_D w_r(P) f(P) dD(P), \quad r = 1, 2, \dots \quad (7.507)$$

are constant modal force amplitudes. The solution of Eqs. (7.506) is simply

$$\eta_r(t) = \frac{N_r e^{i\Omega t}}{(1 + i\gamma)\omega_r^2 - \Omega^2}, \quad r = 1, 2, \dots \quad (7.508)$$

so that, inserting Eqs. (7.508) into Eq. (7.490), we obtain the response

$$w(P, t) = \sum_{r=1}^{\infty} \frac{N_r e^{i\Omega t}}{(1 + i\gamma)\omega_r^2 - \Omega^2} w_r(P) \quad (7.509)$$

As in the case of discrete systems, the concept of structural damping should be used judiciously.

7.19 SYSTEMS WITH NONHOMOGENEOUS BOUNDARY CONDITIONS

In Sec. 7.16, we used modal analysis to obtain the response of boundary-value problems consisting of a nonhomogeneous differential equation of motion and homogeneous boundary conditions. According to this method, we first obtain the solution of the homogeneous boundary-value problem, which is done by separating the time and spatial dependence of the solution. This leads to an eigenvalue problem yielding the natural modes and the associated natural frequencies. Then, the solution of the nonhomogeneous differential equation is obtained by means of the expansion theorem in the form of a linear combination of normal modes.

In many cases, the boundary conditions are nonhomogeneous. In general, in these cases the approach of Sec. 7.16 will not work, and a different approach must be adopted. In this section, we modify the approach of Sec. 7.16 so as to enable us to use modal analysis. This modified approach is based on the fact that a boundary-value problem consisting of a homogeneous differential equation with nonhomogeneous boundary conditions can be transformed into a problem consisting of a nonhomogeneous differential equation with homogeneous boundary conditions (Ref. 8, p. 277). The latter problem can be solved by modal analysis. Actually, the approach can be used also when the differential equation is nonhomogeneous, in which case the nonhomogeneity of the differential equation becomes more involved.

We consider a one-dimensional system described by the differential equation of motion

$$Lw(x, t) + m(x) \frac{\partial^2 w(x, t)}{\partial t^2} = F(x, t), \quad 0 < x < L \quad (7.510)$$

where L is a linear homogeneous differential operator of order $2p$, and by the nonhomogeneous boundary conditions

$$B_i w(x, t) \Big|_{x=0} = e_i(t), \quad i = 1, 2, \dots, p \quad (7.511a)$$

$$B_j w(x, t) \Big|_{x=L} = f_j(t), \quad j = 1, 2, \dots, p \quad (7.511b)$$

where B_i and B_j are linear homogeneous differential operators of order $2p - 1$ or lower. For simplicity, we assume that the initial conditions are zero, or

$$w(x, 0) = 0, \quad \frac{\partial w(x, t)}{\partial t} \Big|_{t=0} = 0 \quad (7.512)$$

It is not difficult to see that the differential equation of motion and the boundary conditions are nonhomogeneous. We attempt a solution of the problem by transforming it into a problem consisting of a nonhomogeneous differential equation with homogeneous boundary conditions. To this end, we assume a solution of the boundary-value problem described by Eqs. (7.510) and (7.511) in the form

$$w(x, t) = v(x, t) + \sum_{i=1}^p g_i(x) e_i(t) + \sum_{j=1}^p h_j(x) f_j(t) \quad (7.513)$$

where the functions $g_i(x)$ and $h_j(x)$ are chosen to render the boundary conditions for the variable $v(x, t)$ homogeneous. In this manner, we transform the boundary-value

problem for the variable $w(x, t)$ into a boundary-value problem for the variable $v(x, t)$. The functions $g_i(x)$ and $h_j(x)$ are not unique and several choices may be acceptable. The corresponding results should be equivalent, however.

Introducing Eq. (7.513) in Eqs. (7.511), we obtain the boundary conditions

$$\begin{aligned} B_r w(x, t)|_{x=0} &= B_r v(x, t)|_{x=0} + \sum_{i=1}^p e_i(t) B_r g_i(x)|_{x=0} + \sum_{j=1}^p f_j(t) B_r h_j(x)|_{x=0} \\ &= e_r(t), \quad r = 1, 2, \dots, p \end{aligned} \quad (7.514a)$$

$$\begin{aligned} B_s w(x, t)|_{x=L} &= B_s v(x, t)|_{x=L} + \sum_{i=1}^p e_i(t) B_s g_i(x)|_{x=L} + \sum_{j=1}^p f_j(t) B_s h_j(x)|_{x=L} \\ &= f_s(t), \quad s = 1, 2, \dots, p \end{aligned} \quad (7.514b)$$

The functions $g_i(x)$ and $h_j(x)$ must be chosen so that the boundary conditions for $v(x, t)$ be homogeneous. Examination of Eqs. (7.514) reveals that, to satisfy these conditions, we must have

$$\begin{aligned} B_r g_i(x)|_{x=0} &= \delta_{ir}, \quad i, j, r = 1, 2, \dots, p \\ B_r h_j(x)|_{x=0} &= 0, \end{aligned} \quad (7.515a)$$

$$\begin{aligned} B_s g_i(x)|_{x=L} &= 0, \\ B_s h_j(x)|_{x=L} &= \delta_{js}, \quad i, j, s = 1, 2, \dots, p \end{aligned} \quad (7.515b)$$

as well as

$$B_i v(x, t)|_{x=0} = 0, \quad i = 1, 2, \dots, p \quad (7.516a)$$

$$B_j v(x, t)|_{x=L} = 0, \quad j = 1, 2, \dots, p \quad (7.516b)$$

Introducing Eq. (7.513) in Eq. (7.510), we obtain the nonhomogeneous differential equation

$$\begin{aligned} Lv(x, t) + m(x) \frac{\partial^2 v(x, t)}{\partial t^2} &= F(x, t) - \sum_{i=1}^p [e_i(t) Lg_i(x) + \ddot{e}_i(t) m(x) g_i(x)] \\ &\quad - \sum_{j=1}^p [f_j(t) Lh_j(x) + \ddot{f}_j(t) m(x) h_j(x)] \end{aligned} \quad (7.517)$$

where $v(x, t)$ is subject to homogeneous boundary conditions, Eqs. (7.516).

Using modal analysis, we first solve the eigenvalue problem consisting of the differential equation

$$Lv(x) = \omega^2 m(x) v(x), \quad 0 < x < L \quad (7.518)$$

and the boundary conditions

$$B_i v(x)|_{x=0} = 0, \quad i = 1, 2, \dots, p \quad (7.519a)$$

$$B_j v(x)|_{x=L} = 0, \quad j = 1, 2, \dots, p \quad (7.519b)$$

The solution of the eigenvalue problem, Eqs. (7.518) and (7.519), yields an infinite set of natural modes $v_r(x)$ and associated natural frequencies ω_r . The modes are orthogonal and, in addition, we normalize them so as to satisfy the orthonormality relations

$$\int_0^L m(x) v_r(x) v_s(x) dx = \delta_{rs}, \quad r, s = 1, 2, \dots \quad (7.520a)$$

$$\int_0^L v_r(x) L v_s(x) dx = \omega_r^2 \delta_{rs}, \quad r, s = 1, 2, \dots \quad (7.520b)$$

Using the expansion theorem, we assume a solution of Eq. (7.517) in the form

$$v(x, t) = \sum_{s=1}^{\infty} v_s(x) \eta_s(t) \quad (7.521)$$

Introducing Eq. (7.521) in Eq. (7.517), we obtain

$$\begin{aligned} & \sum_{s=1}^{\infty} \eta_s(t) L v_s(x) + \ddot{\eta}_s(t) m(x) v_s(x) \\ &= F(x, t) - \sum_{i=1}^p [e_i(t) L g_i(x) + \ddot{e}_i(t) m(x) g_i(x)] \\ & \quad - \sum_{j=1}^p [f_j(t) L h_j(x) + \ddot{f}_j(t) m(x) h_j(x)] \end{aligned} \quad (7.522)$$

and, because $v_s(x)$ and ω_s satisfy Eq. (7.518), Eq. (7.522) reduces to

$$\begin{aligned} & \sum_{s=1}^{\infty} [\ddot{\eta}_s(t) + \omega_s^2 \eta_s(t)] m(x) v_s(x) \\ &= F(x, t) - \sum_{i=1}^p [e_i(t) L g_i(x) + \ddot{e}_i(t) m(x) g_i(x)] \\ & \quad - \sum_{j=1}^p [f_j(t) L h_j(x) + \ddot{f}_j(t) m(x) h_j(x)] \end{aligned} \quad (7.523)$$

Next, we multiply Eq. (7.523) through by $v_r(x)$, integrate with respect to x over the domain, introduce the notation

$$G_{ri} = \int_0^L m(x)v_r(x)g_i(x) dx, \quad i = 1, 2, \dots, p; \quad r = 1, 2, \dots \quad (7.524a)$$

$$G_{ri}^* = \int_0^L v_r(x)Lg_i(x) dx,$$

$$H_{rj} = \int_0^L m(x)v_r(x)h_j(x) dx, \quad j = 1, 2, \dots, p; \quad r = 1, 2, \dots \quad (7.524b)$$

$$H_{rj}^* = \int_0^L v_r(x)Lh_j(x) dx,$$

$$F_r(t) = \int_0^L v_r(x)F(x, t) dx, \quad r = 1, 2, \dots \quad (7.524c)$$

and obtain the infinite set of independent modal equations

$$\ddot{\eta}_r(t) + \omega_r^2 \eta(t) = N_r(t), \quad r = 1, 2, \dots \quad (7.525)$$

where the modal forces have the form

$$N_r(t) = F_r(t) - \sum_{i=1}^p [G_{ri}^* \ddot{e}_i(t) + G_{ri} \ddot{e}_i(t)] - \sum_{j=1}^p [H_{rj}^* \ddot{f}_j(t) + H_{rj} \ddot{f}_j(t)], \quad r = 1, 2, \dots \quad (7.526)$$

The solution of Eqs. (7.525), for zero initial conditions, is given in the form of the convolution integral

$$\eta_r(t) = \frac{1}{\omega_r} \int_0^t N_r(\tau) \sin \omega_r(t - \tau) d\tau, \quad r = 1, 2, \dots \quad (7.527)$$

Equations (7.527), when introduced in Eq. (7.521), yield the solution $v(x, t)$ of the transformed problem and, subsequently, using Eq. (7.513), the solution $w(x, t)$ of the original problem.

Example 7.13

Obtain the solution of the axial vibration problem of a uniform rod clamped at $x = 0$ and with a time-dependent tensile force $P(t)$ at $x = L$.

The differential equation of motion is

$$EA \frac{\partial^2 w(x, t)}{\partial x^2} = m \frac{\partial^2 w(x, t)}{\partial t^2} \quad (a)$$

and the boundary conditions are

$$w(0, t) = 0, \quad EA \frac{\partial w(x, t)}{\partial x} \Big|_{x=L} = P(t) \quad (b)$$

so that we have a homogeneous differential equation with one homogeneous and one nonhomogeneous boundary condition. We assume a solution of Eq. (a) in the form

$$w(x, t) = v(x, t) + h(x)P(t) \quad (c)$$

so that the boundary conditions for $v(x, t)$ are

$$v(0, t) = -h(0)P(t), \quad EA \frac{\partial v(x, t)}{\partial x} \Big|_{x=L} = P(t) - P(t)EA \frac{dh(x)}{dx} \Big|_{x=L} \quad (d)$$

To render boundary conditions (d) homogeneous, we must have

$$h(0) = 0, \quad EA \frac{dh(x)}{dx} \Big|_{x=L} = 1 \quad (e)$$

The second of boundary conditions (e) can be written as

$$\frac{dh(x)}{dx} = \frac{1}{EA} \mu [x - (L - \epsilon)] \quad (f)$$

where $\mu [x - (L - \epsilon)]$ is a spatial step function and ϵ is a small quantity. In view of the first of Eqs. (e), Eq. (f) has the solution

$$h(x) = \frac{1}{EA} [x - (L - \epsilon)] \mu [x - (L - \epsilon)] \quad (g)$$

and we note that $h(x)$ is zero over the domain $0 \leq x \leq L - \epsilon$.

The transformed problem consists of the nonhomogeneous differential equation

$$-EA \frac{\partial^2 v(x, t)}{\partial x^2} + m \frac{\partial^2 v(x, t)}{\partial t^2} = EA \frac{d^2 h(x)}{dx^2} P(t) - mh(x) \ddot{P}(t) \quad (h)$$

and the homogeneous boundary conditions

$$v(0, t) = 0, \quad EA \frac{\partial v(x, t)}{\partial x} \Big|_{x=L} = 0 \quad (i)$$

The corresponding eigenvalue problem consists of the differential equation

$$-EA \frac{d^2 v(x)}{dx^2} = \omega^2 m v(x), \quad 0 < x < L \quad (j)$$

and the boundary conditions

$$v(0) = 0, \quad EA \frac{dv(x)}{dx} \Big|_{x=L} = 0 \quad (k)$$

Its solution was obtained in Sec. 7.6. The eigenfunctions are

$$v_r(x) = \sqrt{\frac{2}{mL}} \sin(2r - 1) \frac{\pi x}{2L}, \quad r = 1, 2, \dots \quad (l)$$

and the corresponding natural frequencies are

$$\omega_r = (2r - 1) \frac{\pi}{2} \sqrt{\frac{EA}{mL^2}}, \quad r = 1, 2, \dots \quad (m)$$

Using Eqs. (7.524b) and dropping the second subscript, we write

$$\begin{aligned} H_r &= \int_0^L m(x) v_r(x) h(x) dx = m \int_0^L v_r(x) h(x) dx \\ &= \frac{m}{EA} \sqrt{\frac{2}{mL}} \int_0^L \sin(2r - 1) \frac{\pi x}{2L} [x - (L - \epsilon)] \mu [x - (L - \epsilon)] dx = 0, \\ & \qquad \qquad \qquad r = 1, 2, \dots \quad (n) \end{aligned}$$

$$\begin{aligned}
 H_r^* &= \int_0^L v_r(x) L h(x) dx = -EA \int_0^L v_r(x) \frac{d^2 h(x)}{dx^2} dx \\
 &= -\sqrt{\frac{2}{mL}} \int_0^L \sin(2r-1) \frac{\pi x}{2L} \cdot \delta[x - (L - \epsilon)] dx \\
 &= (-1)^r \sqrt{\frac{2}{mL}} \cos(2r-1) \frac{\pi \epsilon}{2L}, \quad r = 1, 2, \dots
 \end{aligned} \tag{o}$$

where $\delta[x - (L - \epsilon)]$ is a spatial Dirac delta function. Because Eq. (a) is homogeneous and so is the first of boundary conditions (b), $F_r(t) = 0$ and $e(t) = 0$, so that Eqs. (7.526) yield the modal forces

$$N_r(t) = -H_r^* P(t) = (-1)^{r-1} \sqrt{\frac{2}{mL}} P(t) \cos(2r-1) \frac{\pi \epsilon}{2L}, \quad r = 1, 2, \dots \tag{p}$$

which, when introduced in Eqs. (7.527), give the modal coordinates

$$\eta_r(t) = \frac{(-1)^{r-1} \cos(2r-1) (\pi \epsilon / 2L)}{\omega_r} \sqrt{\frac{2}{mL}} \int_0^t P(\tau) \sin \omega_r(t - \tau) d\tau, \quad r = 1, 2, \dots \tag{q}$$

Finally, using Eq. (7.513), we obtain the desired solution

$$\begin{aligned}
 w(x, t) &= \sum_{r=1}^{\infty} v_r(x) \eta_r(t) + h(x) P(t) \\
 &= \frac{2}{mL} \sum_{r=1}^{\infty} \frac{(-1)^{r-1} \cos(2r-1) (\pi \epsilon / 2L)}{\omega_r} \sin(2r-1) \frac{\pi x}{2L} \int_0^t P(\tau) \sin \omega_r(t - \tau) d\tau \\
 &\quad + \frac{P(t)}{EA} [x - (L - \epsilon)] u[x - (L - \epsilon)]
 \end{aligned} \tag{r}$$

Although the last term in Eq. (r) is zero for $0 \leq x \leq L - \epsilon$ and small for $L - \epsilon \leq x \leq L$, it must be retained, because its derivatives are neither zero nor small for $L - \epsilon < x < L$; its presence ensures the satisfaction of the boundary condition at $x = L$.

7.20 SYNOPSIS

Mathematical models are not unique, and the same system can be modeled in different ways. In fact, modeling is more of an art than an exact science. In the first six chapters, we have been concerned with discrete, or lumped-parameter systems and in this chapter with distributed parameter systems. This division is very significant, as the mathematical techniques for the two types of systems differ drastically. Indeed, discrete systems possess a finite number of degrees of freedom and are governed by ordinary differential equations of motion, whereas distributed systems possess an infinite number of degrees of freedom and are described by boundary-values problems, consisting of partial differential equations and boundary conditions. Of course, the methods for solving ordinary differential equations are appreciably different from the methods for solving partial differential equations. Still, the difference between

the two classes of models is more of form than substance. Indeed, we recall that discrete and distributed models share many of the same characteristics, so that in fact they exhibit similar behavior. This fact is helpful when distributed systems described by boundary-value problems do not admit closed-form solutions, which are more the rule rather than the exception, and the interest lies in approximating them by discrete systems.

In this chapter, the power of analytical mechanics, and in particular the power of the extended Hamilton's principle in deriving boundary-value problems for distributed-parameter systems, is amply demonstrated. The principle can be used to derive the boundary-value problem for a generic system in the form of a partial Lagrange's differential equation of motion and appropriate boundary conditions. As for discrete systems, the free vibration of conservative distributed systems leads to an eigenvalue problem, this time a differential eigenvalue problem, as opposed to an algebraic one. Conservative distributed systems represent a very large and important class of systems referred to as self-adjoint. The discrete counterpart of self-adjoint systems are systems described by real symmetric matrices. Self-adjoint systems possess real eigenvalues and real and orthogonal eigenfunctions, in the same way real symmetric matrices possess real eigenvalues and real and orthogonal eigenvectors. Not surprisingly, systems described by real symmetric matrices are also referred to at times as self-adjoint. Also by analogy with discrete systems, an expansion theorem exists for distributed systems as well. Undamped strings, rods, shafts and beams are all demonstrated to fall in the class of self-adjoint systems. The inclusion in the case of beam vibration of lumped masses at the boundary and rotatory inertia throughout complicates Lagrange's equation and the boundary conditions, as well as the eigenvalue problem, by causing some boundary conditions to depend on the eigenvalue. If shear deformation effects are also included, then a more accurate beam model, known as a Timoshenko beam, is obtained. Two-dimensional systems introduce the shape of the boundary as a factor affecting greatly the nature of the problem. If the shape of the boundary is relatively simple, such as rectangular or circular, then this factor controls the choice of coordinates used to describe the problem. Closed-form solutions are scarce for two-dimensional problems, even when the shape of the boundary is simple. If the shape is irregular, then closed-form solutions do not exist, and approximate solutions are the only viable alternative. In this chapter, we consider rectangular and circular membranes and plates and present some of the few closed-form solutions possible. Approximate solutions are considered in Chapters 8 and 9.

In the case of self-adjoint systems, the differential eigenvalue problem can also be formulated in a weak form by a variational approach, which amounts to rendering Rayleigh's quotient stationary. This approach provides the foundation for some important approximate techniques whereby distributed-parameter systems are discretized (in the spatial variables) by assuming a solution in the form of a series of admissible functions. Reference is made here to the Rayleigh-Ritz method discussed in Chapter 8, and in its premier form, the finite element method, presented in Chapter 9. The net result is to reduce differential eigenvalue problems to algebraic ones. The integral formulation of the eigenvalue problem also provides the basis for some approximate techniques, but on a much more modest scale.

Distributed-parameter system response follows the same pattern as for discrete systems, namely, solve the differential eigenvalue problem, assume the solution of the boundary-value problem in the form of an infinite series of eigenfunctions multiplied by time-dependent modal coordinates and use the orthogonality of the eigenfunctions to obtain an infinite set of independent ordinary differential equations, known as modal equations. The latter can be solved as usual.

From the above, we conclude that, whereas distributed systems differ from discrete systems in form, the basic ideas remain the same.

PROBLEMS

- 7.1 The n -degree-of-freedom system of Fig. 7.30 consists of n beads of mass m_i suspended on a string and subjected to the forces F_i ($i = 1, 2, \dots, n$). The left end of the string is fixed and the right end is supported by a spring of stiffness k . The tension in the portion of the string of length Δx_i between the masses m_i and m_{i+1} is T_i . Derive the differential equations of motion for the transverse vibration of the system. Then, devise a limiting process by letting Δx_i approach zero so as to transform the equations of motion into the boundary-value problem derived in Sec. 7.1.

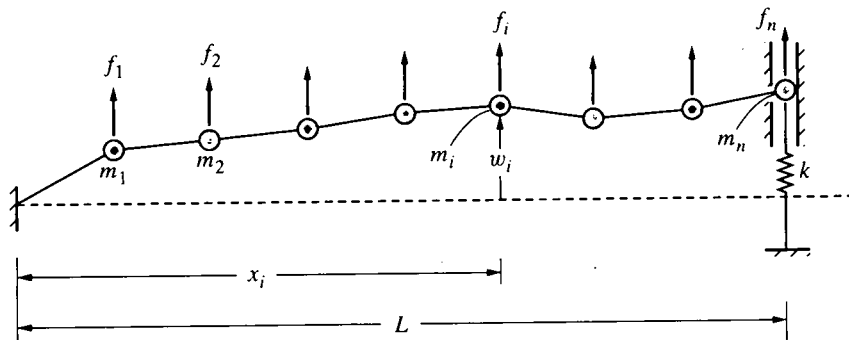


Figure 7.30 String with n masses in transverse vibration

- 7.2 A nonuniform rod in axial vibration has mass per unit length $m(x)$ and axial stiffness $EA(x)$, in which E is the modulus of elasticity and $A(x)$ the cross-sectional area. The left end is connected to a spring of stiffness k and the right end is free, as shown in Fig. 7.31. The rod is subjected to the force density $f(x, t)$. Derive the boundary-value problem in two ways, first by Newton's second law and then by the extended Hamilton's principle.

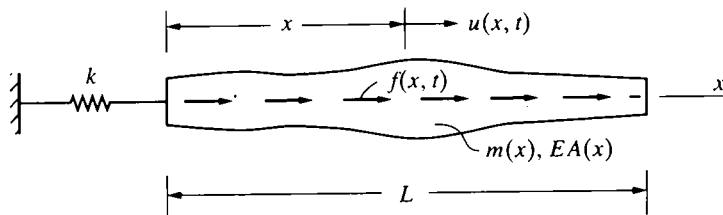


Figure 7.31 Rod in axial vibration connected to a spring at $x = 0$ and free at $x = L$

- 7.3 A nonuniform shaft in torsional vibration has polar mass moment of inertia per unit length $I(x)$ and torsional stiffness $GJ(x)$, in which G is the shear modulus and $J(x)$ is the area polar moment of inertia. The left end is supported by a torsional spring of stiffness k_1 and the right end by a torsional spring of stiffness k_2 , as shown in Fig. 7.32. The shaft is subjected to the moment per unit length $m(x, t)$. Derive the boundary-value problem in two ways, first by Newton's second law and then by the extended Hamilton's principle.

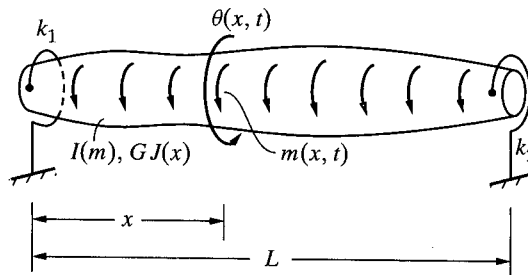


Figure 7.32 Shaft in torsional vibration supported by springs at both ends

- 7.4 A string of mass per unit length $\rho(x)$ hangs freely from a ceiling, as shown in Fig. 7.33. Derive the boundary-value problem for the transverse vibration of the string by the extended Hamilton's principle. Discuss the boundary condition at the lower end.

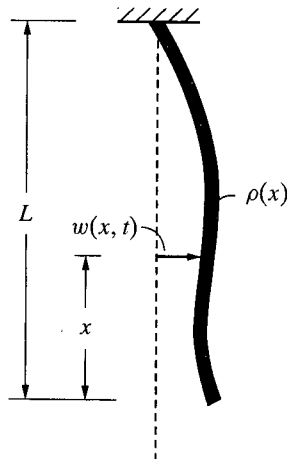


Figure 7.33 String in transverse vibration hanging freely from a ceiling

- 7.5 A beam of mass per unit length $m(x)$ and bending stiffness $EI(x)$, supported by springs of stiffness k_1 and k_2 at the two ends, is subjected to a distributed force $f(x, t)$, as shown in Fig. 7.34. Derive the boundary-value problem for the bending vibration of the beam in two ways, first by Newton's second law and then by the extended Hamilton's principle.

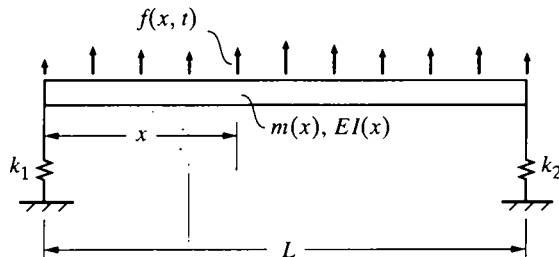


Figure 7.34 Beam in bending vibration supported by springs at both ends

7.6 A beam of mass per unit length $m(x)$ and bending stiffness $EI(x)$, free at both ends, lies on an elastic foundation of distributed stiffness $k(x)$, as shown in Fig. 7.35. Derive the boundary-value problem for the bending vibration of the beam.

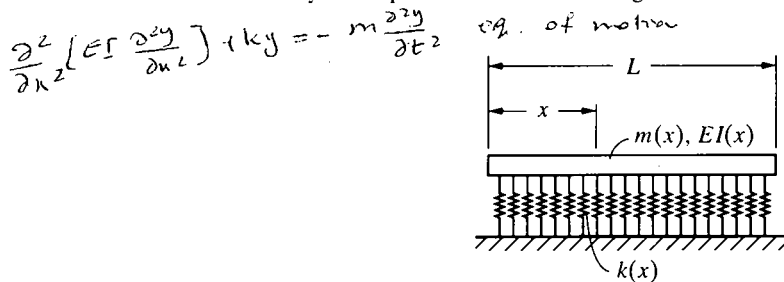


Figure 7.35 Beam in bending free at both ends lying on an elastic foundation

7.7 A beam of mass per unit length $m(x)$ and bending stiffness $EI(x)$, fixed at $x = 0$ and hinged at $x = L$ in a way that the bending slope is restrained by a spring of stiffness k (Fig. 7.36), is acted upon by the distributed force $f(x, t)$. Derive the boundary-value problem for the bending vibration of the beam.

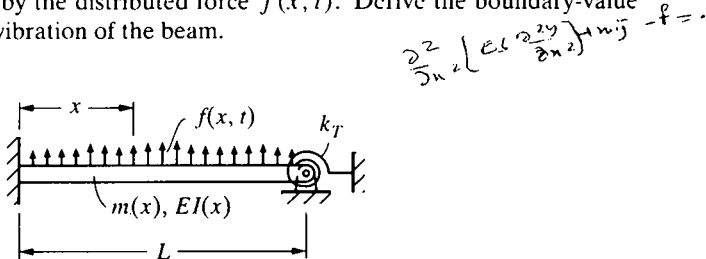


Figure 7.36 Beam in bending free at $x = 0$ and with a slope-restraining spring at $x = L$

7.8 A beam of circular cross section, capable of bending vibration about two orthogonal axes, rotates about the x -axis with the constant angular velocity Ω . The beam has mass per unit length $m(x)$, a disk of mass M at midspan and bending stiffness $EI(x)$, and is hinged at both ends, as shown in Fig. 7.37. Derive the boundary-value problem for the bending vibrations u_y and u_z of the beam about the rotating body axes y and z , respectively. Hint: Note that the lumped mass M of the disk can be treated as a distributed mass having the value $M \delta(x - L/2)$, where $\delta(x - L/2)$ is a spatial Dirac delta function.

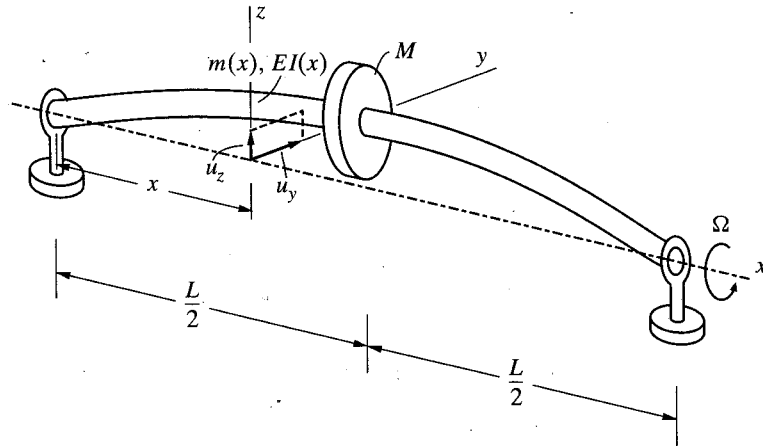


Figure 7.37 Rotating beam in bending hinged at both ends and with a disk at midspan

- 7.9** Derive the boundary-value problem for the rod of Problem 7.2 by the generic Lagrange equation of Sec. 7.3.
- 7.10** Derive the boundary-value problem for the shaft of Problem 7.3 by the generic Lagrange equation of Sec. 7.3.
- 7.11** Derive the boundary-value problem for the beam of Problem 7.5 by the generic Lagrange equation of Sec. 7.3.
- 7.12** Derive the boundary-value problem for the beam of Problem 7.7 by the generic Lagrange equation of Sec. 7.3.
- 7.13** Extend the generic formulation of Sec. 7.3 to the case of bending of a beam about two orthogonal axes. Then, use the formulation to derive the boundary-value problem for the rotating beam of Problem 7.8.
- 7.14** Derive the eigenvalue problem for the rod of Problem 7.2.
- 7.15** Derive the eigenvalue problem for the shaft of Problem 7.3.
- 7.16** Derive the eigenvalue problem for the string of Problem 7.4.
- 7.17** Derive the eigenvalue problem for the beam of Problem 7.5.
- 7.18** Derive the eigenvalue problem for the beam of Problem 7.6.
- 7.19** Derive the eigenvalue problem for the beam of Problem 7.7.
- 7.20** Cast the eigenvalue problem for the beam of Problem 7.14 in the generic form given by Eqs. (7.68) and (7.69), and then check whether the system is self-adjoint and positive definite.
- 7.21** Solve Problem 7.20 for the shaft of Problem 7.15.
- 7.22** Solve Problem 7.20 for the string of Problem 7.16.
- 7.23** Solve Problem 7.20 for the beam of Problem 7.17.
- 7.24** Solve Problem 7.20 for the beam of Problem 7.18.
- 7.25** Solve Problem 7.20 for the beam of Problem 7.19.
- 7.26** Assume that the rod of Problem 7.2 is uniform and solve the eigenvalue problem for the parameter ratio $EA/Lk = 1$. Plot the three lowest modes.
- 7.27** Assume that the shaft of Problem 7.3 is uniform and solve the eigenvalue problem for the parameters $k_1 = k_2 = k$, $GJ = 2kL$.

- 7.28 Solve the eigenvalue problem for a rod in axial vibration clamped at $x = 0$ and free at $x = L$ and with the following mass density and axial stiffness:

$$m(x) = 2m\left(1 - \frac{x}{L}\right), \quad EA(x) = 2EA\left(1 - \frac{x}{L}\right)$$

Plot the three lowest modes. Hints: (1) A suitable transformation reduces the differential equation to a Bessel equation and (2) the boundary condition at $x = L$ is the unorthodox one that the displacement must be finite.

- 7.29 Assume that the mass density of the string of Problem 7.4 is constant and solve the eigenvalue problem. Plot the three lowest modes. Hints: (1) a suitable transformation reduces the differential equation to a Bessel equation and (2) the boundary condition at the lower end is that the displacement must be finite.
- 7.30 Assume that the beam of Problem 7.5 is uniform and solve the eigenvalue problem for the parameters $k_1 = k$, $k_2 = 2k$, $EI = 10kL^3$. Plot the three lowest modes.
- 7.31 Assume that the beam and the elastic foundation of Problem 7.6 are uniform and solve the eigenvalue problem. Plot the three lowest modes. Draw conclusions as to the effect of the elastic foundation on the eigensolutions.
- 7.32 Assume that the beam of Problem 7.7 is uniform and solve the eigenvalue problem for $EI = 5kL$. Plot the three lowest modes.
- 7.33 Derive the boundary-value problem for the shaft of Example 7.5 by Newton's second law.
- 7.34 Derive the boundary-value problem for the rotating beam of Example 7.2 by Newton's second law.
- 7.35 The beam shown in Fig. 7.38 has mass per unit length $m(x)$ and bending stiffness $EI(x)$. The left end is clamped and there is a lumped mass M with mass moment of inertia I_M at the right end. Derive the boundary-value problem by the approach of Sec. 7.4 on the assumption that the rotatory inertia of the beam is negligibly small.

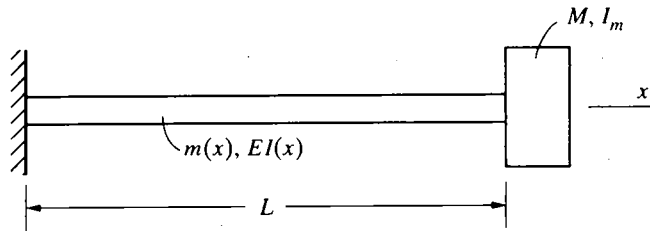


Figure 7.38 Beam in bending clamped at $x = 0$ and with a mass at $x = L$

- 7.36 Derive the eigenvalue problem for the system of Problem 7.35, verify that it fits the formulation of Sec. 7.9 by identifying the differential operators, and check the system self-adjointness and positive definiteness.
- 7.37 Assume that the mass and stiffness of the beam in the system of Problem 7.36 are distributed uniformly, solve the eigenvalue problem for the parameters ratios $M/mL = 2 \times 10^{-1}$ and $I_M/mL^3 = 5 \times 10^{-2}$ and plot the three lowest modes.
- 7.38 A uniform square membrane has repeated natural frequencies $\omega_{mn} = \omega_{nm}$. Any linear combination of the modes W_{mn} and W_{nm} is also a mode. Plot the nodal lines for the mode

$$W(x, y, c) = W_{13}(x, y) + cW_{31}(x, y)$$

for the values $c = 0, \frac{1}{2}, 1$.

- 7.39 Solve the eigenvalue problem for a uniform rectangular membrane fixed at $x = 0, a$ and free at $y = 0, b$. Assume that there are smooth vertical guides at $y = 0, b$ ensuring that the membrane tension is the same at every point and in every direction.
- 7.40 Solve the eigenvalue problem for a uniform rectangular membrane supported by a distributed spring of constant stiffness k at the boundaries $x = 0, a$ and fixed at the boundaries $y = 0, b$ for the parameters $b = 2a, T = 5ak$. The tension can be assumed to be constant at every point and in every direction, as in Problem 7.39.
- 7.41 Solve the eigenvalue problem for a uniform circular membrane supported at the boundary $r = a$ by a uniformly distributed spring of stiffness k for the case in which $T = 5ak$.
- 7.42 Solve the eigenvalue problem for a uniform annular membrane defined over the domain $b < r < a$ and fixed at the boundaries $r = b$ and $r = a$.
- 7.43 Use Eqs. (7.326) in conjunction with the relations

$$\frac{\partial}{\partial n} = \cos \phi \frac{\partial}{\partial x} + \sin \phi \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial s} = -\sin \phi \frac{\partial}{\partial x} + \cos \phi \frac{\partial}{\partial y}, \quad \frac{\delta \phi}{\partial n} = 0, \quad \frac{\partial \phi}{\partial s} = \frac{1}{R}$$

to derive Eqs. (7.336).

- 7.44 Modify the derivation of the generic boundary-value problem for plate vibration of Sec. 7.13 so as to accommodate the case in which the displacement w at every point of the boundary is restrained by a distributed spring of stiffness k .
- 7.45 Repeat Problem 7.44 for the case in which the slope $\partial w / \partial n$ at every point of the boundary is restrained by a distributed spring of stiffness k .
- 7.46 Check the self-adjointness and positive definiteness of a rectangular plate with the following boundaries:
- i. simply supported on all sides
 - ii. clamped on all sides
 - iii. free on all sides
 - iv. all sides supported as in Problem 7.44
 - v. all sides supported as in Problem 7.45
 - vi. any combination of sides as in cases i through v
- 7.47 A uniform rectangular plate is simply supported at the boundaries $y = 0, b$ and free at the boundaries $x = 0, a$. Solve the eigenvalue problem and plot the four lowest modes for the sides ratio $a/b = 1.5$.
- 7.48 Solve the eigenvalue problem for a uniform circular plate simply supported all around.
- 7.49 Calculate the value of Rayleigh's quotient for a uniform string fixed at both ends using the trial function $w = \frac{x}{L} - \left(\frac{x}{L}\right)^3$ and draw conclusions.
- 7.50 Calculate the value of Rayleigh's quotient for the string of Problem 7.22 using the trial function $w = \cos \pi x / 2L$ and draw conclusions.
- 7.51 The natural frequencies and modes of vibration of a uniform string fixed at both ends are $\omega_r = r\pi\sqrt{T/\rho L^2}$ and $W_r(x) = \sqrt{2/\rho L} \sin r\pi x / L$ ($r = 1, 2, \dots$). Construct two trial functions so as to demonstrate that Rayleigh's quotient has a mere stationary value at the second mode.
- 7.52 Use Rayleigh's quotient to estimate the lowest natural frequency of the beam of Problem 7.37.
- 7.53 Use Rayleigh's quotient to estimate the lowest natural frequency of the membrane of Problem 7.41.

- 7.54 Use Rayleigh's quotient to estimate the lowest natural frequency of a uniform rectangular plate clamped on all sides.
- 7.55 Formulate the eigenvalue problem for a uniform rod in axial vibration fixed at $x = 0$ and free at $x = L$ in integral form. Then, use the iteration method of Sec. 7.15 to calculate the lowest eigenvalue.
- 7.56 Use modal analysis to derive the response of a uniform string fixed at both ends to the initial displacement shown in Fig. 7.39. Discuss the mode participation in the response.

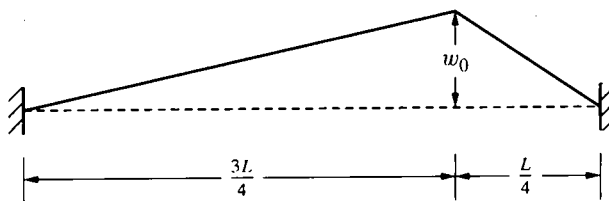


Figure 7.39 Initial displacement of a uniform string fixed at both ends

- 7.57 Derive the response of a uniform beam clamped at both ends to the initial velocity

$$v_0(x, 0) = c \left[\frac{x}{L} - 2 \left(\frac{x}{L} \right)^3 + \left(\frac{x}{L} \right)^4 \right]$$

and discuss the mode participation.

- 7.58 A force $F(t)$ traveling on a bridge in the positive x direction at the constant velocity v can be treated as distributed by writing

$$f(x, t) = \begin{cases} F(t) \delta(x - vt), & 0 \leq vt \leq L \\ 0, & vt > L \end{cases}$$

where L is the length of the bridge. Derive the response to the traveling force if the bridge has the form of a uniform simply supported beam.

- 7.59 A concentrated moment of unit magnitude applied in the clockwise sense at $x = a$ can be represented by two unit impulses acting in opposite directions, as shown in Fig. 7.40. This generalized function, denoted by $\delta'(x - a)$, is called a *spatial unit doublet* and has units length^{-2} . Hence, a concentrated moment $M(t)$ acting in the counterclockwise sense at $x = a$ can be represented as the distributed force $f(x, t) = -M(t) \delta'(x - a)$. Use the unit doublet concept to determine the response of a uniform simply supported beam to a counterclockwise moment applied at the right end.

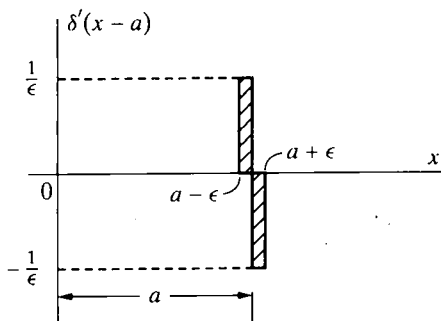


Figure 7.40 Spatial unit doublet at $x = a$

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APPROXIMATE METHODS FOR DISTRIBUTED-PARAMETER SYSTEMS

Chapter 7 contains a wealth of information concerning the vibration of distributed-parameter systems, including a variety of formulations for boundary-value and differential eigenvalue problems, an all-encompassing discussion of the important class of self-adjoint systems and of the properties of the corresponding eigensolutions, solutions to some differential eigenvalue problems and system response. An overview of Chapter 7 leads to the unmistakable conclusion that it contains a preponderance of problem formulations and discussions of the general properties of the solutions, but only a small number of actual solutions to complex problems. The reason for this paucity of solutions lies in the fact that very few differential eigenvalue problems admit closed-form solutions. Indeed, closed-form solutions are possible only in relatively few cases, almost invariably (but not exclusively) involving uniformly distributed parameters and simple boundary conditions. The satisfaction of boundary conditions can be particularly difficult for two-dimensional problems. In many cases, even though closed-form solutions may be possible, the effort in obtaining them may be so great as to discourage all but the most tenacious investigators. Hence, quite often one must be content with an approximate solution.

The difficulty inherent in the solution of boundary-value problems lies in the dependence on spatial variables. Hence, it should come as no surprise that all approximate methods for distributed-parameter problems have one thing in common, namely, the elimination of the spatial dependence. This amounts to reducing a distributed system to a discrete one through spatial discretization. The spatial discretization methods can be divided into two broad classes, lumping procedures and series discretization methods. Lumping methods are physically motivated, intuitive in character. They all amount to lumping the distributed mass at given points of the domain of the system. On the other hand, the stiffness can be treated as distributed

or it can be lumped also. Series discretization methods tend to be more abstract. In such methods, approximate solutions are assumed in the form of series of known space-dependent trial functions multiplied by undetermined coefficients. Certain integrations eliminate the spatial variables and reduce the problem to one of determining these coefficients. In all methods, the net result is to transform differential eigenvalue problems into algebraic ones.

In this chapter, we begin with the *lumped-parameter method using flexibility influence coefficients* whereby, as the name implies, the mass is lumped at discrete points; the stiffness is not lumped but described by means of influence coefficients. The method is applicable to one-dimensional and two-dimensional problems, although it may not be feasible to derive flexibility influence coefficients for the latter. Two other lumped-parameter methods considered are *Holzer's method* for torsional vibration and *Myklestad's method* for bending vibration, and we note that Holzer's method can be adapted to accommodate strings in transverse vibration and rods in axial vibration. These are chain, or step-by-step methods whereby both the mass and the stiffness are lumped and the description of the variables proceeds from one end of the elastic member to the other. The remaining methods in this chapter are all series discretization methods. They can be divided into two classes. The first class is based on variational principles and it amounts to minimization of Rayleigh's quotient. It is identified with the *Rayleigh-Ritz method* and is applicable to self-adjoint systems alone. The second is based on the idea of reducing the error caused by an approximate solution and is known as the *weighted residuals method*. It is in fact not one but a family of methods, the most important one being *Galerkin's method*. The weighted residuals methods are applicable to both self-adjoint and non-self-adjoint systems. Two other methods, component-mode synthesis and substructure synthesis, represent extensions of the Rayleigh-Ritz method to flexible multibody systems.

Conspicuous by its absence from this chapter is the finite element method, a method that rightfully belongs here. Indeed, as the other methods in this chapter, the finite element method seeks approximate solutions to differential eigenvalue problems (and other kinds of problems) not admitting closed-form solutions. Moreover, although the finite element method was believed in the beginning to be entirely new, it was demonstrated later to be another version of the Rayleigh-Ritz method. Still, the procedural details are sufficiently different and the body of literature on the subject has grown to such an extent that presentation of the finite element method in a separate chapter, namely Chapter 9, can be justified.

8.1 LUMPED-PARAMETER METHOD USING FLEXIBILITY INFLUENCE COEFFICIENTS

The lumped-parameter method is arguably the simplest method for the approximate solution of the eigenvalue problem for distributed systems. The approach is based on the integral formulation of the eigenvalue problem, Eq. (7.410). For convenience, we confine ourselves to one-dimensional domains, in which case the eigenvalue problem

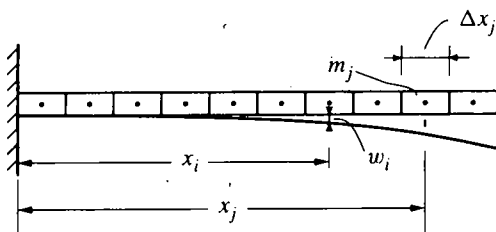


Figure 8.1 Mass lumping in a cantilever beam

has the integral form

$$w(x) = \omega^2 \int_0^L a(x, \xi) m(\xi) w(\xi) d\xi \quad (8.1)$$

where $a(x, \xi)$ is the flexibility influence function (Sec 7.15). Next, we consider the system of Fig. 8.1, divide the domain $0 < x < L$ into n small increments of length Δx_j and denote the center of these increments by $\xi = x_j$ ($j = 1, 2, \dots, n$). Moreover, we let $x = x_i$, as well as

$$w(x_i) = w_i, \quad a(x_i, x_j) = a_{ij}, \quad \int_{\Delta x_j} m(\xi) d\xi = m_j \quad (8.2)$$

and approximate Eq. (8.1) by

$$w_i = \omega^2 \sum_{j=1}^n a_{ij} m_j w_j, \quad i = 1, 2, \dots, n \quad (8.3)$$

Equation (8.3) is the discretized version of Eq. (8.1) and represents an algebraic eigenvalue problem, in which a_{ij} ($i, j = 1, 2, \dots, n$) are known as *flexibility influence coefficients* and denote the displacement at x_i due to a unit force at x_j . They are simply the discrete counterpart of the flexibility influence function $a(x, \xi)$ introduced in Sec. 7.15. Hence, by analogy, *Maxwell's reciprocity theorem* for lumped systems is

$$a_{ij} = a_{ji}, \quad i, j = 1, 2, \dots, n \quad (8.4)$$

indicating that the flexibility influence coefficients are symmetric in i and j .

Equations (8.3) can be cast in matrix form. To this end, we introduce the displacement vector $\mathbf{w} = [w_1 \ w_2 \ \dots \ w_n]^T$, the flexibility matrix $\mathbf{A} = [a_{ij}]$ and the mass matrix $\mathbf{M} = \text{diag}(m_j)$, so that Eqs. (8.3) can be rewritten as

$$\mathbf{w} = \omega^2 \mathbf{A} \mathbf{M} \mathbf{w} \quad (8.5)$$

where we note that $\mathbf{A} \mathbf{M}$ represents a special case of the dynamical matrix first encountered in Sec. 6.3, in the sense that here \mathbf{M} is diagonal. We also note that the flexibility matrix \mathbf{A} is the reciprocal of the stiffness matrix \mathbf{K} , $\mathbf{A} = \mathbf{K}^{-1}$. This is of mere academic interest, however, because computation of stiffness coefficients for distributed-parameter systems is not feasible.

The matrix AM is in general not symmetric, and the most efficient computational algorithms are for real symmetric matrices. In this particular case, the problem can be symmetrized with ease by introducing the vector

$$\mathbf{u} = M^{1/2}\mathbf{w} \quad (8.6)$$

where $M^{1/2} = \text{diag}(\sqrt{m_j})$. Introducing Eq. (8.6) into Eq. (8.5) and premultiplying by $M^{1/2}/\omega^2$, we obtain an eigenvalue problem in the standard form

$$A'\mathbf{u} = \lambda\mathbf{u}, \quad \lambda = 1/\omega^2 \quad (8.7)$$

where

$$A' = M^{1/2}AM^{1/2} \quad (8.8)$$

is a real symmetric matrix. The eigenvalue problem given by Eq. (8.7) can be solved by any of the algorithms presented in Chapter 6.

The accuracy of the results depends on the number and length of the increments Δx_j . Conceivably, the length of each individual increment can be varied to reflect the parameter nonuniformity, but in general the increments are taken equal in length, so that the question reduces to the number of increments. Unfortunately, there are no guidelines permitting a rational choice. Hence, whereas we can conclude on physical grounds that the approximations converge to the actual eigensolutions as $n \rightarrow \infty$, no quantitative convergence statement can be made.

The main appeal of the method is simplicity of the concepts. In fact, Eqs. (8.3) could have been obtained in a more direct manner by regarding the system as lumped from the beginning. But, whereas the concepts are simple, the implementation is not. The reason can be traced to the fact that, except for some simple cases, the evaluation of the influence coefficients a_{ij} can be quite difficult. This is often the case when the boundary conditions are complicated or when the problem is two-dimensional. Methods for obtaining influence coefficients are covered adequately in many textbooks on mechanics of materials.

Flexibility influence coefficients can be defined only when the potential energy is a positive definite function. Hence, the method just described is so restricted. However, the lumped-parameter method using influence coefficients can be extended to positive semidefinite systems. This amounts to eliminating the rigid-body motions from the formulation. To this end, we introduce a reference frame attached to the body in undeflected configuration and measure elastic displacements relative to the reference frame. The translation and rotation of the reference frame play the role of rigid-body modes, so that the elastic displacements are measured relative to the rigid-body modes. For convenience, we place the origin of the reference frame at the mass center C of the system. As an example, we consider the free vibration of the unrestrained beam in bending shown in Fig. 8.2 and denote the rigid-body translation of the origin C of the reference frame x, y by w_C , the rigid-body rotation of the reference frame by ψ_C , the elastic displacements of m_i relative to x, y by w_i and the total displacement of m_i relative to the inertial frame X, Y by W_i . We assume that the rotation ψ_C is relatively small, so that axes x, y are nearly parallel to axes X, Y . Under these circumstances, the absolute displacement of m_i has the form

$$W_i = w_C + x_i\psi_C + w_i, \quad i = 1, 2, \dots, n \quad (8.9)$$

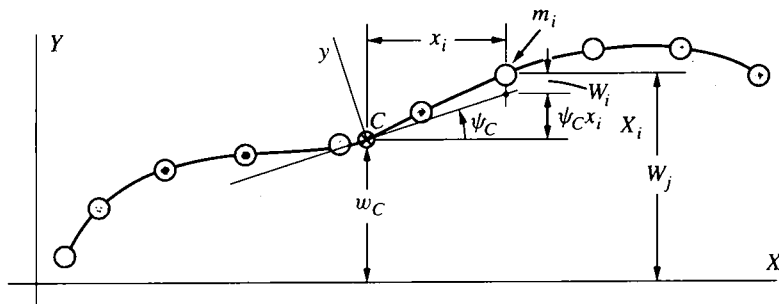


Figure 8.2 Lumped model of an unrestrained beam in bending

where x_i is the nominal position of m_i relative to C . To eliminate the rigid-body motions, we recognize that in free vibration the linear and angular momenta must vanish, or

$$\sum_{i=1}^n m_i \dot{W}_i = \sum_{i=1}^n m_i (\dot{w}_C + x_i \dot{\psi}_C + \dot{w}_i) = m \dot{w}_C + \mathbf{1}^T M \dot{\mathbf{w}} = 0 \quad (8.10a)$$

$$\sum_{i=1}^n m_i x_i \dot{W}_i = \sum_{i=1}^n m_i x_i (\dot{w}_C + x_i \dot{\psi}_C + \dot{w}_i) = I_C \dot{\psi}_C + \mathbf{x}^T M \dot{\mathbf{w}} = 0 \quad (8.10b)$$

where $m = \sum_{i=1}^n m_i$ is the total mass of the beam, $I_C = \sum_{i=1}^n m_i x_i^2$ the moment of inertia of the beam about C , $\mathbf{1} = [1 \ 1 \ \dots \ 1]^T$, $\mathbf{x} = [x_1 \ x_2 \ \dots \ x_n]^T$, $\dot{\mathbf{w}}$ the elastic velocity vector and M the diagonal mass matrix. Moreover, $\sum_{i=1}^n m_i x_i = 0$ by virtue of the fact that C is the mass center of the beam. Equations (8.10) yield simply

$$\dot{w}_C = -\frac{1}{m} \mathbf{1}^T M \dot{\mathbf{w}}, \quad \dot{\psi}_C = -\frac{1}{I_C} \mathbf{x}^T M \dot{\mathbf{w}} \quad (8.11)$$

Next, we use Newton's second law, recognize that in free vibration all forces are internal and write the equation of motion for each of the lumped masses in the form

$$m_i \ddot{W}_i = m_i (\ddot{w}_C + x_i \ddot{\psi}_C + \ddot{w}_i) = -f_i = -\sum_{j=1}^n k_{ij} w_j, \quad i = 1, 2, \dots, n \quad (8.12)$$

where k_{ij} are the stiffness coefficients. Writing Eqs. (8.12) in matrix form and using Eqs. (8.11), we obtain the equations for the elastic motions alone

$$M' \ddot{\mathbf{w}} + K \mathbf{w} = 0 \quad (8.13)$$

in which

$$M' = M - \frac{1}{m} M \mathbf{1} \mathbf{1}^T M - \frac{1}{I_C} M \mathbf{x} \mathbf{x}^T M \quad (8.14)$$

is a modified mass matrix. Because free vibration is harmonic, $\ddot{\mathbf{w}} = -\omega^2 \mathbf{w}$, Eq. (8.13) in conjunction with the usual operations yields the eigenvalue problem for the elastic modes

$$AM' \mathbf{w} = \lambda \mathbf{w}, \quad \lambda = 1/\omega^2 \quad (8.15)$$

where $A = K^{-1}$ is the flexibility matrix, and we note that the stiffness coefficients are not really required. We also note that the flexibility matrix is block-diagonal, as it consists of two independent submatrices on the main diagonal, one for a cantilever beam extending to the right of C and one for a cantilever beam extending to the left of C .

Although the eigenvalue problem (8.15) is of order n , there are only $n - 2$ valid solutions. The reason lies in the fact that the modified matrix M' is singular. Indeed, it is not difficult to verify that the rigid-body translation modal vector $\mathbf{1}$ and the rigid-body rotation modal vector \mathbf{x} are in the nullspace (Appendix B) of M' , and hence they satisfy $M'\mathbf{1} = \mathbf{0}$ and $M'\mathbf{x} = \mathbf{0}$. Of course, the system has a full complement of n eigenvectors, as the two missing eigenvectors are simply the two rigid-body modal vectors $\mathbf{1}$ and \mathbf{x} .

Equation (8.15) yields only the elastic part of the eigenvectors. To recover the contribution of the rigid-body modes to the elastic modes, we consider Eqs. (8.9), (8.11) and (8.14) and write in matrix form

$$\mathbf{W} = \mathbf{1}w_C + \mathbf{x}\psi_C + \mathbf{w} = \left(I - \frac{1}{m} \mathbf{1}\mathbf{1}^T M - \frac{1}{I_C} \mathbf{x}\mathbf{x}^T M \right) \mathbf{w} = M^{-1} M' \mathbf{w} \quad (8.16)$$

Finally, there is the question of lack of symmetry of the matrix AM' . Because M' is singular, the symmetrization process resulting in Eq. (8.8) is not possible. Fortunately, however, the flexibility matrix is positive definite. Hence, using the Cholesky decomposition, we can write

$$A = LL^T \quad (8.17)$$

Introducing Eq. (8.17) into Eq. (8.15), using the linear transformation

$$\mathbf{w} = L\mathbf{u}, \quad L^{-1}\mathbf{w} = \mathbf{u} \quad (8.18a, b)$$

and premultiplying the result by L , we can reduce the eigenvalue problem to the standard form

$$M^* \mathbf{u} = \lambda \mathbf{u} \quad (8.19)$$

where

$$M^* = L^T M' L = M^{*T} \quad (8.20)$$

is a symmetric matrix. Because M^* is related to M' by an orthogonal transformation, the eigenvalue problem (8.19) retains the characteristics of the eigenvalue problem (8.15), i.e., it possesses only $n - 2$ valid solutions. In this regard, we recognize that the nullspace of M^* consists of the two vectors $L^{-1}\mathbf{1}$ and $L^{-1}\mathbf{x}$. Upon solving eigenvalue problem (8.19), we must use Eq. (8.18a) in conjunction with the eigenvectors \mathbf{u}_r to compute the elastic eigenvectors \mathbf{w}_r for the original problem, Eq. (8.15).

Example 8.1

Obtain the natural frequencies and natural modes of vibration associated with the free-free beam shown in Fig. 8.3. The mass and stiffness distributions are

$$m(\xi) = \frac{4M}{5L} \left(1 + \frac{\xi}{2L}\right), \quad EI(\xi) = \frac{4}{5}EI \left(1 + \frac{\xi}{2L}\right) \quad (a)$$

where ξ is the distance from the left end, M the total mass and L the length of the beam.

The center of mass of the system is located at a distance $\bar{\xi}$ from the left end given by

$$\bar{\xi} = \frac{1}{M} \int_0^L \xi m(\xi) d\xi = \frac{8}{15}L \quad (b)$$

Next, we assume that the mass of the beam is lumped into n discrete masses such that the mass m_i ($i = 1, 2, \dots, n$) is equal to the mass in the segment $(i-1)(L/n) \leq \xi \leq i(L/n)$ and is located at the corresponding center of mass. Hence, the masses m_i have the values

$$m_i = \int_{(i-1)L/n}^{iL/n} m(\xi) d\xi = \frac{M}{5n^2}(4n + 2i - 1), \quad i = 1, 2, \dots, n \quad (c)$$

and are located at distances ξ_i from the left end given by

$$\xi_i = \frac{1}{m_i} \int_{(i-1)L/n}^{iL/n} \xi m(\xi) d\xi = \frac{2L}{3n(4n + 2i - 1)} [3n(2i - 1) + 3i(i - 1) + 1], \quad i = 1, 2, \dots, n \quad (d)$$

When measured from the center of mass C of the beam, instead of the left end, these locations are

$$x_i = \xi_i - \bar{\xi}, \quad i = 1, 2, \dots, n \quad (e)$$

Similarly, in terms of the distance x from C , the stiffness has the form

$$EI(x) = \frac{2EI}{5L}(2L + \bar{\xi} - x) \quad (f)$$

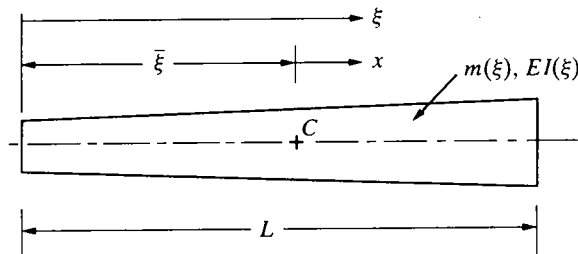


Figure 8.3 Nonuniform free-free beam in bending

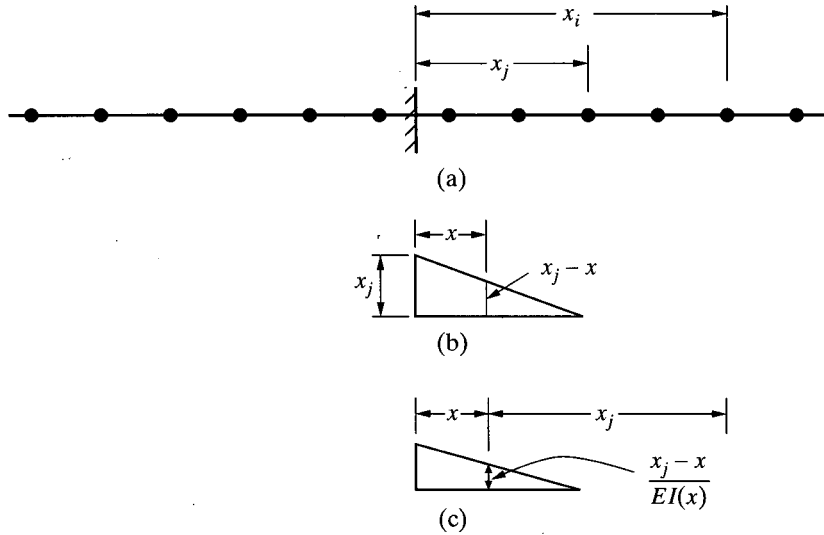


Figure 8.4 (a) Lumped-parameter model of a beam cantilevered on both sides (b) Bending moment diagram due to a unit force (c) Bending moment diagram divided by the bending stiffness

The influence coefficients a_{ij} can be determined by the moment-area method. To this end, we regard the beam as fixed at the mass center and cantilevered on each side, as shown in Fig. 8.4a. Figure 8.4b shows the bending moment diagram due to a unit load applied at $x = x_j$, and Fig. 8.4c shows the bending moment divided by the stiffness $EI(x)$. The slope of the deflection curve is zero at the fixed end, $x = 0$, so the moment-area method gives the deflection at the point x_i due to a unit load at x_j in the form of the moment with respect to point x_i of the area of the bending moment divided by $EI(x)$. Hence, for $x_i > x_j > 0$, we obtain

$$\begin{aligned}
 a_{ij} &= \int_0^{x_j} \frac{(x_j - x)(x_i - x)}{EI(x)} dx \\
 &= \frac{5L}{2EI} \left\{ [(x_i + x_j + 2L + \bar{\xi})(2L + \bar{\xi}) + x_i x_j] \ln \frac{x_j + 2L + \bar{\xi}}{2L + \bar{\xi}} \right. \\
 &\quad \left. - (x_i + x_j + 2L + \bar{\xi}) x_j - \frac{1}{2} x_j^2 \right\}, \quad i \leq j
 \end{aligned} \tag{g}$$

and a similar expression can be written for the case in which $x_i < 0$ and $x_j < 0$. Furthermore, the influence coefficients are symmetric, $a_{ij} = a_{ji}$. For $x_i > 0$ and $x_j < 0$, or for $x_i < 0$ and $x_j > 0$, we have

$$a_{ij} = 0 \tag{h}$$

The coefficients a_{ij} can be arranged in the form of the symmetric flexibility matrix A . The diagonal mass matrix M is obtained from Eq. (c) and the vector \mathbf{x} , as well as the mass moment of inertia I_C , from Eqs. (b), (d) and (e). This, in turn, allows us to evaluate the modified mass matrix M' according to Eq. (8.14). The natural frequencies and natural modes are obtained by solving the eigenvalue problem, Eq. (8.15). The solution consists of 18 eigenvalues ω_r^2 and purely elastic eigenvectors \mathbf{w}_r ($r = 3, 4, \dots, 20$), i.e., excluding the contribution from the rigid-body modes. The eigenvectors can be

inserted into Eq. (8.16) to obtain the absolute natural modes \mathbf{W}_r ($r = 3, 4, \dots, 20$), in the sense that they are measured relative to the inertial space. It should be recalled that the remaining two modes are the rigid-body modes $\mathbf{W}_1 = \mathbf{1}$ and $\mathbf{W}_2 = \mathbf{x}$ with corresponding natural frequencies equal to zero. The elastic modes \mathbf{W}_3 , \mathbf{W}_4 and \mathbf{W}_5 are displayed in Fig. 8.5.

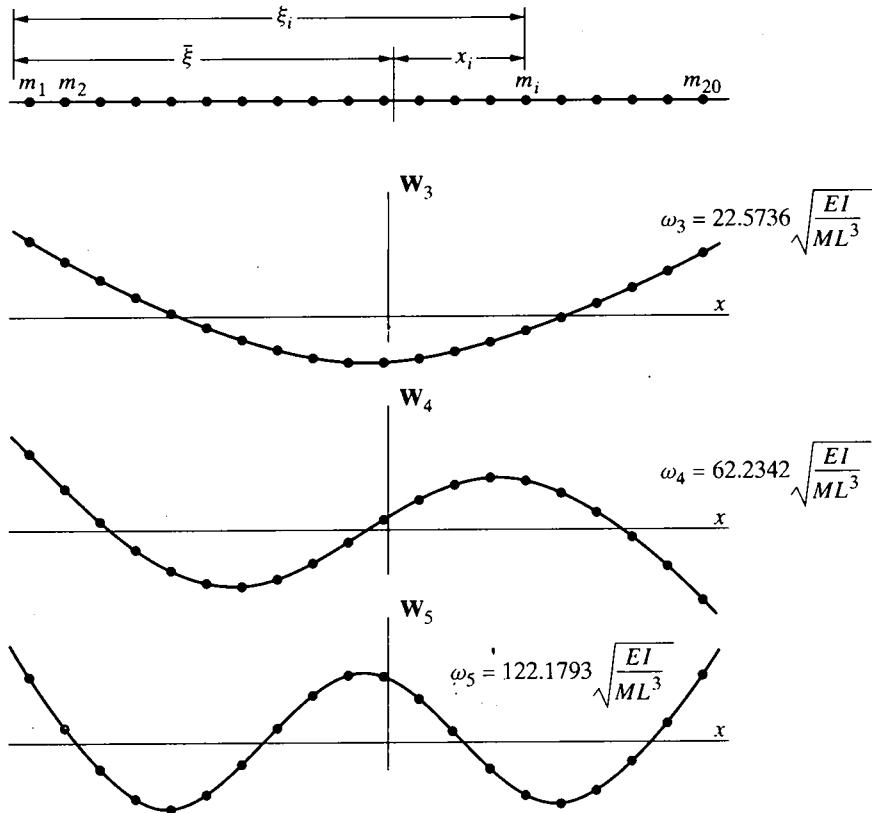


Figure 8.5 Lumped-parameter model of a free-free beam in bending and the three lowest elastic modes

It should be noted that the natural frequencies and natural modes are reasonably close to the ones of a uniform free-free beam of total mass M and stiffness EI , as can be expected. We must also note that smaller displacements occur at the heavier, stiffer end of the beam, which agrees with the expectation.

8.2 HOLZER'S METHOD FOR TORSIONAL VIBRATION

In the lumped-parameter method using flexibility influence coefficients, the mass properties are approximated by lumping the distributed mass at individual discrete points. If the flexibility coefficients are evaluated by regarding the stiffness as distributed, as was done in Example 8.1, then the stiffness properties are accounted for exactly. This is the strength of the method, but also its main drawback, as the

evaluation of flexibility coefficients for systems with distributed stiffness tends to be difficult. Even if the stiffness between any two adjacent lumps is assumed to be uniform, the situation is not significantly better, except for some simple systems, such as strings, rods and shafts characterized by second-order differential eigenvalue problems (Sec. 7.6).

Another lumped-parameter approach consists of lumping the mass at discrete points and regarding the portion between the lumped masses as being massless and of uniform stiffness, as suggested in the preceding paragraph, and does not rely on influence coefficients to characterize the stiffness properties. According to this approach, the eigenvalue problem is derived in a step-by-step process, advancing from one end of the member to the other. Clearly, this is a chain method, suitable for structures described by only one spatial variable. In this section, we apply the approach to a nonuniform shaft in torsional vibration, and in the next section we extend it to nonuniform beams in bending.

From mechanics of materials, the relation between the angle of twist $\theta(x, t)$ and the twisting moment $M(x, t)$ for a shaft in torsion is

$$\frac{\partial \theta(x, t)}{\partial x} = \frac{M(x, t)}{GJ(x)} \quad (8.21)$$

where $GJ(x)$ is the torsional stiffness, in which G is the shear modulus and $J(x)$ is the area polar moment of inertia of the cross section. Using the right-hand rule, M is positive if the vector indicating the sense of the moment is in the same direction as the normal to the cross section. For free vibration, the equation of motion is

$$\frac{\partial M(x, t)}{\partial x} = I(x) \frac{\partial^2 \theta(x, t)}{\partial t^2} \quad (8.22)$$

where $I(x)$ is the mass moment of inertia density.

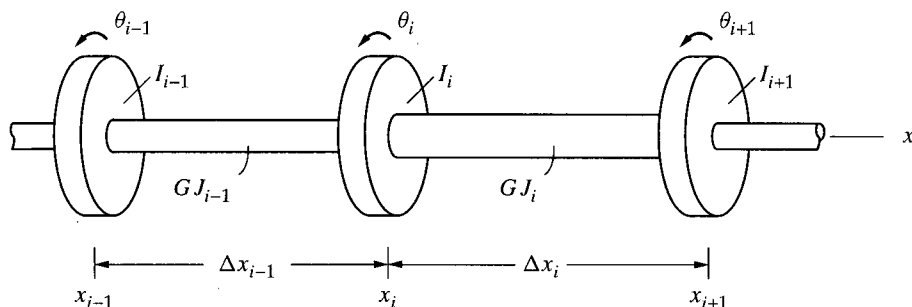


Figure 8.6 Lumped-parameter model of a shaft in torsion

Next, we consider a nonuniform shaft modeled as a lumped system consisting of a number of rigid disks connected by massless circular shafts of uniform stiffness, as shown in Fig. 8.6. Consistent with this, we approximate the differential expressions (8.21) and (8.22) by some recursive relations, which can be done through an incremental procedure combined with a good dose of ingenuity. It turns out that it is simpler to derive the recursive relations from the lumped model directly. To this

end, we denote the angular displacement and torque *on the left side of disk i* by θ_i^L and M_i^L , respectively, and the same quantities *on the right side of disk i* by θ_i^R and M_i^R . Moreover, in keeping with the tradition, we refer to disk i as *station i* and to the segment between station i and station $i + 1$ as *field i* . To derive the desired relations, we consider two free-body diagrams, one for station i and the other for field i ; they are displayed in Figs. 8.7a and 8.7b, respectively. Because the disks are rigid, the rotations on both sides of disk i must be the same, or

$$\theta_i^R(t) = \theta_i^L(t) = \theta_i(t) \quad (8.23)$$

Then, referring to Fig. 8.7a, the moment equation of motion is

$$M_i^R(t) - M_i^L(t) = I_i \ddot{\theta}_i(t) \quad (8.24)$$

Moreover, considering the shaft segment in Fig. 8.7b, we can interpret

$$a_i = \frac{\Delta x_i}{GJ_i} \quad (8.25)$$

as a torsional flexibility coefficient representing the angular displacement at the right end of the shaft due to a unit torque at the same end, where J_i is the polar moment of inertia of the cross section. Hence, the relation between the rotations at the two ends of field i is

$$\theta_{i+1}^L(t) = \theta_i^R(t) + a_i M_{i+1}^L(t) \quad (8.26)$$

Finally, considering the fact that the shaft segment has no inertia, we have

$$M_{i+1}^L(t) = M_i^R(t) \quad (8.27)$$

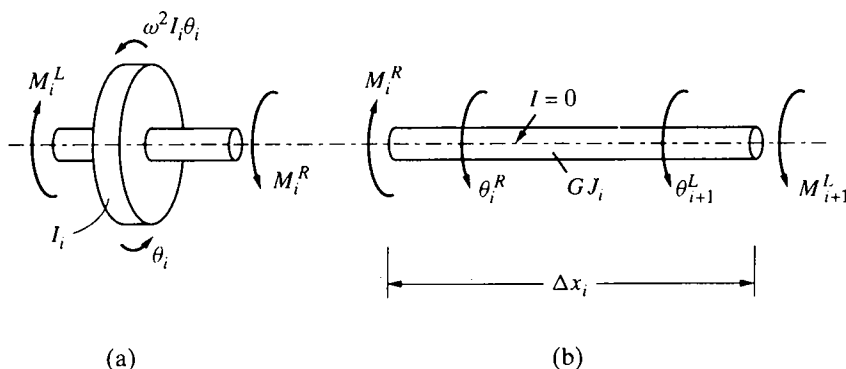


Figure 8.7 (a) Station i for a shaft in torsion (b) Field i for a shaft in torsion

At this point, we recall that free vibration is harmonic, so that $\ddot{\theta}_i(t) = -\omega^2\theta_i \cos(\omega t - \phi)$, where θ_i is a constant amplitude, ω the frequency of oscillation and ϕ an inconsequential phase angle. Introducing this expression into Eqs. (8.23) and (8.24), dividing through by $\cos(\omega t - \phi)$ and rearranging, we obtain relations in terms of amplitudes alone in the form

$$\theta_i^R = \theta_i^L, \quad M_i^R = -\omega^2 I_i \theta_i^L + M_i^L \quad (8.28a, b)$$

and we observe that Eqs. (8.28) carry us across station i , i.e., they provide us with θ_i^R and M_i^R in terms of θ_i^L and M_i^L . Similarly, Eqs. (8.26) and (8.27) can be rewritten as

$$\theta_{i+1}^L = \theta_i^R + a_i M_i^R, \quad M_{i+1}^L = M_i^R \quad (8.29a, b)$$

which carry us across the field i .

Equations (8.28) and (8.29) can be used recursively to relate the angle and torque at the left end to the angle and torque at the right end. The *method of Holzer* (Ref. 16) consists of using these recursive relations to solve the eigenvalue problem. The method represents a trial and error procedure, assuming values for θ_0^L and M_0^L consistent with the boundary condition at the left end and then assigning values for ω^2 repeatedly until the boundary condition at the right end is satisfied.

Instead of using Eqs. (8.28) and (8.29) on a recursive basis in conjunction with a trial and error procedure, it is possible to derive a characteristic equation in ω^2 and solve the equation by a root-finding algorithm. This approach is better explained by means of matrix notation. To this end, we express Eqs. (8.28) in the compact form

$$\mathbf{v}_i^R = T_{Si} \mathbf{v}_i^L \quad (8.30)$$

where $\mathbf{v}_i^R = [\theta_i^R \ M_i^R]^T$ and $\mathbf{v}_i^L = [\theta_i^L \ M_i^L]^T$ are referred to as *station vectors*¹ for the right side and left side of station i and

$$T_{Si} = \begin{bmatrix} 1 & 0 \\ -\omega^2 I_i & 1 \end{bmatrix} \quad (8.31)$$

is a *station transfer matrix* relating angular displacements and torques on both sides of station i . Similarly, Eqs. (8.29) can be expressed as

$$\mathbf{v}_{i+1}^L = T_{Fi} \mathbf{v}_i^R \quad (8.32)$$

where $\mathbf{v}_{i+1}^L = [\theta_{i+1}^L \ M_{i+1}^L]^T$ is a station vector for the left side of station $i + 1$ and

$$T_{Fi} = \begin{bmatrix} 1 & a_i \\ 0 & 1 \end{bmatrix} \quad (8.33)$$

is a *field transfer matrix* relating the angular displacement and torque on the left end of field i (right side of station i) to the angular displacement and torque on the right end of field i (left side of station $i + 1$). It should be stressed here that *the superscripts*

¹ In various discussions of the subject, the vectors \mathbf{v}_i^R and \mathbf{v}_i^L are referred to as "state vectors". In view of the fact that state vectors generally refer to vectors consisting of displacements and velocities, the term "station vectors" for \mathbf{v}_i^R and \mathbf{v}_i^L seems more appropriate.

R and L refer to the right side and left side of a station, not a field. Equations (8.30) and (8.32) can be combined into

$$\mathbf{v}_{i+1}^L = T_i \mathbf{v}_i^L \quad (8.34)$$

where

$$T_i = T_{Fi} T_{Si} = \begin{bmatrix} 1 - \omega^2 a_i I_i & a_i \\ -\omega^2 I_i & 1 \end{bmatrix} \quad (8.35)$$

is a transfer matrix relating the station vector on the left side of station $i + 1$ to the station vector on the left side of station i .

Equations (8.30), (8.32) and (8.34) can be used to derive an *overall transfer matrix* relating the station vector at the left boundary to the station vector at the right boundary. Embedded in this matrix is the characteristic polynomial. To illustrate the procedure, we consider the following cases:

1. *Clamped-free shaft.* In this case, we have n fields $i = 0, 1, \dots, n - 1$ and n stations, $i = 1, 2, \dots, n$, so that the recursive relations are

$$\begin{aligned} \mathbf{v}_1^L &= T_{F0} \mathbf{v}_0 \\ \mathbf{v}_2^L &= T_1 \mathbf{v}_1^L = T_1 T_{F0} \mathbf{v}_0 \\ &\vdots \\ \mathbf{v}_n^L &= T_{n-1} \mathbf{v}_{n-1}^L = T_{n-1} T_{n-2} \cdots T_2 T_1 T_{F0} \mathbf{v}_0 \\ \mathbf{v}_n^R &= T_{Sn} \mathbf{v}_n^L = T_{Sn} T_{n-1} T_{n-2} \cdots T_2 T_1 T_{F0} \mathbf{v}_0 = T \mathbf{v}_0 \end{aligned} \quad (8.36)$$

where

$$T = T_{Sn} \left(\prod_{i=n-1}^1 T_i \right) T_{F0} \quad (8.37)$$

is the *overall transfer matrix* for the case at hand. But, at the clamped end, we have the boundary condition

$$\theta_0 = 0 \quad (8.38a)$$

and at the right end the boundary condition is

$$M_n^R = 0 \quad (8.38b)$$

Hence, the last of Eqs. (8.36), in conjunction with Eqs. (8.38), yields

$$\begin{bmatrix} \theta_n^R \\ 0 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} 0 \\ M_0 \end{bmatrix} \quad (8.39)$$

which requires that

$$T_{22} = T_{22}(\omega^2) = 0 \quad (8.40)$$

Equation (8.40) represents the *frequency equation*, in which T_{22} is a polynomial of degree n in ω^2 . It has n roots, $\omega_1^2, \omega_2^2, \dots, \omega_n^2$, which can be found by a root-finding technique, such as the *secant method*, the *Newton-Raphson method*, or *Graeffe's root squaring method* (Ref. 38).

The question remains as to how to determine the eigenvectors $\boldsymbol{\theta}_r$ belonging to the eigenvalue ω_r^2 ($r = 1, 2, \dots, n$). To this end, we return to the recursive relations, Eqs. (8.36), and write

$$\mathbf{v}_i^L = T_{i-1}(\omega_r^2) T_{i-2}(\omega_r^2) \cdots T_2(\omega_r^2) T_{F0} \mathbf{v}_0, \quad r = 1, 2, \dots, n \quad (8.41)$$

where we let for simplicity $M_0 = 1$, so that $\mathbf{v}_0 = [0 \ 1]^T$. The eigenvector $\boldsymbol{\theta}_r$ has as its components the top component of \mathbf{v}_i^L ($i = 1, 2, \dots, n$) in the mode r . Note that the bottom component can be used to compute the vector \mathbf{M}_i^L , which represents the torque vector on the left side of every disk in the mode r . Also note that, by taking $M_0 = 1$ arbitrarily, the eigenvector $\boldsymbol{\theta}_r$ has been normalized in a certain sense. As soon as $\boldsymbol{\theta}_r$ is computed, it can be normalized according to any other scheme, if desired.

2. *Free-free shaft.* In this case the shaft is only positive semidefinite. Hence, to approximate the shaft by an n -degree-of-freedom system, we model the system by $n + 1$ stations $i = 0, 1, \dots, n$ and n in-between fields $i = 0, 1, \dots, n - 1$. Then, the relation between the station vectors to the left of station 0 and to the right of station n is simply

$$\mathbf{v}_n^R = T \mathbf{v}_0^L \quad (8.42)$$

where in this case the overall transfer matrix is

$$T = T_{Sn} \prod_{i=n-1}^0 T_i \quad (8.43)$$

Because now the boundary conditions are

$$M_0^L = 0, \quad M_n^R = 0 \quad (8.44a, b)$$

the frequency equation is

$$T_{21}(\omega^2) = 0 \quad (8.45)$$

where T_{21} is a polynomial of degree $n + 1$ in ω^2 . However, ω^2 can be factored out, so that T_{21} is the product of ω^2 and a polynomial of degree n in ω^2 . Hence, there is one natural frequency equal to zero. This is consistent with the fact that the system is only positive semidefinite, so that there is one rigid-body mode with zero frequency and n elastic modes.

As the number of degrees of freedom of the system increases, the task of deriving the characteristic polynomial and finding its roots becomes more and more tedious. This task can be avoided altogether by returning to the idea of Holzer's method. For example, in the case of the clamped-free shaft, we begin with the arbitrary station vector $\mathbf{v}_0 = [0 \ 1]^T$, choose some value for ω^2 and compute M_n^R by means of the recursive relations

$$\begin{aligned} \mathbf{v}_1^L &= T_{F0} \mathbf{v}_0 \\ \mathbf{v}_{i+1}^L &= T_i \mathbf{v}_i^L, \quad i = 1, 2, \dots, n - 1 \\ \mathbf{v}_n^R &= T_{Sn} \mathbf{v}_n^L \end{aligned} \quad (8.46)$$

which are clearly based on Eqs. (8.36). If we begin with a very low value for ω^2 , then the first value for which the bottom component of \mathbf{v}_n^R becomes zero is the lowest eigenvalue ω_1^2 . The procedure can be rendered more systematic by plotting the curve $M_n^R(\omega^2)$ versus ω^2 . Then, the eigenvalues $\omega_1^2, \omega_2^2, \dots, \omega_n^2$ are the points at which $M_n^R(\omega^2)$ intersects the axis ω^2 .

Although the method was developed in connection with the torsional vibration of shafts, the approach can be clearly applied to the axial vibration of rods and the transverse vibration of strings.

8.3 MYKLESTAD'S METHOD FOR BENDING VIBRATION

Myklestad's method (Ref. 36) for the bending vibration of beams represents an extension of the ideas introduced in Sec. 8.2 for the torsional vibration of shafts. Although the extension appears natural, it is not trivial, as witnessed by the fact that it took over 20 years to complete. The presentation in this section parallels closely the one in Sec. 8.2. Here, however, we begin directly with the lumped-parameter system, instead of beginning with a distributed-parameter system and using an incremental procedure to derive a lumped-parameter model.

The differential eigenvalue problem for shafts in torsion is of degree two. Consistent with this, the station vectors are two-dimensional, with the components being the angular displacement and torsional moment. In contrast, the differential eigenvalue problem for beams in bending is of degree four, so that the station vectors must be of order four. Extrapolating from second-order problems, it is possible to conclude that the components of the station vectors must be the displacement, slope, bending moment and shearing force.

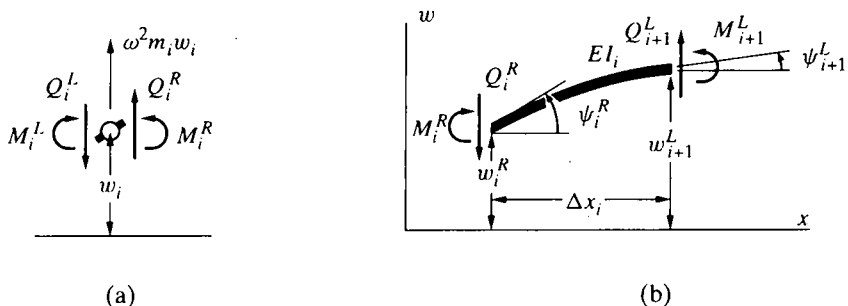


Figure 8.8 (a) Station i for a beam in bending (b) Field i for a beam in bending

By analogy with Sec. 8.2, we assume that a nonuniform Euler-Bernoulli beam is modeled as a set of lumped masses connected by massless uniform beams of length Δx_i . Free-body diagrams for a typical station i and field i are depicted in Figs. 8.8a and 8.8b, respectively. From Fig. 8.8a, due to continuity, we must have

$$w_i^R(t) = w_i^L(t) = w_i(t), \quad \psi_i^R(t) = \psi_i^L(t) = \psi_i(t) \quad (8.47a, b)$$

where ψ_i is the slope, i.e., the tangent to the deflection curve. Two other relations consist of the two equations of motion, one force and one moment equation. But, an Euler-Bernoulli beam implies that the rotatory inertia is negligibly small, so that the moment equation yields simply

$$M_i^R(t) = M_i^L(t) \quad (8.48)$$

On the other hand, the force equation is

$$Q_i^R(t) - Q_i^L(t) = m_i \ddot{w}_i(t) \quad (8.49)$$

Because beam segments possess flexibility, we can refer to Fig. 8.8b to obtain relations between translational and rotational displacements on the one hand and forces and moments on the other. To this end, it is convenient to regard station i as clamped and introduce several definitions of flexibility influence coefficients, as follows:

a_i^{wQ} is the translation at $i + 1$ due to a unit force at $i + 1$, $Q_{i+1}^L = 1$

a_i^{wM} is the translation at $i + 1$ due to a unit moment at $i + 1$, $M_{i+1}^L = 1$

$a_i^{\psi Q}$ is the rotation at $i + 1$ due to a unit force at $i + 1$, $Q_{i+1}^L = 1$

$a_i^{\psi M}$ is the rotation at $i + 1$ due to a unit moment at $i + 1$, $M_{i+1}^L = 1$.

Then, from Fig. 8.8b, we can write

$$w_{i+1}^L(t) = w_i^R(t) + \Delta x_i \psi_i^R(t) + a_i^{wM} M_{i+1}^L(t) + a_i^{wQ} Q_{i+1}^L(t) \quad (8.50a)$$

$$\psi_{i+1}^L(t) = \psi_i^R(t) + a_i^{\psi M} M_{i+1}^L(t) + a_i^{\psi Q} Q_{i+1}^L(t) \quad (8.50b)$$

Moreover, because beam segments are massless, we can write from Fig. 8.8b

$$M_{i+1}^L(t) = M_i^R(t) - \Delta x_i Q_i^R(t) \quad (8.51a)$$

$$Q_{i+1}^L(t) = Q_i^R(t) \quad (8.51b)$$

But, the right side of Eqs. (8.50) contains terms corresponding to both ends of the field. It is convenient, however, that all the terms on the right side of Eqs. (8.50) correspond to the right side only, so that we introduce Eqs. (8.51) into Eqs. (8.50) and obtain

$$w_{i+1}^L(t) = w_i^R(t) + \Delta x_i \psi_i^R(t) + a_i^{wM} M_i^R(t) + \left(a_i^{wQ} - \Delta x_i a_i^{wM} \right) Q_i^R(t) \quad (8.52a)$$

$$\psi_{i+1}^L(t) = \psi_i^R(t) + a_i^{\psi M} M_i^R(t) + \left(a_i^{\psi Q} - \Delta x_i a_i^{\psi M} \right) Q_i^R(t) \quad (8.52b)$$

At this point, following the pattern established in Sec. 8.2, we invoke the fact that free vibration is harmonic, eliminate the time dependence and express the equations in matrix form. In view of the fact that elimination of the time dependence is quite obvious, we proceed directly to the matrix formulation. To this end, we define the station vectors at i as $\mathbf{v}_i^R = [w_i^R \ \psi_i^R \ M_i^R \ Q_i^R]^T$ and $\mathbf{v}_i^L = [w_i^L \ \psi_i^L \ M_i^L \ Q_i^L]^T$,

where the various components represent constant amplitudes, so that the time-independent version of Eqs. (8.47)–(8.49) can be written as

$$\mathbf{w}_i^R = T_{Si} \mathbf{w}_i^L \quad (8.53)$$

where T_{Si} is a station transfer matrix carrying us from the left side to the right side of station i and having the form

$$T_{Si} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\omega^2 m_i & 0 & 0 & 1 \end{bmatrix} \quad (8.54)$$

In a similar fashion, Eqs. (8.51) and (8.52) can be used to write a matrix expression carrying us from the left end to the right end of field i . Before writing this expression, however, we recall from mechanics of materials that the various flexibility coefficients defined earlier have the values

$$\begin{aligned} a_i^{wQ} &= \frac{(\Delta x_i)^3}{3EI_i} = \frac{a_i (\Delta x_i)^2}{3}, & a_i^{wM} &= \frac{(\Delta x_i)^2}{2EI_i} = \frac{a_i \Delta x_i}{2} \\ a_i^{\psi Q} &= \frac{(\Delta x_i)^2}{2EI_i} = \frac{a_i \Delta x_i}{2}, & a_i^{\psi M} &= \frac{\Delta x_i}{EI_i} = a_i \end{aligned} \quad (8.55)$$

where I_i is the area moment of inertia of the beam for field i . Then, inserting Eqs. (8.55) into Eqs. (8.51) and (8.52), we can write the expression relating the station vector at the left end of field i to the one at the right end in the compact matrix form

$$\mathbf{v}_{i+1}^L = T_{Fi} \mathbf{v}_i^R \quad (8.56)$$

where

$$T_{Fi} = \begin{bmatrix} 1 & \Delta x_i & a_i \Delta x_i / 2 & -a_i (\Delta x_i)^2 / 6 \\ 0 & 1 & a_i & -a_i \Delta x_i / 2 \\ 0 & 0 & 1 & -\Delta x_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (8.57)$$

is the associated field transfer matrix.

Following the process established in Sec. 8.2, it is possible to derive an overall transfer matrix relating the station vector \mathbf{v}_n^R on the right of the beam to the station vector \mathbf{v}_0^L on the left. Before the overall transfer matrix can be derived, we must specify the boundary conditions. Because the process is the same as in Sec. 8.2, except that here the matrix T is 4×4 , the reader is referred to Sec. 8.2 for details.

As an illustration, we consider a *cantilever beam* in bending, in which case the relation between the station vectors \mathbf{v}_0 and \mathbf{v}_n^R is given by the last of Eqs. (8.36), with the overall matrix T being given by Eq. (8.37). To derive the frequency equation, we must invoke the boundary conditions. The beam is clamped at the left end, so that the boundary conditions there are

$$w_0 = 0, \quad \psi_0 = 0 \quad (8.58a)$$

On the other hand, the beam is free at the right end, so that the boundary conditions there are

$$M_n^R = 0, \quad Q_n^R = 0 \quad (8.58b)$$

Hence, the last of Eqs. (8.36) for the case at hand has the form

$$\begin{bmatrix} w_n^R \\ \psi_n^R \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} \\ T_{21} & T_{22} & T_{23} & T_{24} \\ T_{31} & T_{32} & T_{33} & T_{34} \\ T_{41} & T_{42} & T_{43} & T_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ M_0 \\ Q_0 \end{bmatrix} \quad (8.59)$$

The satisfaction of the two bottom equations requires that

$$\det \begin{bmatrix} T_{33}(\omega^2) & T_{34}(\omega^2) \\ T_{43}(\omega^2) & T_{44}(\omega^2) \end{bmatrix} = 0 \quad (8.60)$$

which is recognized as the *frequency equation*, an equation of degree n in ω^2 . Its solutions are the eigenvalues $\omega_1^2, \omega_2^2, \dots, \omega_n^2$, which can be obtained by one of the methods mentioned in Sec. 8.2. Then, the eigenvectors \mathbf{w}_r can be obtained by assuming arbitrarily that $\mathbf{v}_0 = [0 \ 0 \ M_0 \ Q_0]^T = [0 \ 0 \ 1 \ 1]^T$, using Eqs. (8.41) and retaining the top component w_i of the vectors \mathbf{v}_i^L ($i = 1, 2, \dots, n$); these values of w_i represent the n components of the eigenvector \mathbf{w}_r . It should be pointed out that the second component ψ_i of the vector \mathbf{v}_i^L represents the slope at station i . Although it may seem that a plot of the eigenvector \mathbf{w}_r using both displacements and slopes is likely to be more accurate, for sufficiently large n , displacements alone should suffice.

Myklestad (Ref. 36) suggested a solution of the problem in tabular form, based on some scalar recursive formulas. Thomson (Ref. 45) was the first to set up the problem in matrix form using transfer matrices, without introducing the concept of station and field transfer matrices. The formulation presented here is closer to the treatment of Pestel and Leckie (Ref. 37), who applied the concept of transfer matrices to a large number of problems, including branched torsional systems and framed structures.

8.4 RAYLEIGH'S ENERGY METHOD

The lumped-parameter methods for approximating distributed systems presented in Secs. 8.1–8.3 are all characterized by the fact that the distributed mass is concentrated at given points. On the other hand, the methods differ in the treatment of the stiffness. Indeed, in the lumped-parameter method using influence coefficients the stiffness involves no approximation. In this regard, it should be noted that the use of flexibility influence coefficients instead of the flexibility influence function does not mean that the stiffness has been lumped. It simply means that the flexibility coefficients are merely obtained by evaluating the flexibility influence function at points coinciding with the nominal position of the lumped masses. By contrast, in Holzer's method and in Myklestad's method the stiffness is approximated by regarding it as uniform over the field between any two lumped masses. Satisfaction of the boundary conditions is not a major concern in lumped-parameter methods, as they are taken into account

in the lumping process. Although convergence of the approximate eigenvalues to the actual ones can be assumed, no mathematical proof exists. Perhaps even more disquieting is that there are no clues as to the nature of convergence.

Beginning in this section, we turn our attention to an entirely different, and considerably more satisfying approach to the spatial discretization of distributed-parameter systems than the lumped-parameter approach. Indeed, this new approach addresses virtually all the concerns expressed in the preceding paragraph. The approach has a solid mathematical foundation permitting a clear statement concerning the nature of convergence, as well as the formulation of stability criteria. Moreover, the parameter discretization process is consistent, in the sense that the mass and stiffness distributions are treated in the same manner. The approach is based on Rayleigh's principle (Secs. 5.2 and 7.14).

To introduce the ideas, we consider first the free vibration of a conservative discrete system and write the kinetic energy and potential energy in the matrix form

$$T(t) = \frac{1}{2} \dot{\mathbf{q}}^T(t) M \dot{\mathbf{q}}(t), \quad V(t) = \frac{1}{2} \mathbf{q}^T(t) K \mathbf{q}(t) \quad (8.61a, b)$$

where $\mathbf{q}(t) = [q_1(t) \ q_2(t) \ \dots \ q_n(t)]^T$ is the displacement vector, M the mass matrix and K the stiffness matrix. Both M and K are real symmetric and positive definite. As shown in Sec. 4.6, the free vibration of positive definite conservative systems is harmonic, so that the vector $\mathbf{q}(t)$ can be expressed as

$$\mathbf{q}(t) = \mathbf{u} \cos(\omega t - \phi) \quad (8.62)$$

where \mathbf{u} is a constant vector, ω the frequency of oscillation and ϕ a phase angle. It follows that Eqs. (8.61) can be rewritten as

$$T(t) = \frac{1}{2} \omega^2 \mathbf{u}^T M \mathbf{u} \sin^2(\omega t - \phi), \quad V(t) = \frac{1}{2} \mathbf{u}^T K \mathbf{u} \cos^2(\omega t - \phi) \quad (8.63a, b)$$

When $\sin(\omega t - \phi) = \pm 1$, $\cos(\omega t - \phi) = 0$, the kinetic energy reaches the maximum value

$$T_{\max} = \frac{1}{2} \omega^2 \mathbf{u}^T M \mathbf{u} \quad (8.64a)$$

and the potential energy is zero. On the other hand, when $\cos(\omega t - \phi) = \pm 1$, $\sin(\omega t - \phi) = 0$, the potential energy reaches the maximum value

$$V_{\max} = \frac{1}{2} \mathbf{u}^T K \mathbf{u} \quad (8.64b)$$

and the kinetic energy is zero. But, according to the conservation of energy principle (Sec. 2.5), we can write

$$E = T + V = T_{\max} + 0 = 0 + V_{\max} \quad (8.65)$$

where E is the total energy. Hence, inserting Eqs. (8.64) into Eq. (8.65) and introducing the *reference kinetic energy*

$$T_{\text{ref}} = \frac{1}{2} \mathbf{u}^T M \mathbf{u} \quad (8.66)$$

we obtain

$$\omega^2 = R(\mathbf{u}) = \frac{\mathbf{u}^T \mathbf{K} \mathbf{u}}{\mathbf{u}^T \mathbf{M} \mathbf{u}} = \frac{V_{\max}}{T_{\text{ref}}} \quad (8.67)$$

which is recognized as Rayleigh's quotient for discrete systems, first encountered in Sec. 5.2. In this regard, it should be noted that Rayleigh's quotient is expressed here for the first time as the ratio of one term proportional to the potential energy and another term proportional to the kinetic energy.

As demonstrated in Sec. 5.2, Rayleigh's quotient possesses the stationarity property, which can be stated in physical terms as follows: *The frequency of vibration of a conservative system oscillating about an equilibrium position has a stationary value in the neighborhood of a natural mode* (Ref. 39). This statement is known as *Rayleigh's principle*. As a special case, but by far the most important one, it can be stated: *The frequency of vibration of a conservative system has a minimum value in the neighborhood of the fundamental mode, or*

$$\omega_1^2 = \min R(\mathbf{u}) = \min \frac{\mathbf{u}^T \mathbf{K} \mathbf{u}}{\mathbf{u}^T \mathbf{M} \mathbf{u}} = \min \frac{V_{\max}}{T_{\text{ref}}} \quad (8.68)$$

where ω_1 is the lowest natural frequency. This statement alone is at times referred to as *Rayleigh's principle* (Ref. 13).

In many cases of practical interest, it is necessary to estimate the lowest natural frequency of a structure. In view of the stationarity property, Rayleigh's principle, Eq. (8.68), is ideally suited for the task. Indeed, if a trial vector \mathbf{u} differing from the fundamental mode \mathbf{u}_1 by a small quantity of order ϵ can be found, then Eq. (8.68) can be used to produce an estimate ω^2 differing from ω_1^2 by a small quantity of order ϵ^2 . In this regard, it should be noted that if $\omega^2 = (1 + \epsilon^2)\omega_1^2$, then $\omega \cong (1 + \epsilon^2/2)\omega_1$, where the binomial approximation $(1 + \epsilon^2)^{1/2} \cong 1 + \epsilon^2/2$ has been used. This procedure for estimating the fundamental frequency is known as *Rayleigh's energy method*. Clearly, Rayleigh's energy method is applicable to discrete models of distributed-parameter systems, such as models derived by the lumped-parameter method using flexibility influence coefficients. We should note, however, that Rayleigh's quotient, as given by Eq. (8.68), involves the stiffness matrix, and it was pointed out in Sec. 8.1 that the evaluation of stiffness coefficients for distributed systems is not practical. This slight inconvenience can be overcome by recognizing that the force vector \mathbf{f} is related to the displacement vector \mathbf{u} by

$$\mathbf{f} = \mathbf{K} \mathbf{u}, \quad \mathbf{u} = \mathbf{K}^{-1} \mathbf{f} = \mathbf{A} \mathbf{f} \quad (8.69a, b)$$

where \mathbf{A} is the flexibility matrix. Inserting Eqs. (8.69) into Eq. (8.68), we can rewrite Rayleigh's principle in the form

$$\omega_1^2 = \min \frac{\mathbf{f}^T \mathbf{A} \mathbf{f}}{\mathbf{u}^T \mathbf{M} \mathbf{u}} \quad (8.70)$$

The question remains as to how to obtain a vector \mathbf{u} resembling the fundamental mode \mathbf{u}_1 , as well as the associated force vector \mathbf{f} . Quite often a good choice for \mathbf{u} is the static displacement vector due to loads proportional to the system lumped masses, which implies that the force vector \mathbf{f} is proportional to the vector $[m_1 \ m_2 \ \dots \ m_n]^T$.

Then, the static displacement vector \mathbf{u} can be obtained by simply inserting \mathbf{f} into Eq. (8.69b).

As indicated in the beginning of this section, our objective is to develop a method for the spatial discretization of distributed-parameter systems not involving parameter lumping. In this regard, it must be pointed out that, although we demonstrated Rayleigh's energy method on the basis of a discrete system, the method is equally applicable to distributed-parameter systems. By analogy with Eq. (8.68), Rayleigh's principle for distributed-parameter systems, Eq. (7.389), can be written in the form

$$\omega_1^2 = \min R(w) = \min \frac{V_{\max}}{T_{\text{ref}}} = \min \frac{[w, w]}{(\sqrt{mw}, \sqrt{mw})} \quad (8.71)$$

where $[w, w]$ is an energy inner product (Sec. 7.5) and (\sqrt{mw}, \sqrt{mw}) is a weighted inner product (Sec. 7.14). Here, however, the question of choosing a trial function w is more involved than in the discrete case. Of course, a function w resembling closely the lowest mode of vibration w_1 is always a good choice. Quite often, the static deflection curve due to a distributed load proportional to the mass density is likely to yield excellent estimates of w_1 . Unfortunately, for complex mass and stiffness distributions the task of obtaining the static deflection curve is not trivial. Perhaps a good approach is to use as a trial function the first eigenfunction of a related but simpler system, such as one with uniform mass and stiffness distributions.

Example 8.2

Use Rayleigh's energy method to estimate the fundamental frequency of the tapered clamped-free rod in axial vibration shown in Fig. 8.9. The mass per unit length is given by

$$m(x) = 2m\left(1 - \frac{x}{L}\right) \quad (a)$$

and the stiffness distribution is

$$EA(x) = 2EA\left(1 - \frac{x}{L}\right) \quad (b)$$

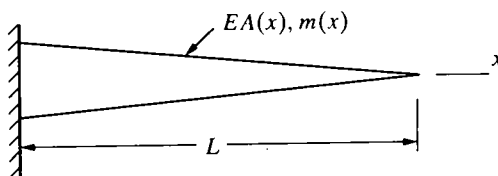


Figure 8.9 Tapered clamped-free rod in axial vibration

Using the analogy with the string in transverse vibration of Example 7.3, Rayleigh's quotient, Eq. (7.399), has the explicit expression

$$\omega^2 = R(U) = \frac{[U, U]}{(\sqrt{m}U, \sqrt{m}U)} = \frac{\int_0^L EA(x) [dU(x)/dx]^2 dx}{\int_0^L m(x) U^2(x) dx} \quad (c)$$

As a trial function, we use the fundamental mode of a clamped-free uniform rod in axial vibration, which can be shown to be

$$U(x) = \sin \frac{\pi x}{2L} \quad (d)$$

It is clear that this trial function represents an admissible function for the problem at hand. Inserting Eq. (d) into the numerator and denominator of Rayleigh's quotient, we obtain

$$\begin{aligned} \int_0^L EA(x) \left[\frac{dU(x)}{dx} \right]^2 dx &= 2EA \left(\frac{\pi}{2L} \right)^2 \int_0^L \left(1 - \frac{x}{L} \right) \cos^2 \frac{\pi x}{2L} dx \\ &= \left(1 + \frac{\pi^2}{4} \right) \frac{EA}{2L} \end{aligned} \quad (e)$$

and

$$\int_0^L m(x)U^2(x) dx = 2m \int_0^L \left(1 - \frac{x}{L} \right) \sin^2 \frac{\pi x}{2L} dx = \left(1 - \frac{4}{\pi^2} \right) \frac{mL}{2} \quad (f)$$

respectively. Hence, introducing Eqs. (e) and (f) in Eq. (c), we have

$$\omega^2 = \frac{(1 + \pi^2/4) EA}{(1 - 4/\pi^2) mL^2} = 5.8304 \frac{EA}{mL^2} \quad (g)$$

from which we obtain the estimated fundamental frequency

$$\omega = 2.4146 \sqrt{\frac{EA}{mL^2}} \quad (h)$$

As it turns out, the eigenvalue problem for the system under consideration can be solved in closed form (see Problem 7.28). The actual fundamental frequency has the value

$$\omega_1 = 2.4048 \sqrt{\frac{EA}{mL^2}} \quad (i)$$

from which we conclude that Rayleigh's energy method yields an estimate about 0.4% higher than the actual fundamental frequency. This is a remarkable result, which can be attributed to the fact that the chosen trial function resembles the actual fundamental mode very closely.

8.5 THE RAYLEIGH-RITZ METHOD

According to Rayleigh's principle (Sec. 7.14), for a self-adjoint distributed-parameter system Rayleigh's quotient $R(w)$ has stationary values at the system eigenfunctions. Most importantly, the stationary value at the lowest eigenfunction w_1 is a minimum equal to the lowest eigenvalue λ_1 , or

$$\lambda_1 = \min_w R(w) \quad (8.72)$$

where w is a trial function from the space \mathcal{K}_B^{2p} of comparison functions or from the space \mathcal{K}_G^p of admissible functions (Sec. 7.5), depending on the particular form of Rayleigh's quotient. This extremal property is very useful in estimating the lowest eigenvalue in cases in which no closed-form solution of the differential eigenvalue

problem is possible. Indeed, Rayleigh's energy method (Sec. 8.4) consists of using Rayleigh's quotient in the form $R(w) = V_{\max}/T_{\text{ref}}$ in conjunction with an admissible function w differing from w_1 by a small quantity of order ϵ to obtain an estimate λ differing from λ_1 by a small quantity of order ϵ^2 .

The extremal characterization can be extended to higher eigenvalues by restricting w to the space orthogonal to the lowest s eigenfunctions w_i ($i = 1, 2, \dots, s$) and writing (Sec. 7.14)

$$\lambda_{s+1} = \min_w R(w), \quad (w, w_i) = 0, \quad i = 1, 2, \dots, s \quad (8.73)$$

Of course, if the objective is to obtain estimates of the eigenvalues $\lambda_2, \lambda_3, \dots, \lambda_{s+1}$, then this characterization has no practical value, as the eigenfunctions w_1, w_2, \dots, w_s are generally not available.

A characterization independent of the lower eigenfunctions is provided by the Courant and Fischer maximin theorem (Sec. 7.14) in the form

$$\lambda_{s+1} = \max_{v_i} \min_w R(w), \quad (w, v_i) = 0, \quad i = 1, 2, \dots, s \quad (8.74)$$

where v_i ($i = 1, 2, \dots, s$) are s independent, but otherwise arbitrary functions. Whereas the maximin theorem by itself does not represent a computational tool, it has significant implications in numerical solutions of the eigenvalue problem for distributed systems.

The Rayleigh-Ritz method is a technique for the computation of approximate solutions of the eigenvalue problem for self-adjoint distributed-parameter systems. It consists of replacing the eigenvalue problem for distributed systems by a sequence of algebraic eigenvalue problems. To introduce the ideas, it is convenient to specify the form of Rayleigh's quotient. Hence, from Eq. (7.390), we write

$$\lambda = R(w) = \frac{\int_D w L w dD}{\int_D m w^2 dD} \quad (8.75)$$

where L is a self-adjoint differential operator of order $2p$, so that the trial function w must be from the space \mathcal{K}_B^{2p} . Next, we select a set of comparison functions $\phi_1(P), \phi_2(P), \dots, \phi_n(P), \dots$ satisfying the two conditions: (i) any n members $\phi_1, \phi_2, \dots, \phi_n$ are linearly independent and (ii) the set of functions $\phi_1, \phi_2, \dots, \phi_n, \dots$ is complete (Sec. 7.5), where P denotes a nominal point in the domain D . Then, we determine $\min R(w)$ not from the entire space \mathcal{K}_B^{2p} but for functions of the form

$$w^{(n)}(P) = a_1 \phi_1(P) + a_2 \phi_2(P) + \dots + a_n \phi_n(P) = \sum_{i=1}^n a_i \phi_i(P) \quad (8.76)$$

The functions $\phi_i(P)$ are referred to as *coordinate functions* and they span a function space \mathcal{R}_n , referred to as a *Ritz space*. In fact, there is a sequence of Ritz spaces, $\mathcal{R}_1, \mathcal{R}_2, \dots, \mathcal{R}_n$, each being a subspace of the next and with \mathcal{R}_n being a subspace of \mathcal{K}_B^{2p} . For functions $w^{(n)}$ in \mathcal{R}_n , the coefficients a_i ($i = 1, 2, \dots, n$) are constants yet to be determined. This amounts to approximating the variational problem for $R(w)$

by a sequence of variational problems for $R(w^{(n)})$ corresponding to $n = 1, 2, \dots$, or

$$\lambda^{(n)} = R(w^{(n)}) = \frac{(w^{(n)}, Lw^{(n)})}{(\sqrt{m}w^{(n)}, \sqrt{m}w^{(n)})} \quad (8.77)$$

Of course, the case $n = 1$ merely represents Rayleigh's energy method, so that the variational approach applies to the cases in which $n \geq 2$. But, because the functions $\phi_1, \phi_2, \dots, \phi_n$ are known, after carrying out the indicated integrations, Rayleigh's quotient reduces to a function of the undetermined coefficients, or

$$\lambda^{(n)}(a_1, a_2, \dots, a_n) = R(a_1, a_2, \dots, a_n) = \frac{\sum_{i=1}^n \sum_{j=1}^n k_{ij} a_i a_j}{\sum_{i=1}^n \sum_{j=1}^n m_{ij} a_i a_j} \quad (8.78)$$

where, because L is self-adjoint,

$$k_{ij} = k_{ji} = (\phi_i, L\phi_j) = \int_D \phi_i L\phi_j dD, \quad i, j = 1, 2, \dots, n \quad (8.79a)$$

$$m_{ij} = m_{ji} = (\sqrt{m}\phi_i, \sqrt{m}\phi_j) = \int_D m\phi_i \phi_j dD, \quad i, j = 1, 2, \dots, n \quad (8.79b)$$

are symmetric *stiffness* and *mass coefficients*, respectively.

The condition for the stationarity of Rayleigh's quotient is simply

$$\delta\lambda^{(n)} = \delta R = \sum_{r=1}^n \frac{\partial R}{\partial a_r} \delta a_r = 0 \quad (8.80)$$

Observing that the coefficients a_1, a_2, \dots, a_n are independent, we conclude that Eq. (8.80) is satisfied if and only if the following conditions are satisfied:

$$\frac{\partial R}{\partial a_r} = 0, \quad r = 1, 2, \dots, n \quad (8.81)$$

Equations (8.81) involve the term

$$\begin{aligned} \frac{\partial}{\partial a_r} \sum_{i=1}^n \sum_{j=1}^n k_{ij} a_i a_j &= \sum_{i=1}^n \sum_{j=1}^n k_{ij} \left(\frac{\partial a_i}{\partial a_r} a_j + a_i \frac{\partial a_j}{\partial a_r} \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n k_{ij} (\delta_{ir} a_j + a_i \delta_{jr}) \\ &= \sum_{j=1}^n k_{rj} a_j + \sum_{i=1}^n k_{ir} a_i = 2 \sum_{j=1}^n k_{rj} a_j, \quad r = 1, 2, \dots, n \end{aligned} \quad (8.82a)$$

where we considered the symmetry of the stiffness coefficients, as well as the fact that i and j are dummy indices. In the same fashion, we obtain

$$\frac{\partial}{\partial a_r} \sum_{i=1}^n \sum_{j=1}^n m_{ij} a_i a_j = 2 \sum_{j=1}^n m_{rj} a_j, \quad r = 1, 2, \dots, n \quad (8.82b)$$

Hence, conditions (8.81) in conjunction with Eqs. (8.78) and (8.82) reduce to

$$\frac{\left(\frac{\partial}{\partial a_r} \sum_{i=1}^n \sum_{j=1}^n k_{ij} a_i a_j \right) \left(\sum_{i=1}^n \sum_{j=1}^n m_{ij} a_i a_j \right) - \left(\sum_{i=1}^n \sum_{j=1}^n k_{ij} a_i a_j \right) \left(\frac{\partial}{\partial a_r} \sum_{i=1}^n \sum_{j=1}^n m_{ij} a_i a_j \right)}{\left(\sum_{i=1}^n \sum_{j=1}^n m_{ij} a_i a_j \right)^2} = \frac{2}{\sum_{i=1}^n \sum_{j=1}^n m_{ij} a_i a_j} \left(\sum_{j=1}^n k_{rj} a_j - \lambda^{(n)} \sum_{j=1}^n m_{rj} a_j \right) = 0, \quad r = 1, 2, \dots, n \quad (8.83)$$

which can be satisfied provided

$$\sum_{j=1}^n k_{rj} a_j = \lambda^{(n)} \sum_{j=1}^n m_{rj} a_j, \quad r = 1, 2, \dots, n \quad (8.84)$$

For $n = 1$, we obtain

$$\lambda_1^{(1)} = k_{11}/m_{11} \quad (8.85)$$

directly. On the other hand, letting $n = 2, 3, \dots$, we obtain a sequence of algebraic eigenvalue problems of order n .

Before we proceed with a discussion of the eigenvalue problem, Eqs. (8.84), it will prove convenient to reformulate the problem in matrix form. To this end, we rewrite Eq. (8.76) as

$$w^{(n)}(P) = \Phi^T(P) \mathbf{a} \quad (8.86)$$

where $\Phi = [\phi_1 \phi_2 \dots \phi_n]^T$ is an n -vector with components depending on the spatial position P and $\mathbf{a} = [a_1 a_2 \dots a_n]^T$ is a constant n -vector. Then, the sequence of algebraic eigenvalue problems, Eqs. (8.84), takes the form

$$K^{(n)} \mathbf{a} = \lambda^{(n)} M^{(n)} \mathbf{a} \quad (8.87)$$

in which

$$K^{(n)} = K^{(n)T} = \int_D \Phi L \Phi^T dD, \quad M^{(n)} = M^{(n)T} = \int_D m \Phi \Phi^T dD \quad (8.88a, b)$$

are $n \times n$ symmetric *stiffness* and *mass matrices*. Each eigenvalue problem in the sequence represented by Eq. (8.87) is entirely analogous to that of a conservative n -degree-of-freedom discrete system, Eq. (4.81). Hence, *the Rayleigh-Ritz method*

is a discretization technique replacing a differential eigenvalue problem by a sequence of algebraic eigenvalue problems of increasing order.

At this point, we consider Eq. (7.399) and write the second form of Rayleigh's quotient as

$$\lambda = R(w) = \frac{[w, w]}{(\sqrt{m}w, \sqrt{m}w)} \quad (8.89)$$

where $[w, w]$ is an energy inner product (Sec. 7.5) and $(\sqrt{m}w, \sqrt{m}w)$ is a weighted inner product (Sec. 7.14), and we note that this form of Rayleigh's quotient was used in Eq. (8.71) in conjunction with Rayleigh's energy method. Then, following the established pattern, we obtain the same eigenvalue problem as that given by Eq. (8.87), except that the stiffness matrix is given by

$$K^{(n)} = [\Phi, \Phi^T] \quad (8.90)$$

The mass matrix remains in the form of Eq. (8.88b). There is another difference, however. As discussed in Sec. 7.5, the formulation of the eigenvalue problem based on Eq. (8.89) requires that the trial functions $\phi_1, \phi_2, \dots, \phi_n$ be only from the space \mathcal{K}_G^p of admissible functions.

The solution of the algebraic eigenvalue problem, Eq. (8.87) consists of n eigenvalues $\lambda_r^{(n)}$ and eigenvectors \mathbf{a}_r ($r = 1, 2, \dots, n$). The eigenvalues $\lambda_r^{(n)}$ provide approximations to the actual eigenvalues λ_r ($r = 1, 2, \dots, n$). On the other hand, the eigenvectors \mathbf{a}_r can be inserted into Eq. (8.86) to obtain the estimates

$$w_r^{(n)}(P) = \Phi^T(P)\mathbf{a}_r, \quad r = 1, 2, \dots, n \quad (8.91)$$

of the eigenfunctions w_r . We refer to $\lambda_r^{(n)}$ as *Ritz eigenvalues* and to $w_r^{(n)}$ as *Ritz eigenfunctions*.

A question of particular interest is how the Ritz eigenvalues and eigenfunctions relate to the actual eigenvalues and eigenfunctions. In earlier discussions, it was implied that the Ritz eigenvalues are ordered so as to satisfy $\lambda_1^{(n)} \leq \lambda_2^{(n)} \leq \dots \leq \lambda_n^{(n)}$, while the actual eigenvalues satisfy $\lambda_1 \leq \lambda_2 \leq \dots$. Because the coordinate functions $\phi_1, \phi_2, \dots, \phi_n$ are from a complete set, it can be assumed that the solution to the differential eigenvalue problem can be obtained by letting $n \rightarrow \infty$. For finite n , the approximate solution $w^{(n)}$ lies in the Ritz space \mathcal{R}_n , which can be interpreted as stating that the solution is subject to the constraints

$$a_{n+1} = a_{n+2} = \dots = 0 \quad (8.92)$$

But, according to Rayleigh's principle (Sec. 7.14), the lowest eigenvalue λ_1 is the minimum value Rayleigh's quotient can take as w varies over the space \mathcal{K}_B^{2p} . On the other hand, $\lambda_1^{(n)}$ is the minimum value Rayleigh's quotient can take for functions confined to the Ritz space \mathcal{R}_n . It follows that

$$\lambda_1 \leq \lambda_1^{(n)} \quad (8.93)$$

To examine how the higher Ritz eigenvalues relate to the actual eigenvalues, we invoke the maximin theorem (Sec. 7.14). If we impose on the solution w of the

actual system the requirement that it be orthogonal to the function v_1 , then from Eq. (7.404) with $s = 1$ we can write

$$\lambda_2 = \max_{v_1} \min_w R(w), \quad (w, v_1) = 0 \quad (8.94)$$

On the other hand, by imposing the same constraint on the Ritz system, we have

$$\lambda_2^{(n)} = \max_{v_1} \min_w R(w), \quad (w, v_1) = 0, \quad (w, \phi_j) = 0, \quad j = n + 1, n + 2, \dots \quad (8.95)$$

Because the space of constraint \mathcal{R}_{n-1} for calculating $\lambda_2^{(n)}$ is only a small subspace of the still infinite-dimensional space of constraint for calculating λ_2 , we can write

$$\lambda_2 \leq \lambda_2^{(n)} \quad (8.96)$$

Inequality (8.96) can be generalized by writing

$$\lambda_r \leq \lambda_r^{(n)}, \quad r = 1, 2, \dots, n \quad (8.97)$$

Hence, the Ritz eigenvalues represent upper bounds for the actual eigenvalues.

Next, we address the question as to how the Ritz eigenvalues behave as the order n of the discrete model increases. To answer this question, we add one more term to series (8.76), so that Eq. (8.86) must be replaced by

$$w^{(n+1)}(P) = \Phi^T(P)\mathbf{a} \quad (8.98)$$

where now Φ and \mathbf{a} are $(n + 1)$ -vectors. Consistent with this, Eq. (8.87) must be replaced by

$$K^{(n+1)}\mathbf{a} = \lambda^{(n+1)}M^{(n+1)}\mathbf{a} \quad (8.99)$$

and we observe that $K^{(n+1)}$ and $M^{(n+1)}$ are obtained through the addition of one row and one column to $K^{(n)}$ and $M^{(n)}$ without disturbing the elements of the latter two matrices. Hence, $K^{(n)}$ and $K^{(n+1)}$ on the one hand and $M^{(n)}$ and $M^{(n+1)}$ on the other hand possess the *embedding property*, or

$$K^{(n+1)} = \left[\begin{array}{c|c} K^{(n)} & \mathbf{k} \\ \hline \mathbf{k}^T & k \end{array} \right], \quad M^{(n+1)} = \left[\begin{array}{c|c} M^{(n)} & \mathbf{m} \\ \hline \mathbf{m}^T & m \end{array} \right] \quad (8.100)$$

in which $\mathbf{k} = [k_{n+1,1} \ k_{n+1,2} \ \dots \ k_{n+1,n}]^T$ and $\mathbf{m} = [m_{n+1,1} \ m_{n+1,2} \ \dots \ m_{n+1,n}]^T$ are n -vectors and $k = k_{n+1,n+1}$ and $m = m_{n+1,n+1}$ are scalars. It follows from Sec. 5.4 that the two sets of eigenvalues corresponding to the two eigenvalue problems, Eqs. (8.87) and (8.99), satisfy the *separation theorem*, Eq. (5.73), which in the case at hand can be expressed as

$$\lambda_1^{(n+1)} \leq \lambda_1^{(n)} \leq \lambda_2^{(n+1)} \leq \lambda_2^{(n)} \leq \lambda_3^{(n+1)} \leq \dots \leq \lambda_n^{(n+1)} \leq \lambda_n^{(n)} \leq \lambda_{n+1}^{(n+1)} \quad (8.101)$$

We observe that, by increasing the order of the eigenvalue problem from n to $n + 1$, the n lowest newly computed eigenvalues decrease relative to the corresponding n previously computed eigenvalues, or at least they do not increase. At the same time, one more approximate eigenvalue at the higher end of the spectrum is obtained. But, inequalities (8.97) state that the approximate eigenvalues are higher than (or equal

to) the corresponding actual eigenvalues. In view of the fact that the admissible functions $\phi_1, \phi_2, \dots, \phi_n, \dots$ are from a complete set, it follows from the preceding statements that, as $n \rightarrow \infty$, the Ritz eigenvalues converge to the actual eigenvalues monotonically from above. Hence, we can write

$$\lim_{n \rightarrow \infty} \lambda_r^{(n)} = \lambda_r, \quad r = 1, 2, \dots, n \quad (8.102)$$

The convergence process can be illustrated by means of the triangular array

$$\begin{array}{ccccccccccc} \lambda_1^{(1)} & \geq & \lambda_1^{(2)} & \geq & \lambda_1^{(3)} & \geq & \dots & \geq & \lambda_1^{(n)} & \geq & \lambda_1^{(n+1)} & \geq & \lambda_1 \\ & & \lambda_2^{(2)} & \geq & \lambda_2^{(3)} & \geq & \dots & \geq & \lambda_2^{(n)} & \geq & \lambda_2^{(n+1)} & \geq & \lambda_2 \\ & & & & \lambda_3^{(3)} & \geq & \dots & \geq & \lambda_3^{(n)} & \geq & \lambda_3^{(n+1)} & \geq & \lambda_3 \\ & & & & & & \dots & & & & & & \\ & & & & & & & & \lambda_n^{(n)} & \geq & \lambda_n^{(n+1)} & \geq & \lambda_n \\ & & & & & & & & & & & & \vdots \\ & & & & & & & & & & & & \infty \end{array} \quad (8.103)$$

or more pictorially by the diagram of Fig. 8.10. Because each step of the process brings about a reduction in the computed eigenvalues, or at least not an increase, the sequence of approximate solutions $w^{(1)}, w^{(2)}, \dots, w^{(n)}$ is referred to as a *minimizing sequence*.

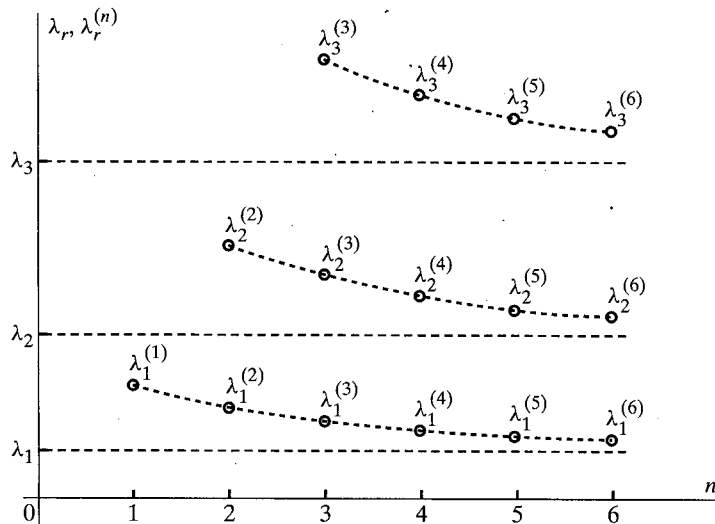


Figure 8.10 The three lowest computed eigenvalues versus the number of terms in the approximating series

The fact that the approximate eigenvalues computed by the Rayleigh-Ritz method converge to the actual eigenvalues as $n \rightarrow \infty$, provided the coordinate functions $\phi_1, \phi_2, \dots, \phi_n, \dots$ are from a complete set, is very reassuring and from a mathematical point of view perhaps sufficient. From a computational point of view,

however, n cannot approach infinity. In fact, in computations the objective is to achieve convergence with as few terms as possible, so that the question of convergence rate is important. Before addressing this question, it is appropriate to mention a paradox characterizing not only the Rayleigh-Ritz method but all methods approximating distributed-parameter systems by discrete ones. It is no coincidence that we choose to discuss the paradox here, because the Rayleigh-Ritz method has the best convergence characteristics of all spatial discretization techniques. The paradox is that *no discrete model of a distributed system is able to yield a full set of accurate approximate eigenvalues*. As a rule of thumb, the lower computed eigenvalues tend to be more accurate than the higher eigenvalues and reach convergence first, as can be verified by means of Fig. 8.10. Increasing the order of the model will not change the state of affairs, because as the order is increased more eigenvalues are added, and these higher eigenvalues can have significant error, as shown in Fig. 8.10. In fact, at times the error in the higher computed eigenvalues can be so large as to render them meaningless. The situation is not as critical as it may appear, however, because in most practical cases the interest lies only in a given number of lower eigenvalues, as higher modes are difficult to excite and tend not to participate in the motion. To ensure that the lower eigenvalues of interest are accurate, the order of the discrete model must be at times more than twice as large as the number of these lower eigenvalues.

The rate of convergence of the Rayleigh-Ritz process depends on the quality of the trial functions $\phi_1, \phi_2, \dots, \phi_n$, and in particular how well linear combinations of these functions can approximate the actual eigenfunctions. Of course, the actual eigenfunctions are not known a priori, so that an assessment of the choice of trial functions can be made only after an examination of the numerical results. Still, some guidelines for the selection of the trial functions can be formulated.

The issue of selecting the trial functions $\phi_1, \phi_2, \dots, \phi_n$ is broadly connected with the particular form of Rayleigh's quotient used. As indicated earlier in this section, *in using Rayleigh's quotient in the form given by Eq. (8.75), the trial functions must be comparison functions*, i.e., they must be from the space \mathcal{K}_B^{2p} . By definition, comparison functions must be $2p$ -times differentiable and satisfy all the boundary conditions of the problem. In problems involving natural boundary conditions, which tend to be more complicated than geometric boundary conditions (Sec. 7.1), comparison functions may not be readily available. In some cases, they can be generated by solving a related but simpler eigenvalue problem. As an example, we consider a nonuniform string in transverse vibration with one end fixed and with the other end attached to a spring, as shown in Fig. 7.1a, so that one boundary condition is geometric and the other is natural. A suitable set of comparison functions for this system consists of the eigenfunctions of a uniform string with the same boundary conditions. Another example is the rotating cantilever beam of Fig. 7.4a. The beam has two geometric boundary conditions at $x = 0$ and two natural boundary conditions at $x = L$. Even when the bending stiffness and the mass are distributed uniformly, the differential equation of motion contains terms depending on the spatial position, and no closed-form solution is possible. In this case, the eigenfunctions of a uniform nonrotating cantilever beam represent a suitable set of comparison functions. Clearly, eigenfunctions are independent and form a complete set by definition, albeit for a

somewhat different system. In the first example it is relatively easy to generate a set of comparison functions and in the second example they are readily available; this is not the case in general. Indeed, there are many cases in which the natural boundary conditions cannot be satisfied exactly, particularly for two-dimensional systems. For this reason, *it is more common to base the Rayleigh-Ritz discretization process on the energy form of Rayleigh's quotient given by Eq. (8.89), in which case, the trial functions $\phi_1, \phi_2, \dots, \phi_n$ need be only from the space of admissible functions, i.e., from the space \mathcal{K}_G^p .* We recall that functions from this space must be only p -times differentiable and satisfy only the geometric boundary conditions, a significantly larger space than the space of comparison functions. Independence and completeness are required of all sets of coordinate functions, so that they are assumed throughout. Differentiability is seldom a problem, so that admissible functions are only required to satisfy the geometric boundary conditions, which for the most part are very easy to satisfy. This opens the choice to a large number of sets of functions known to be independent, such as power series, trigonometric functions, Bessel functions, Legendre polynomials, Tchebycheff polynomials, etc. In fact, most of these sets are orthogonal in some sense, but this only guarantees independence, as they are not likely to be orthogonal with respect to the mass density of the particular system under consideration. Of course, independent functions can always be rendered orthogonal with respect to any mass distribution by the Gram-Schmidt orthogonalization process, which is akin to the process for discrete systems discussed in Appendix B, but the benefits of working with a diagonal mass matrix may pale compared with the effort required by the orthogonalization process. This is particularly true when the interest lies in shifting routine work from the analyst to the computer. Completeness in energy of a given set of functions is in general more difficult to ascertain, but it can be assumed for the most part. Of course, it is always possible to use comparison functions, as they are admissible by definition. In this case, the two forms of Rayleigh's quotient, Eqs. (8.75) and (8.89), yield exactly the same results, because the two forms can be derived from one another with due consideration to all boundary conditions. Even if comparison functions are used, Eq. (8.89) is still to be preferred, as it involves lower-order derivatives.

From the above discussion, it appears that using the eigenfunctions of a related but simpler system as admissible functions remains an attractive alternative, particularly in view of the fact that natural boundary conditions can be ignored. Care must be exercised in violating natural boundary conditions, however, as certain violations can slow down convergence significantly. We examine some of the implications of violating natural boundary conditions in Sec. 8.6.

The Rayleigh-Ritz theory is perhaps the best exponent of a variational approach used to approximate distributed-parameter systems by discrete ones. Indeed, the estimation of the Ritz eigenvalues is so satisfying mathematically that the associated theory has few equals. Unfortunately, the estimation of the Ritz eigenfunctions is a different matter. Although it may be reasonable to expect that the minimizing sequences representing the Ritz eigenfunctions $w_r^{(n)}$ converge to the actual eigenfunctions w_r , proof of convergence is not simple (Ref. 5). Moreover, because of the stationarity of Rayleigh's quotient, it can be stated that in general the Ritz eigenvalues approximate the actual eigenvalues one order of magnitude better than the

Ritz eigenfunctions approximate the actual eigenfunctions. In spite of that, it is common practice to assume that the eigenfunctions converge when the corresponding eigenvalues do.

In the case of two-dimensional problems, factors frustrating closed-form solutions are likely to frustrate approximate solutions by the Rayleigh-Ritz method as well. Indeed, if approximate solutions are at all possible, then they tend to be confined to systems with regular boundary shape, such as rectangular or circular. There is a significant difference in complexity between membranes and plates, making approximate solutions much more difficult for plates than for membranes. In the case of rectangular plates, approximate solutions can often be obtained in the form of products of beam eigenfunctions. A variety of such solutions can be found in Ref. 23. For two-dimensional systems with nonuniform mass and stiffness distributions, as for one-dimensional systems, the eigenfunctions of systems with the same boundary conditions but with uniform parameter distributions can serve as suitable admissible functions, even when these eigenfunctions are only approximations to the actual ones.

Before we conclude this section, perhaps a poignant historical note is in order. The approach was first used by Rayleigh on various occasions beginning in 1870 (Ref. 13) in connection with the vibration of air in organ pipes closed at one end and open at the other, but the approach did not receive much attention. The method became widely known as the *Ritz method* following publication of two papers by Ritz (Refs. 40 and 41). The wide attention received by these two papers can be attributed to two reasons, the "masterly" exposition of the theory by Ritz and the tragic circumstances under which Ritz wrote the papers (he was dying of consumption). In view of the fact that Ritz's work was independent of Rayleigh's, referring to the approach as the Rayleigh-Ritz method is quite appropriate. It is perhaps interesting to note that in Ref. 41 Ritz himself used products of beam eigenfunctions to solve the eigenvalue problem for a rectangular plate free on all sides.

Example 8.3

Derive and solve the appropriate algebraic eigenvalue problems for the tapered bar of Example 8.2 by means of the Rayleigh-Ritz method using Rayleigh's quotient in the form of Eq. (8.89). Use a minimizing sequence through $n = 3$ in terms of the admissible functions

$$\phi_i(x) = \sin \frac{(2i-1)\pi x}{2L}, \quad i = 1, 2, \dots, n \quad (a)$$

Then, verify the separation theorem.

Rayleigh's quotient for the problem at hand has the form

$$\lambda = \frac{[U, U]}{(\sqrt{m}U, \sqrt{m}U)} = \frac{\int_0^L EA(x) [dU(x)/dx]^2 dx}{\int_0^L m(x)U^2(x) dx} \quad (b)$$

where, from Example 8.2,

$$EA(x) = 2EA \left(1 - \frac{x}{L}\right), \quad m(x) = 2m \left(1 - \frac{x}{L}\right) \quad (c)$$

Note that, whereas the boundary condition at $x = 0$ is geometric, $w(0) = 0$, the boundary condition at $x = L$ would be natural. However, because the axial stiffness reduces to zero there, the boundary condition at $x = L$ is somewhat unorthodox and it amounts to the displacement being finite.

Using Eqs. (a)–(c), the stiffness coefficients for $i = j$ are

$$\begin{aligned} k_{ii} &= \int_0^L EA(x)(\phi'_i)^2 dx \\ &= \frac{EA(2i-1)^2\pi^2}{2L^2} \int_0^L \left(1 - \frac{x}{L}\right) \cos^2 \frac{(2i-1)\pi x}{2L} dx \\ &= \frac{EA}{2L} \left[1 + \frac{(2i-1)^2\pi^2}{4}\right] \end{aligned} \quad (d)$$

and for $i \neq j$ they are

$$\begin{aligned} k_{ij} &= \int_0^L EA(x)\phi'_i\phi'_j dx \\ &= \frac{EA(2i-1)(2j-1)\pi^2}{2L^2} \int_0^L \left(1 - \frac{x}{L}\right) \cos \frac{(2i-1)\pi x}{2L} \cos \frac{(2j-1)\pi x}{2L} dx \\ &= \frac{EA(2i-1)(2j-1)}{4L^2} \left[\frac{1 + (-1)^{i+j}}{(i+j-1)^2} + \frac{1 - (-1)^{i-j}}{(i-j)^2} \right] \end{aligned} \quad (e)$$

Moreover, the mass coefficients for $i = j$ are

$$\begin{aligned} m_{ii} &= \int_0^L m(x)\phi_i^2(x) dx = 2m \int_0^L \left(1 - \frac{x}{L}\right) \sin^2 \frac{(2i-1)\pi x}{2L} dx \\ &= \frac{mL}{2} \left[1 - \frac{4}{(2i-1)^2\pi^2}\right] \end{aligned} \quad (f)$$

and for $i \neq j$ they are

$$\begin{aligned} m_{ij} &= \int_0^L m(x)\phi_i(x)\phi_j(x) dx \\ &= 2m \int_0^L \left(1 - \frac{x}{L}\right) \sin \frac{(2i-1)\pi x}{2L} \sin \frac{(2j-1)\pi x}{2L} dx \\ &= \frac{mL}{\pi^2} \left[\frac{1 - (-1)^{i-j}}{(i-j)^2} - \frac{1 + (-1)^{i+j}}{(i+j-1)^2} \right] \end{aligned} \quad (g)$$

For $n = 1$, we insert Eqs. (d) and (f) with $i = 1$ into Eq. (8.85) and obtain the estimate of the lowest eigenvalue

$$\lambda_1^{(1)} = \frac{k_{11}}{m_{11}} = \frac{\frac{EA}{2L} \left(1 + \frac{\pi^2}{4}\right)}{\frac{mL}{2} \left(1 - \frac{4}{\pi^2}\right)} = 5.8304 \frac{EA}{mL^2} \quad (h)$$

from which we obtain the approximation to the lowest natural frequency

$$\omega_1^{(1)} = 2.4146 \sqrt{\frac{EA}{mL^2}} \quad (i)$$

This is the result obtained in Example 8.2 by means of Rayleigh's energy method.

For $n = 2$, we use Eqs. (d)–(g) with $i, j = 1, 2$ to derive the 2×2 stiffness and mass matrices

$$K^{(2)} = \frac{EA}{8L} \begin{bmatrix} 4 + \pi^2 & 12 \\ 12 & 4 + 9\pi^2 \end{bmatrix}, \quad M^{(2)} = \frac{mL}{2\pi^2} \begin{bmatrix} \pi^2 - 4 & 4 \\ 4 & \pi^2 - 4/9 \end{bmatrix} \quad (j)$$

Inserting Eqs. (j) into Eq. (8.87) with $n = 2$, we obtain a 2×2 eigenvalue problem, which has the eigenvalues and eigenvectors

$$\begin{aligned} \lambda_1^{(2)} &= 5.7897 \frac{EA}{mL^2}, & \lambda_2^{(2)} &= 30.5717 \frac{EA}{mL^2} \\ \mathbf{a}_1 &= \begin{bmatrix} 1.0000 \\ -0.0369 \end{bmatrix}, & \mathbf{a}_2 &= \begin{bmatrix} 1.0000 \\ -1.5651 \end{bmatrix} \end{aligned} \quad (k)$$

and we note that the eigenvectors have been normalized so that the top component is unity. The eigenvalues can be used to compute an improved approximation to the lowest natural frequency and an estimate of the second natural frequency in the form

$$\omega_1^{(2)} = 2.4062 \sqrt{\frac{EA}{mL^2}}, \quad \omega_2^{(2)} = 5.5292 \sqrt{\frac{EA}{mL^2}} \quad (l)$$

respectively. Moreover, inserting the eigenvectors into Eqs. (8.91), we obtain the approximate eigenfunctions

$$w_1^{(2)} = \sin \frac{\pi x}{2L} - 0.0369 \sin \frac{3\pi x}{2L}, \quad w_2^{(2)} = \sin \frac{\pi x}{2L} - 1.5651 \sin \frac{3\pi x}{2L} \quad (m)$$

Following the same pattern for $n = 3$, we obtain the stiffness and mass matrices

$$\begin{aligned} K^{(3)} &= \frac{EA}{8L} \begin{bmatrix} 4 + \pi^2 & 12 & 20/9 \\ 12 & 4 + 9\pi^2 & 60 \\ 20/9 & 60 & 4 + 25\pi^2 \end{bmatrix} \\ M^{(3)} &= \frac{mL}{2\pi^2} \begin{bmatrix} \pi^2 - 4 & 4 & -4/9 \\ 4 & \pi^2 - 4/9 & 4 \\ -4/9 & 4 & \pi^2 - 4/25 \end{bmatrix} \end{aligned} \quad (n)$$

The corresponding eigenvalue problem has the solutions

$$\begin{aligned} \lambda_1^{(3)} &= 5.7837 \frac{EA}{mL^2}, & \lambda_2^{(3)} &= 30.4878 \frac{EA}{mL^2}, & \lambda_3^{(3)} &= 75.0751 \frac{EA}{mL^2} \\ \mathbf{a}_1 &= \begin{bmatrix} 1.0000 \\ -0.0319 \\ -0.0072 \end{bmatrix}, & \mathbf{a}_2 &= \begin{bmatrix} 1.0000 \\ -1.5540 \\ 0.0666 \end{bmatrix}, & \mathbf{a}_3 &= \begin{bmatrix} 1.0000 \\ -1.2110 \\ 2.0270 \end{bmatrix} \end{aligned} \quad (o)$$

and we note that $\lambda_1^{(3)}$ and $\lambda_2^{(3)}$ represent improved approximations to the actual eigenvalues λ_1 and λ_2 and $\lambda_3^{(3)}$ is a first estimate of the third actual eigenvalue λ_3 . From Eqs. (o), we obtain the approximate natural frequencies

$$\omega_1^{(3)} = 2.4049 \sqrt{\frac{EA}{mL^2}}, \quad \omega_2^{(3)} = 5.5216 \sqrt{\frac{EA}{mL^2}}, \quad \omega_3^{(3)} = 8.6646 \sqrt{\frac{EA}{mL^2}} \quad (p)$$

and approximate eigenfunctions

$$w_1^{(3)} = 0.9759 \left(\sin \frac{\pi x}{2L} - 0.0319 \sin \frac{3\pi x}{2L} - 0.0072 \sin \frac{5\pi x}{2L} \right)$$

$$w_2^{(3)} = 0.3816 \left(\sin \frac{\pi x}{2L} - 1.5540 \sin \frac{3\pi x}{2L} + 0.0666 \sin \frac{5\pi x}{2L} \right) \quad (q)$$

$$w_3^{(3)} = 0.2360 \left(\sin \frac{\pi x}{2L} - 1.2110 \sin \frac{3\pi x}{2L} + 2.0270 \sin \frac{5\pi x}{2L} \right)$$

The approximate eigenfunctions, normalized so that $w_r^{(3)}(L) = 1$ ($r = 1, 2, 3$), are displayed in Fig. 8.11.

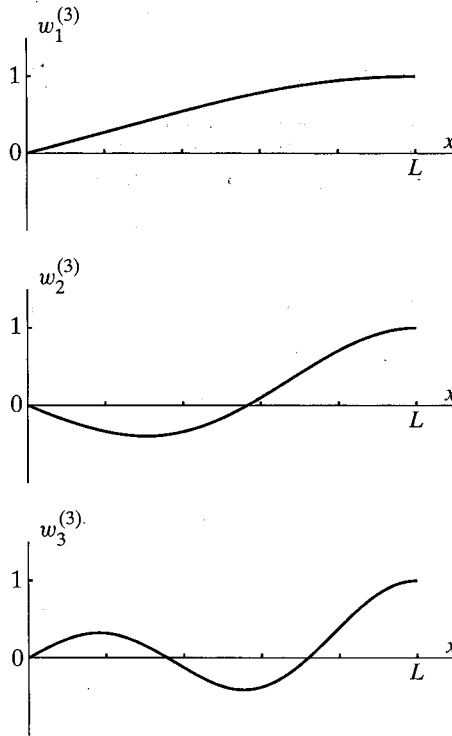


Figure 8.11 The three lowest approximate eigenfunctions for a tapered rod in axial vibration fixed at $x = 0$ and free at $x = L$.

Omitting the parameter ratio EA/mL^2 , the computed eigenvalues can be verified to satisfy the separation theorem as follows:

$$\begin{aligned} \lambda_1^{(2)} = 5.7897 < \lambda_1^{(1)} = 5.8304 < \lambda_2^{(2)} = 30.5717 \\ \lambda_1^{(3)} = 5.7837 < \lambda_1^{(2)} = 5.7897 < \lambda_2^{(3)} = 30.4878 \\ < \lambda_2^{(2)} = 30.5717 < \lambda_3^{(3)} = 75.0751 \end{aligned} \quad (r)$$

Moreover, they form a triangular array as given by Eq. (8.103). To this end, we note that the differential eigenvalue problem admits a closed-form solution, which permits

us to write

$$\begin{aligned}\lambda_1^{(1)} = 5.8304 > \lambda_1^{(2)} = 5.7897 > \lambda_1^{(3)} = 5.7837 > \dots \rightarrow \lambda_1 = 5.7831 \\ \lambda_2^{(2)} = 30.5717 > \lambda_2^{(3)} = 30.4878 > \dots \rightarrow \lambda_2 = 30.4715 \\ \lambda_3^{(3)} = 75.0751 > \dots \rightarrow \lambda_3 = 74.8865\end{aligned}\tag{s}$$

From the above array, we conclude that the computed eigenvalues are remarkably accurate. Indeed, plots of the computed eigenvalues versus the number of admissible functions in the series for the approximate solution are relatively flat and close to the horizontal lines representing the asymptotes. The reason for this is that the admissible functions are in fact comparison functions and capable of approximating the actual eigenfunctions quite accurately, at least the first three. In general, such good accuracy with so few terms should not be expected.

8.6 THE CLASS OF QUASI-COMPARISON FUNCTIONS: AN ENHANCED RAYLEIGH-RITZ METHOD

As pointed out in Sec. 8.5, the Rayleigh-Ritz method is a technique for approximating a finite number of eigensolutions for a distributed-parameter system whereby the solution of a differential eigenvalue problem is replaced by a variational problem consisting of the minimization of Rayleigh's quotient. To this end, the solution is assumed to have the form of a minimizing sequence, with each term in the sequence consisting of a linear combination of trial functions, thus leading to a sequence of algebraic eigenvalue problems of increasing order. If the numerator of Rayleigh's quotient has the form of an inner product involving the stiffness operator L , then the trial functions must be from the space \mathcal{K}_B^{2p} of comparison functions. A more common and more desirable version of Rayleigh's quotient is that in which the numerator represents a measure of the potential energy, in which case the trial functions need be from the space \mathcal{K}_G^p of admissible functions alone.

The energy version of Rayleigh's quotient, Eq. (8.89), is equivalent to the version involving the stiffness operator, Eq. (8.75), only when the trial function w is from the space of comparison functions. Clearly, in using the energy version of Rayleigh's quotient in conjunction with admissible functions, the natural boundary conditions are violated, so that the question arises as to whether something that should not be sacrificed is in fact sacrificed. The answer depends on the character of the natural boundary conditions and what is potentially sacrificed is the speed of convergence.

The question of convergence speed is related to the completeness of the set of admissible functions. The concept of completeness is more qualitative than quantitative in nature (see Sec. 7.5). Whereas many will agree that $\epsilon = 10^{-6}$ is a small number, there is far less agreement as to what constitutes a sufficiently large number n of terms in the linear combination. It is precisely this number that defines the speed of convergence. A set of admissible functions can be complete in energy and still exhibit poor convergence characteristics. This can happen when eigenfunctions of a related simpler system are used as admissible functions for a system with natural boundary conditions more complicated than the free boundary of Example 8.5.

To investigate the convergence question raised above, we consider a nonuniform rod in axial vibration fixed at $x = 0$ and restrained by a spring at $x = L$, as shown in Fig. 8.12. Rayleigh's quotient for the problem at hand is

$$R(U) = \frac{[U, U]}{[\sqrt{m}U, \sqrt{m}U]} = \frac{\int_0^L EA(x) [dU(x)/dx]^2 dx + kU^2(L)}{\int_0^L m(x)U^2(x) dx} \quad (8.104)$$

where the parameters are as follows:

$$EA(x) = \frac{6EA}{5} \left[1 - \frac{1}{2} \left(\frac{x}{L} \right)^2 \right], \quad m(x) = \frac{6m}{5} \left[1 - \frac{1}{2} \left(\frac{x}{L} \right)^2 \right], \quad k = \frac{EA}{L} \quad (8.105)$$

in which $EA(x)$ is the axial stiffness, $m(x)$ the mass density and k the spring constant. For future reference, the boundary conditions are as follows:

$$U(0) = 0, \quad EA(x) \frac{dU(x)}{dx} + kU(x) = 0 \text{ at } x = L. \quad (8.106a, b)$$

and we note that Eq. (8.106a) represents a geometric boundary condition and Eq. (8.106b) a natural one.

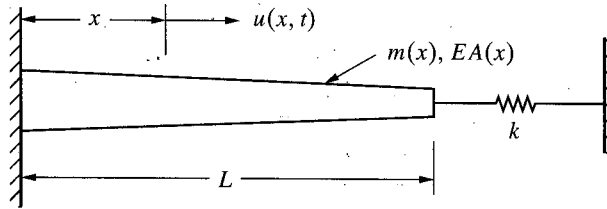


Figure 8.12 Nonuniform rod in axial vibration fixed at $x = 0$ and restrained by a spring at $x = L$

In accordance with the Rayleigh-Ritz method, we consider an approximate solution in the form

$$U^{(n)}(x) = \Phi^T(x) \mathbf{a} \quad (8.107)$$

where $\Phi = [\phi_1 \phi_2 \dots \phi_n]^T$ is an n -vector of trial functions and $\mathbf{a} = [a_1 a_2 \dots a_n]^T$ an n -vector of undetermined coefficients. Inserting Eq. (8.107) into Eq. (8.104), we obtain the discretized Rayleigh quotient

$$R(\mathbf{a}) = \frac{\mathbf{a}^T K^{(n)} \mathbf{a}}{\mathbf{a}^T M^{(n)} \mathbf{a}} \quad (8.108)$$

in which

$$K^{(n)} = [\Phi, \Phi^T] = \int_0^L EA(x) \frac{d\Phi(x)}{dx} \frac{d\Phi^T(x)}{dx} dx + k\Phi(L)\Phi^T(L) \quad (8.109a)$$

$$M^{(n)} = (\sqrt{m}\Phi, \sqrt{m}\Phi^T) = \int_0^L m(x)\Phi(x)\Phi^T(x) dx \quad (8.109b)$$

are the stiffness matrix and mass matrix, respectively. Following the approach of Sec. 8.5, minimization of Rayleigh's quotient leads to the sequence of eigenvalue problems given by Eq. (8.87), which can be solved for the approximate eigenvalues $\lambda_r^{(n)}$ and eigenvectors \mathbf{a}_r ($r = 1, 2, \dots, n$). The approximate eigenfunctions $U_r^{(n)}$ are obtained by inserting the eigenvectors into Eq. (8.107).

In view of the Rayleigh-Ritz theory, in using Rayleigh's quotient in the form of Eq. (8.104), the trial functions need be only admissible functions. We use as admissible functions the eigenfunctions of a uniform fixed-free rod, or

$$\phi_i(x) = \sin \frac{(2i-1)\pi x}{2L}, \quad i = 1, 2, \dots, n \quad (8.110)$$

Following introduction of Eq. (8.110) into Eqs. (8.109) and evaluation of the matrices $K^{(n)}$ and $M^{(n)}$, the eigenvalue problem (8.87) has been solved for $n = 1, 2, \dots, 30$ (see Ref. 30). The resulting approximate natural frequencies $\omega_r^{(n)}$, related to the eigenvalues by $\omega_r^{(n)} = \sqrt{\lambda_r^{(n)} EA/mL^2}$, are displayed in Table 8.1. It is clear from Table 8.1 that convergence is painfully slow, as convergence to six significant digits accuracy has not been reached with $n = 30$ in the approximate solution, Eq. (8.107). The culprit can be easily identified as the inability to satisfy the natural boundary condition, Eq. (8.106b). Indeed, all admissible functions have zero derivative at $x = L$ and, according to Eq. (8.106b), the derivative at $x = L$ must be different from zero. In theory, for the derivative of a linear combination of n terms with zero derivative at $x = L$ to be different from zero there, the number n must approach infinity. This is unacceptable for an approximate solution, for which the number n must be not only finite but also as small as possible.

TABLE 8.1 The Three Lowest Approximate Natural Frequencies Using Admissible Functions

n	$\omega_1^{(n)} \sqrt{mL^2/EA}$	$\omega_2^{(n)} \sqrt{mL^2/EA}$	$\omega_3^{(n)} \sqrt{mL^2/EA}$
1	2.32965	—	—
2	2.27291	5.13905	—
3	2.25352	5.12823	8.13148
4	2.24369	5.12158	8.13028
5	2.23781	5.11727	8.12835
\vdots	\vdots	\vdots	\vdots
28	2.21920	5.10253	8.11855
29	2.21907	5.10242	8.11847
30	2.21895	5.10232	8.11840

Next, we wish to examine convergence with a solution in terms of comparison functions. To this end, we replace the admissible functions of Eq. (8.110) by the comparison functions

$$\phi_i(x) = \sin \beta_i x, \quad i = 1, 2, \dots, n \quad (8.111)$$

where, according to Eq. (8.106b), β_i must satisfy the transcendental equation

$$EA(L)\beta_i \cos \beta_i L + k \sin \beta_i L = 0, \quad i = 1, 2, \dots, n \quad (8.112)$$

The approximate natural frequencies $\omega_r^{(n)}$ using comparison functions have been computed in Ref. 30 following the established pattern and the results are exhibited in Table 8.2. Clearly, the approximate natural frequencies computed by means of comparison functions have superior convergence characteristics compared with those computed by means of admissible functions. Indeed, $\omega_1^{(n)}$ reaches convergence with $n = 11$ and $\omega_2^{(n)}$ with $n = 18$. Whereas $\omega_3^{(n)}$ has not reached convergence yet with $n = 30$, convergence is not far away.

TABLE 8.2 The Three Lowest Approximate Natural Frequencies Using Comparison Functions

n	$\omega_1^{(n)} \sqrt{mL^2/EA}$	$\omega_2^{(n)} \sqrt{mL^2/EA}$	$\omega_3^{(n)} \sqrt{mL^2/EA}$
1	2.22297	—	—
2	2.21647	5.10630	—
3	2.21573	5.10070	8.12426
4	2.21559	5.09984	8.11790
5	2.21555	5.09964	8.11680
⋮	⋮	⋮	⋮
10	2.21553	5.09953	8.11633
11	2.21552	5.09953	8.11633
⋮	⋮	⋮	⋮
17		5.09953	8.11632
18		5.09952	8.11632
⋮			⋮
30			8.11632

From the preceding results, it is tempting to conclude that, when the system involves natural boundary conditions at boundaries that are not free, the argument has been settled in favor of using comparison functions. There is one problem with this conclusion, however. Whereas in the case at hand it was easy to generate comparison functions, this is not the case in general. In fact, in many cases it may not even be possible to generate comparison functions. Hence, the question is whether a way out of this seeming impasse exists. The answer is affirmative, but this requires breaking away from some of the thinking conditioned by the Rayleigh-Ritz method.

Before addressing the aspects of the Rayleigh-Ritz method in need of rethinking, it would help reviewing the points on which there is no dispute. There is general agreement that, when the differential equation cannot be satisfied exactly, approximate solutions are to be optimized by a variational process involving Rayleigh's

quotient. Moreover, there is no question that the geometric boundary conditions must be satisfied. This leaves satisfaction of the natural boundary conditions as the only issue to be settled. The Rayleigh-Ritz method offers two choices, satisfy the natural boundary conditions exactly through the use of comparison functions, or abandon any attempt to satisfy them and use admissible functions. The first choice may not be an option and the second choice can lead to very poor convergence, as amply demonstrated here. But, the number of points inside the domain D of the system is infinitely larger than the number of points on the boundary S . If the boundary includes points at which the geometric integrity must be preserved, then the solution must reflect this requirement. On the other hand, if the boundary includes points involving force and moment balance, then there is no reason to insist that force and moment balance be satisfied at such boundary points while the differential equation is violated in the interior of the domain. It follows that the points inside D and the points on S involving natural boundary conditions should be afforded equal status. This implies that the same degree of approximation of the solution should extend to all points of the system, with the exception of boundary points involving the system geometry, which must be respected. This further implies that the degree of completeness required of the admissible functions should cover all the points in D and all the points on S in question. To this end, a new class of functions has been conceived in Ref. 30, namely, the class of *quasi-comparison functions*, defined as *linear combinations of admissible functions capable of approximating the differential equation and the natural boundary conditions to any degree of accuracy by merely increasing the number n of terms in the approximating solution*. In practice, *the linear combinations must be capable of satisfying the natural boundary conditions by simply adjusting the coefficients a_i ($i = 1, 2, \dots, n$)*. This is not to say that the coefficients should be adjusted so as to satisfy the natural boundary conditions. On the contrary, the adjustment of the coefficients should be left to the variational process. The quasi-comparison functions can also be defined as linear combinations of admissible functions acting like comparison functions. It should be pointed out that there is a minimum number of admissible functions required before the linear combination becomes capable of satisfying all the boundary conditions of the system, including the natural boundary conditions. In another break with the Rayleigh-Ritz tradition, *the approximating solution $w^{(n)}$ must be constructed using members from different families of admissible functions*, each family having different dynamic characteristics. It is this variety of admissible functions that enhances the minimization process, thus permitting accurate approximations to the differential equation and the natural boundary conditions with only a relatively small number of terms. Such a feat cannot be duplicated with admissible functions from a single family, as in the ordinary Rayleigh-Ritz practice.

To illustrate this point, we return to the rod in axial vibration investigated earlier in this section and consider a set of quasi-comparison functions in the form

$$\phi_i(x) = \sin \frac{i\pi x}{2L}, \quad i = 1, 2, \dots, n \quad (8.113)$$

and we note that individually none of these admissible functions satisfies the natural

boundary condition at $x = L$. However, as an example, the linear combination

$$w^{(2)} = \sin \frac{\pi x}{2L} + c \sin \frac{\pi x}{L} \quad (8.114)$$

can be made to satisfy the natural boundary condition, Eq. (8.106b), by merely adjusting the coefficient c . Following the same pattern as with the other two classes of functions, Eqs. (8.110) and (8.111), the three lowest approximate natural frequencies were computed in Ref. 30 using the quasi-comparison functions given by Eq. (8.113). They are given in Table 8.3. As can be concluded from Table 8.3, convergence is extremely rapid.

TABLE 8.3 The Three Lowest Approximate Natural Frequencies Using Quasi-Comparison Functions

n	$\omega_1^{(n)} \sqrt{mL^2/EA}$	$\omega_2^{(n)} \sqrt{mL^2/EA}$	$\omega_3^{(n)} \sqrt{mL^2/EA}$
1	2.32965	—	—
2	2.22359	5.98485	—
3	2.21615	5.10007	11.09264
4	2.21557	5.09957	8.15365
5	2.21555	5.09953	8.11632
6	2.21552	5.09952	8.11632
7			8.11632
8			8.11632
9			8.11632
10			8.11632
11			8.11632
12			8.11632
13			8.11631

Contrasting the results in Tables 8.2 and 8.3, we conclude that the solution in terms of quasi-comparison functions converges faster than even the solution in terms of comparison functions, except for small n . Although this may come as a surprise to some, the explanation lies in the fact that quasi-comparison functions are capable of approximating the actual solution throughout the domain $0 < x < L$ somewhat better than comparison functions, because a larger variety of functions permits a better minimization. As far as the poor results for small n are concerned, it should be recalled that there is a minimum number of admissible functions required before the linear combination becomes a quasi-comparison function. Consistent with the faster convergence of the eigenvalues, Ref. 30 gives a plot of the logarithm of the mean-square error of the first approximate eigenfunction showing that the error for the solution in terms of quasi-comparison functions drops faster with n than the corresponding error for comparison functions.

Finally, we should explain the statement that there is larger variety in the quasi-comparison functions than in comparison functions. We observe that the set of

functions in Eq. (8.113) consists of two families. The functions corresponding to $i = 1, 3, \dots$ are really the admissible functions given by Eq. (8.110). They represent the eigenfunctions of a uniform fixed-free rod and guarantee that the displacement is different from zero at $x = L$, although the slope there is zero. On the other hand, the functions corresponding to $i = 2, 4, \dots$ represent the eigenfunctions of a uniform fixed-fixed rod and guarantee that the slope is different from zero at $x = L$, although the displacement there is zero. When the functions from the two families are combined to form an approximate solution, then the solution is such that both the displacement and force, where the latter is proportional to the slope, are different from zero at $x = L$. Hence, a linear combination of admissible functions from two families with different dynamic characteristics is able to satisfy the natural boundary condition at $x = L$ and can provide a better approximation throughout the domain $0 < x < L$ than admissible functions from a single family, or even than comparison functions. It should be pointed out that, although the two families represent eigenfunctions, they do not represent vibration modes in a physical sense, as a system cannot be fixed-free and fixed-fixed at the same time. In this regard, it should be stressed that the admissible functions used to generate quasi-comparison functions can be any functions with the proper characteristics, and need not be eigenfunctions of a related system at all (Ref. 30). In this particular example, the admissible functions must be such that at $x = 0$ any linear combination of these functions is zero and its derivative with respect to x is different from zero, thus guaranteeing that the displacement is zero and the force is not zero at $x = 0$. On the other hand, at $x = L$ both the linear combination and its derivative must be different from zero, thus ensuring that neither the displacement nor the force is zero at $x = L$. In the case of a fourth-order problem, such as a beam in bending, the required characteristics are more involved. This explains why, to secure the characteristics required to qualify as quasi-comparison functions, the linear combination of admissible functions must include members from two or more distinct families and must contain a minimum number of functions.

Care must be exercised in choosing the various families of admissible functions to ensure linear independence. Indeed, if each family forms a complete set, then a member from one family can be approximated by a sufficiently large linear combination of members from another family, which implies that, as the number of terms increases, the independence tends to be lost. This is actually the case with the families of fixed-free and fixed-fixed functions combining into the set given by Eq. (8.113). Because convergence was achieved with a relatively small number of terms, this dependence did not have an opportunity to materialize. The loss of independence can become a problem when computing higher eigenvalues, which require a larger number of terms in the linear combination. Related to this is the fact that the eigenfunctions of a uniform rod fixed at $x = 0$ and with a spring attached at $x = L$ tend to coincide with the eigenfunctions of a uniform fixed-free rod as the eigenfunction number increases. As this happens, the contribution of the eigenfunctions of the fixed-fixed rod to the accuracy of the approximate solution tends to wane, and in fact it can cause the mass matrix to become singular. If linear combinations from two families of admissible functions experience difficulties in yielding a desired number

of accurate eigenvalues, then the possibility of using linear combinations from three or more families should be considered.

In the case of two-dimensional systems, the generation of quasi-comparison functions can encounter serious difficulties, and quite often may not even be possible. Still, the idea of approximating the solution to the differential eigenvalue problem by means of linear combinations from several families of admissible functions remains valid, even when they do not constitute quasi-comparison functions, and such linear combinations are likely to yield more accurate eigensolutions than linear combinations from a single family.

8.7 THE ASSUMED-MODES METHOD

The assumed-modes method is a procedure for the discretization of distributed systems closely related to the Rayleigh-Ritz method. In fact, at times it is referred to as such (Ref. 3). Although the motivation and details are different, the results are the same as those obtained by the Rayleigh-Ritz method using the energy form of Rayleigh's quotient. The main advantage of the assumed-modes method is that it is perhaps easier to grasp. Of course, the method is quite heuristic, so that if the interest lies in the finer points of analysis, such as convergence, then it is necessary to refer to the Rayleigh-Ritz theory.

In contrast with the Rayleigh-Ritz method, in the assumed-modes method we begin with the free vibration of distributed systems prior to the elimination of the time variable. First, by analogy with the Rayleigh-Ritz method, the system is discretized in space by means of a series of space-dependent trial functions multiplied by time-dependent generalized coordinates. Then, the equations of motion for the discretized system are derived by means of Lagrange's equations. The associated eigenvalue problem is precisely the same as that obtained by the Rayleigh-Ritz method.

To illustrate the approach, we assume an approximate solution in the form

$$w(P, t) \cong w^{(n)}(P, t) = \sum_{i=1}^n \phi_i(P) q_i(t) = \boldsymbol{\phi}^T(P) \mathbf{q}(t) \quad (8.115)$$

where $\boldsymbol{\phi} = [\phi_1 \phi_2 \dots \phi_n]^T$ is an n -vector of trial functions depending on the spatial position P and $\mathbf{q} = [q_1 q_2 \dots q_n]^T$ is an n -vector of time-dependent generalized coordinates. Using Eq. (8.115), we discretize the kinetic energy as follows:

$$\begin{aligned} T(t) &= \frac{1}{2} \int_D m(P) \dot{w}^2(P, t) dD(P) \cong \frac{1}{2} \int_D m(P) [\dot{w}^{(n)}(P, t)]^2 dD(P) \\ &= \frac{1}{2} \int_D m(P) \dot{\mathbf{q}}^T(t) \boldsymbol{\phi}(P) \boldsymbol{\phi}^T(P) \dot{\mathbf{q}}(t) dD(P) = \frac{1}{2} \dot{\mathbf{q}}^T(t) M^{(n)} \dot{\mathbf{q}}(t) \end{aligned} \quad (8.116)$$

in which

$$M^{(n)} = \int_D m(P) \boldsymbol{\phi}(P) \boldsymbol{\phi}^T(P) dD(P) \quad (8.117)$$

is recognized as the mass matrix obtained by the Rayleigh-Ritz method, Eq. (8.88b). Moreover, by analogy with Eq. (7.86), the discretized potential energy has the generic form

$$\begin{aligned} V(t) &= \frac{1}{2} [w(P, t), w(P, t)] \cong \frac{1}{2} [w^{(n)}(P, t), w^{(n)}(P, t)] \\ &= \frac{1}{2} [\mathbf{q}^T(t) \boldsymbol{\Phi}(P), \boldsymbol{\Phi}^T(P) \mathbf{q}(t)] = \frac{1}{2} \mathbf{q}^T(t) K^{(n)} \mathbf{q}(t) \end{aligned} \quad (8.118)$$

where

$$K^{(n)} = [\boldsymbol{\Phi}(P), \boldsymbol{\Phi}^T(P)] \quad (8.119)$$

is the same stiffness matrix as that obtained by the Rayleigh-Ritz method, Eq. (8.90).

Lagrange's equations of motion for conservative discrete systems can be written in the symbolic form

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{\mathbf{q}}} \right) - \frac{\partial T}{\partial \mathbf{q}} + \frac{\partial V}{\partial \mathbf{q}} = \mathbf{0} \quad (8.120)$$

Hence, inserting Eqs. (8.116) and (8.118) into Eq. (8.120), we obtain the equations of motion

$$M^{(n)} \ddot{\mathbf{q}}(t) + K^{(n)} \mathbf{q}(t) = \mathbf{0} \quad (8.121)$$

Then, letting the solution be harmonic, so that $\mathbf{q}(t) = e^{i\omega^{(n)}t} \mathbf{a}$, $\omega^{(n)} = \sqrt{\lambda^{(n)}}$, Eq. (8.121) yields the algebraic eigenvalue problem

$$K^{(n)} \mathbf{a} = \lambda^{(n)} M^{(n)} \mathbf{a} \quad (8.122)$$

which is identical to that obtained by the Rayleigh-Ritz method, Eq. (8.87).

Although it may not be immediately obvious, the eigenvalue problem derived by the assumed-modes method, Eq. (8.122), can also be regarded as being based on a variational approach. In this regard, we observe that the generic Lagrange's equations themselves were derived by means of a variational approach, namely, Hamilton's principle.

Finally, there remains the question of the trial functions selection. The term "assumed modes" connotes certain eigenfunctions. But, the assumed-modes method is equivalent to the Rayleigh-Ritz method with Rayleigh's quotient in energy form, Eq. (8.89). Hence, the Rayleigh-Ritz theory applies equally well here, so that the same guidelines for the selection of the trial functions can be used for the assumed-modes method as for the Rayleigh-Ritz method. It follows from Sec. 8.5 that the trial functions need be admissible functions only, and need not be modes at all. In fact, in accordance to the enhanced Rayleigh-Ritz method of Sec. 8.6, improved accuracy can be realized by assuming solutions in the form of quasi-comparison functions.

8.8 THE METHOD OF WEIGHTED RESIDUALS

The Rayleigh-Ritz method is a technique for deriving approximate solutions to differential eigenvalue problems not permitting closed-form solutions. It is a variational approach based on the stationarity of Rayleigh's quotient, which restricts its use to self-adjoint systems. Whereas the class of self-adjoint systems is very large indeed, and it includes most systems of interest, there are many important systems not falling in this class. In view of this, a broader approach, applicable to both self-adjoint and non-self-adjoint systems is highly desirable. Such an approach is the *method of weighted residuals*, which is not just one method but an umbrella for a number of seemingly disparate approximate techniques. In contrast with the Rayleigh-Ritz method, the weighted residuals method works directly with the differential equation.

We are interested in eigenvalue problems of the type

$$Lw(x) = \lambda m(x)w(x) \quad (8.123)$$

where L is a generally non-self-adjoint differential operator of order $2p$ and m is the mass density. The solution $w(x)$ is subject to given boundary conditions. The assumption is that the problem does not admit a closed-form solution, so that we consider an approximate solution in the form

$$w(x) \cong w^{(n)}(x) = \sum_{j=1}^n a_j \phi_j(x) \quad (8.124)$$

in which $\phi_1, \phi_2, \dots, \phi_n$ are n independent comparison functions from a complete set. Hence, we confine our approximate solution $w^{(n)}(x)$ to the n -dimensional subspace S_n of \mathcal{K}_B^{2p} , where S_n is referred to as the *trial space*. Because $w^{(n)}$ does not satisfy Eq. (8.123) exactly, there is an error at every point x . We refer to the error as a *residual* and denote it by

$$R(w^{(n)}, x) = Lw^{(n)} - \lambda^{(n)}mw^{(n)} \quad (8.125)$$

At the same time, we choose n independent functions $\psi_1(x), \psi_2(x), \dots, \psi_n(x)$ from a different complete set and regard them as a basis for an n -dimensional subspace \mathcal{T}_n of \mathcal{K}^0 , referred to as the *test space*, and define the *weighted residual* as

$$\psi_i R = \psi_i (Lw^{(n)} - \lambda^{(n)}mw^{(n)}), \quad i = 1, 2, \dots, n \quad (8.126)$$

Clearly, our objective is to reduce the error to the largest extent possible. To this end, we insist that the coefficients a_j in Eq. (8.124) ($j = 1, 2, \dots, n$) be such that the integral of the weighted residual be zero for every i , or

$$\int_0^L \psi_i R dx = \int_0^L \psi_i (Lw^{(n)} - \lambda^{(n)}mw^{(n)}) dx = 0, \quad i = 1, 2, \dots, n \quad (8.127)$$

This is equivalent to requiring that the residual be orthogonal to every weighting function ψ_i ($i = 1, 2, \dots, n$). Inserting Eq. (8.124) into Eq. (8.127), we obtain the

algebraic eigenvalue problem

$$\begin{aligned} (\psi_i, R) &= \int_0^L \psi_i \left(\sum_{j=1}^n a_j L\phi_j - \lambda^{(n)} \sum_{j=1}^n a_j m\phi_j \right) dx \\ &= \sum_{j=1}^n (k_{ij} - \lambda^{(n)} m_{ij}) a_j = 0, \quad i = 1, 2, \dots, n \end{aligned} \quad (8.128)$$

where

$$k_{ij} = (\psi_i, L\phi_j) = \int_0^L \psi_i L\phi_j dx, \quad i, j = 1, 2, \dots, n \quad (8.129a)$$

$$m_{ij} = (\psi_i, m\phi_j) = \int_0^L \psi_i m\phi_j dx, \quad i, j = 1, 2, \dots, n \quad (8.129b)$$

are constant coefficients, generally nonsymmetric.

It remains to show that the solution of Eqs. (8.127) converges to the solution of the differential eigenvalue problem, Eq. (8.123). To this end, we recall that the weighting functions ψ_i ($i = 1, 2, \dots, n$) are from a complete set, and Eqs. (8.127) state that the residual R is orthogonal to every one of these functions. As the number n of comparison functions ϕ_j and weighting functions ψ_i is allowed to approach infinity, the only way the residual function R can be orthogonal to a complete set of functions ψ_i is for R itself to approach zero, or

$$\lim_{n \rightarrow \infty} R = \lim_{n \rightarrow \infty} (Lw^{(n)} - \lambda^{(n)} m w^{(n)}) = Lw - \lambda m w = 0 \quad (8.130)$$

Convergence arising from the vanishing of inner products, such as Eqs. (8.127), represents *weak convergence*.

It should be mentioned at this point that the weighting functions ψ_i ($i = 1, 2, \dots, n$) can actually be from the class \mathcal{K}^{-1} . It should also be mentioned that under certain circumstances the requirement that ϕ_j ($j = 1, 2, \dots, n$) be from the class of comparison functions can be relaxed. Indeed, integrating Eq. (8.129a) by parts and considering the boundary conditions, we conclude that ϕ_j can be from the class \mathcal{K}_G^p of admissible functions. Then, as a result of the integrations by parts, the weighting functions ψ_i must also be from \mathcal{K}_G^p . Here too, the convergence can be vastly improved through the use of quasi-comparison functions (Refs. 14 and 32), instead of ordinary admissible functions.

The eigenvalue problem can be cast in matrix form. To this end, we introduce the n -vectors $\boldsymbol{\phi} = [\phi_1 \phi_2 \dots \phi_n]^T$, $\boldsymbol{\psi} = [\psi_1 \psi_2 \dots \psi_n]^T$ and $\mathbf{a} = [a_1 a_2 \dots a_n]^T$, so that Eqs. (8.128) can be rewritten in the compact form

$$K^{(n)} \mathbf{a} = \lambda^{(n)} M^{(n)} \mathbf{a} \quad (8.131)$$

in which

$$K^{(n)} = (\boldsymbol{\psi}, L\boldsymbol{\phi}^T) = \int_0^L \boldsymbol{\psi} L\boldsymbol{\phi}^T dx, \quad M^{(n)} = (\boldsymbol{\psi}, m\boldsymbol{\phi}^T) = \int_0^L m\boldsymbol{\psi}\boldsymbol{\phi}^T dx \quad (8.132a, b)$$

are nonsymmetric $n \times n$ matrices. It can be safely assumed that the matrix $M^{(n)}$ is nonsingular, so that Eq. (8.131) can be cast in the standard, single matrix form

$$A^{(n)} \mathbf{a} = \lambda^{(n)} \mathbf{a} \quad (8.133)$$

where

$$A^{(n)} = (M^{(n)})^{-1} K^{(n)} \quad (8.134)$$

is a nonsymmetric $n \times n$ matrix.

Because the coefficient matrix $A^{(n)}$ is nonsymmetric, the eigensolutions can be complex, although real solutions are possible, depending on the nature of the system. Methods for computing the eigensolutions of nonsymmetric matrices are presented in Secs. 6.13-6.16.

As indicated in the beginning of this section, the method of weighted residuals is a generic name for a family of methods based on the theory just presented. The various methods differ from one another in the nature of the test functions ψ_i . In the sequel, we discuss two methods of particular interest in vibrations.

i. Galerkin's method

Galerkin's method is the most widely used of the weighted residual methods. In fact, the method is better known under its own name than as a weighted residual method. In Galerkin's method, the weighting functions coincide with the trial functions. In vector form, we have

$$\boldsymbol{\psi} = \boldsymbol{\phi} \quad (8.135)$$

so that the coefficient matrices, Eqs. (8.132), become

$$K^{(n)} = (\boldsymbol{\phi}, L\boldsymbol{\phi}^T) = \int_0^L \boldsymbol{\phi} L \boldsymbol{\phi}^T dx, \quad M^{(n)} = (\boldsymbol{\phi}, m\boldsymbol{\phi}^T) = \int_0^L m \boldsymbol{\phi} \boldsymbol{\phi}^T dx \quad (8.136a, b)$$

and we observe that, whereas $M^{(n)}$ is symmetric, $K^{(n)}$ is in general not symmetric, because L is non-self-adjoint.

The operator L in Eq. (8.136a) is of order $2p$ and, consistent with this, the trial functions $\phi_1, \phi_2, \dots, \phi_n$ are from the space \mathcal{K}_B^{2p} of comparison functions. As indicated earlier, the requirements on the trial functions can be lowered by integrating Eq. (8.136a) by parts p times with due consideration to the boundary conditions. Then, the trial functions need be from the space \mathcal{K}_G^p of admissible functions alone. Even if these integrations are carried out, the matrix $K^{(n)}$ remains nonsymmetric because the operator L is non-self-adjoint. As an example, we consider the non-self-adjoint eigenvalue problem defined by the differential equation

$$-\frac{d}{dx} \left(s \frac{dw}{dx} \right) + r \frac{dw}{dx} = \lambda m w, \quad 0 < x < L \quad (8.137)$$

and the boundary conditions

$$w(0) = 0, \quad \left. \frac{dw}{dx} \right|_{x=L} = 0 \quad (8.138)$$

so that the order of the operator L is $2p = 2$. Hence, carrying out one integration by parts and considering the boundary conditions, Eqs. (8.138), a typical element of the matrix $K^{(n)}$ can be reduced as follows:

$$\begin{aligned} k_{ij} &= (\phi_i, L\phi_j) = \int_0^L \phi_i \left[-\frac{d}{dx} \left(s \frac{d\phi_j}{dx} \right) + r \frac{d\phi_j}{dx} \right] dx \\ &= -\phi_i s \frac{d\phi_j}{dx} \Big|_0^L + \int_0^L \left(s \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} + r \phi_i \frac{d\phi_j}{dx} \right) dx \\ &= \int_0^L \left(s \frac{d\phi_i}{dx} \frac{d\phi_j}{dx} + r \phi_i \frac{d\phi_j}{dx} \right) dx \end{aligned} \quad (8.139)$$

which is clearly not symmetric in ϕ_i and ϕ_j and their first derivative. We observe that the symmetry, and hence the self-adjointness, is destroyed by the term $r\phi_i d\phi_j/dx$.

In the special case in which the operator L is self-adjoint, p integrations by parts of Eq. (8.136a) with due consideration to the boundary conditions yield

$$K^{(n)} = [\Phi, \Phi^T] = K^{(n)T} \quad (8.140)$$

which is identical to Eq. (8.90) obtained by the Rayleigh-Ritz method. Clearly, now the trial functions $\phi_1, \phi_2, \dots, \phi_n$ need be from the energy space \mathcal{K}_G^p only, i.e., they need be admissible functions. Although the Galerkin method is based on a different idea than the Rayleigh-Ritz method, because it yields the same matrices $K^{(n)}$ and $M^{(n)}$ as the Rayleigh-Ritz method, *in the case of self-adjoint systems, the Galerkin and the Rayleigh-Ritz methods are equivalent.*

ii. The collocation method

The collocation method is another widely used weighted residuals method, although it is not commonly recognized as such. In this case, the weighting functions are spatial Dirac delta functions located at various preselected points x_i of the system, or

$$\psi_i(x) = \delta(x - x_i), \quad i = 1, 2, \dots, n \quad (8.141)$$

and we note that the Dirac delta functions are from the class \mathcal{K}^{-1} . This is permissible, because the functions ψ_i not only are not differentiated, but they are part of an integrand. Indeed, inserting Eqs. (8.141) into Eqs. (8.127), we obtain

$$\begin{aligned} \int_0^L \psi_i R dx &= \int_0^L \delta(x - x_i) (Lw^{(n)} - \lambda^{(n)} m w^{(n)}) dx \\ &= Lw^{(n)}(x_i) - \lambda^{(n)} m(x_i) w^{(n)}(x_i) = 0, \quad i = 1, 2, \dots, n \end{aligned} \quad (8.142)$$

Introducing Eq. (8.124) into Eqs. (8.142), we obtain the algebraic eigenvalue problem given by Eq. (8.131), in which the matrices $K^{(n)}$ and $M^{(n)}$ have the elements

$$k_{ij} = \int_0^L \delta(x - x_i) L\phi_j dx = L\phi_j(x_i), \quad i, j = 1, 2, \dots, n \quad (8.143a)$$

and

$$m_{ij} = \int_0^L \delta(x - x_i) m \phi_j dx = m(x_i) \phi_j(x_i), \quad i, j = 1, 2, \dots, n \quad (8.143b)$$

respectively. It is obvious that the advantage of the collocation method over any other method is its simplicity, as manifested by the fact that *the evaluation of the coefficients k_{ij} and m_{ij} does not involve integrations*. The evaluation of k_{ij} requires differentiations and substitutions and the evaluation of m_{ij} involves mere substitutions, both relatively simple operations.

Equations (8.142) indicate that the differential equation is satisfied at the n preselected locations $x = x_i$ ($i = 1, 2, \dots, n$) throughout the domain $0 < x < L$, which explains the name of the method. Convergence can be argued in a heuristic fashion. To this end, we reinterpret the process defined by Eqs. (8.142) as driving the error to zero at the points $x = x_i$ ($i = 1, 2, \dots, n$). Convergence is achieved as $n \rightarrow \infty$, when the number of points at which the error has been annihilated becomes infinitely large, thus covering the entire domain $0 < x < L$.

One drawback of the collocation method is that it requires the solution of a nonsymmetric eigenvalue problem. Indeed, matrices $K^{(n)}$ and $M^{(n)}$ are nonsymmetric, and remain so even when the operator L is self-adjoint. This disadvantage is mitigated somewhat in the case of self-adjoint systems, in that the approximate eigenvalues $\lambda_r^{(n)}$ ($r = 1, 2, \dots, n$) retain the self-adjointness characteristic of being real. The self-adjointness characteristics do not extend to the eigenvectors, which are not mutually orthogonal, albeit they are real. In this regard, we recall from Sec. 4.8 that in the case of nonsymmetric eigenvalue problems there are two sets of eigenvectors, right and left eigenvectors, and one set is orthogonal to the other set, i.e., they are biorthogonal. Another disadvantage of the collocation method is that it requires the use of comparison functions. It should be noted that, unlike Galerkin's method, carrying out p integrations by parts to obviate the use of comparison functions is not an option here, as Dirac delta functions cannot be differentiated as required.

Other weighted residual methods include the method of least squares, the method of subdomains and the method of moments. In the method of least squares, the objective is to minimize the norm of the residual squared. It has a serious drawback in that the resulting algebraic eigenvalue problem is of order $2n$, i.e., twice the order in the Galerkin method or the collocation method. The method of subdomains is somewhat similar to the collocation method, except that the weighting functions are defined over subdomains D_i of D rather than at points. The method has the same disadvantages as the collocation method without the advantage of simplicity, as the evaluation of the coefficients k_{ij} and m_{ij} requires integrations. In the method of moments, the weighting functions represent powers of x . The method is more suitable for boundary-layer problems, and there seem to be no applications from the area of vibrations. Details of these methods can be found in Ref. 28.

8.9 FLUTTER OF CANTILEVER WINGS

A classical example of non-self-adjoint problems consists of the combined bending and torsional vibration of a cantilever aircraft wing in steady air flow (Fig. 8.13).

Before we begin describing the problem, we should define the two axes shown in Fig. 8.13a, the inertia axis and the elastic axis. The inertia axis is defined as the locus of the mass centers of the cross sections and the elastic axis as the locus of the shear centers, where a shear center is a point such that a shearing force passing through it produces pure bending and a moment about it produces pure torsion. We denote the bending deflection of the elastic axis by $w(x, t)$ and the torsional rotation about the elastic axis by $\theta(x, t)$, where w is positive if it acts downward and θ is positive if the leading edge is up (Fig. 8.13b). The angle θ is referred to as the local angle of attack. We take the x -axis to coincide with the elastic axis, which is assumed to be straight, and denote the distance between the leading edge and the elastic axis at any point x by $y_0(x)$, the distance between the elastic axis and the inertia axis by $y_\theta(x)$ and the chord length by $c(x)$. The bending deflection of the elastic axis is shown in Fig. 8.13c. The speed of the air flow relative to the wing, denoted by U , is assumed to be constant.

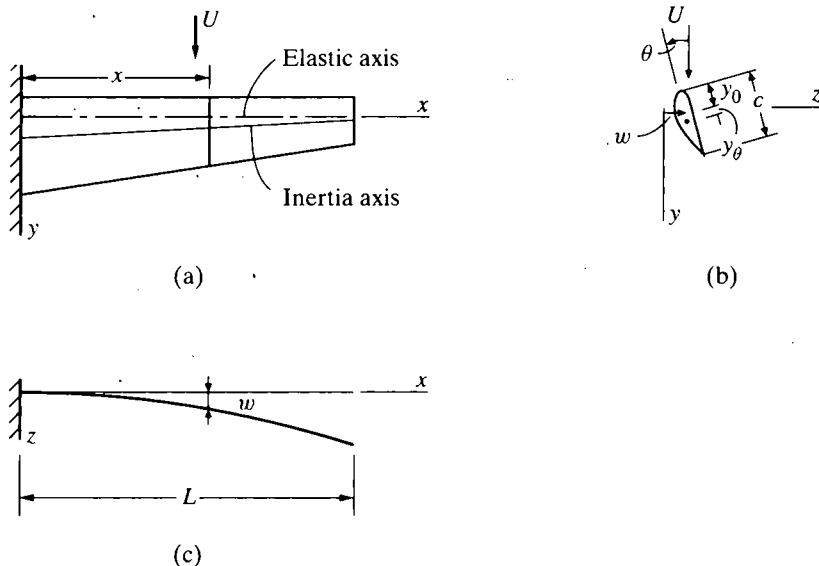


Figure 8.13 (a) Elastic axis and inertia axis for a cantilever aircraft wing in steady air flow (b) Wing cross section (c) Bending deflection of the elastic axis

From Ref. 10, the boundary-value problem for the free vibration of the wing in the presence of aerodynamic forces is described by the differential equations

$$\begin{aligned}
 \frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 w}{\partial x^2} \right) + m \frac{\partial^2 w}{\partial t^2} + m y_\theta \frac{\partial^2 \theta}{\partial t^2} + \frac{\rho U^2}{2} c \frac{dC_L}{d\theta} \left[\theta + \frac{1}{U} \frac{\partial w}{\partial t} \right. \\
 \left. + \frac{c}{U} \left(\frac{3}{4} - \frac{y_0}{c} \right) \frac{\partial \theta}{\partial t} \right] = 0, \quad 0 < x < L \quad (8.144a) \\
 - \frac{\partial}{\partial x} \left(GJ \frac{\partial \theta}{\partial x} \right) + m y_\theta \frac{\partial^2 w}{\partial t^2} + I_\theta \frac{\partial^2 \theta}{\partial t^2} + \frac{\rho U^2}{2} c^2 \left\{ \frac{c\pi}{8U} \frac{\partial \theta}{\partial t} \right.
 \end{aligned}$$

$$+ \left(\frac{1}{4} - \frac{y_0}{c} \right) \frac{dC_L}{d\theta} \left[\theta + \frac{1}{U} \frac{\partial w}{\partial t} + \frac{c}{U} \left(\frac{3}{4} - \frac{y_0}{c} \right) \frac{\partial \theta}{\partial t} \right] = 0, \quad 0 < x < L \quad (8.144b)$$

and the boundary conditions

$$w = 0, \quad \frac{\partial w}{\partial x} = 0, \quad \theta = 0 \quad \text{at } x = 0 \quad (8.145)$$

$$EI \frac{\partial^2 w}{\partial x^2} = 0, \quad \frac{\partial}{\partial x} \left(EI \frac{\partial^2 w}{\partial x^2} \right) = 0, \quad GJ \frac{\partial \theta}{\partial x} = 0 \quad \text{at } x = L$$

where EI is the bending stiffness, GJ the torsional stiffness, m the mass per unit length, I_θ the mass moment of inertia per unit length, ρ the air density and C_L the local lift coefficient. The aerodynamic forces and moments were derived by means of the so-called quasi-steady "strip theory" whereby the local lift coefficient C_L is proportional to the instantaneous angle of attack θ . The derivative $dC_L/d\theta$ is assumed to be constant, with a theoretical value of 2π for incompressible flow and an experimental value of somewhat less than 2π . The quasi-steady assumption implies that the aerodynamic forces and moments depend only on the instantaneous deformations and prior history of the motion can be ignored (Ref. 10), which simplifies the equations of motion greatly. Indeed, the resulting equations of motion and boundary conditions are linear and homogeneous. Still, the system is non-self-adjoint.

The boundary-value problem admits a solution in the exponential form

$$w(x, t) = W(x)e^{\lambda t}, \quad \theta(x, t) = \Theta(x)e^{\lambda t} \quad (8.146)$$

where $W(x)$, $\Theta(x)$ and λ are in general complex. Inserting Eqs. (8.146) into Eqs. (8.144) and (8.145) and dividing through by $e^{\lambda t}$, we obtain the differential eigenvalue problem consisting of the differential equations

$$\frac{d^2}{dx^2} \left(EI \frac{d^2 W}{dx^2} \right) + \frac{\rho U^2}{2} c \frac{dC_L}{d\theta} \Theta + \lambda \frac{\rho U}{2} c \frac{dC_L}{d\theta} \left[W + c \left(\frac{3}{4} - \frac{y_0}{c} \right) \Theta \right] + \lambda^2 m (W + y_\theta \Theta) = 0, \quad 0 < x < L \quad (8.147a)$$

$$-\frac{d}{dx} \left(GJ \frac{d\Theta}{dx} \right) + \frac{\rho U^2}{2} c^2 \left(\frac{1}{4} - \frac{y_0}{c} \right) \frac{dC_L}{d\theta} \Theta + \lambda \frac{\rho U}{2} c^2 \left\{ \left(\frac{1}{4} - \frac{y_0}{c} \right) \frac{dC_L}{d\theta} W + c \left[\left(\frac{1}{4} - \frac{y_0}{c} \right) \left(\frac{3}{4} - \frac{y_0}{c} \right) \frac{dC_L}{d\theta} + \frac{\pi}{8} \right] \Theta \right\} + \lambda^2 (m y_\theta W + I_\theta \Theta) = 0, \quad 0 < x < L \quad (8.147b)$$

and the boundary conditions

$$W = 0, \quad \frac{dW}{dx} = 0, \quad \Theta = 0 \quad \text{at } x = 0 \quad (8.148)$$

$$EI \frac{d^2W}{dx^2} = 0, \quad \frac{d}{dx} \left(EI \frac{d^2W}{dx^2} \right) = 0, \quad GJ \frac{d\Theta}{dx} = 0 \quad \text{at } x = L$$

The differential eigenvalue problem, Eqs. (8.147) and (8.148), has no closed-form solution, so that we consider an approximate solution by means of Galerkin's method. To this end, we assume a solution in the form

$$W(x) = \Phi_1^T(x) \mathbf{a}_1, \quad \Theta(x) = \Phi_2^T(x) \mathbf{a}_2 \quad (8.149)$$

in which Φ_1 and Φ_2 are vectors of comparison functions and \mathbf{a}_1 and \mathbf{a}_2 are vectors of undetermined coefficients, where Φ_1 and \mathbf{a}_1 are of dimension n_1 and Φ_2 and \mathbf{a}_2 of dimension n_2 , $n_1 + n_2 = n$. The vector Φ_1 satisfies the boundary conditions

$$\Phi_1(0) = \mathbf{0}, \quad \Phi_1' \Big|_{x=0} = \mathbf{0}, \quad EI \Phi_1'' \Big|_{x=L} = \mathbf{0}, \quad (EI \Phi_1'')' \Big|_{x=L} = \mathbf{0} \quad (8.150a)$$

and the vector Φ_2 satisfies the boundary conditions

$$\Phi_2(0) = \mathbf{0}, \quad GJ \Phi_2' \Big|_{x=L} = \mathbf{0} \quad (8.150b)$$

in which primes denote the customary derivatives with respect to x . Inserting Eqs. (8.149) into Eqs. (8.147), premultiplying Eq. (8.147a) by Φ_1 and Eq. (8.147b) by Φ_2 and integrating over the length of the beam, we obtain the algebraic eigenvalue problem

$$[K + U^2 H + \lambda UL + \lambda^2 M] \mathbf{a} = \mathbf{0} \quad (8.151)$$

where $\mathbf{a} = [\mathbf{a}_1^T \ \mathbf{a}_2^T]^T$ and the various matrices have the submatrices

$$K_{11} = \int_0^L \Phi_1 (EI \Phi_1''^T)' dx = \int_0^L EI \Phi_1'' \Phi_1''^T dx, \quad K_{12} = 0$$

$$K_{21} = 0, \quad K_{22} = - \int_0^L \Phi_2 (GJ \Phi_2'^T)' dx = \int_0^L GJ \Phi_2' \Phi_2'^T dx$$

$$H_{11} = 0, \quad H_{12} = \frac{\rho}{2} \frac{dC_L}{d\theta} \int_0^L c \Phi_1 \Phi_2^T dx$$

$$H_{21} = 0, \quad H_{22} = \frac{\rho}{2} \frac{dC_L}{d\theta} \int_0^L c^2 \left(\frac{1}{4} - \frac{y_0}{c} \right) \Phi_2 \Phi_2^T dx$$

$$L_{11} = \frac{\rho}{2} \frac{dC_L}{d\theta} \int_0^L c \Phi_1 \Phi_1^T dx, \quad L_{12} = \frac{\rho}{2} \frac{dC_L}{d\theta} \int_0^L c^2 \left(\frac{3}{4} - \frac{y_0}{c} \right) \Phi_1 \Phi_2^T dx$$

$$L_{21} = \frac{\rho}{2} \frac{dC_L}{d\theta} \int_0^L c^2 \left(\frac{1}{4} - \frac{y_0}{c} \right) \Phi_2 \Phi_1^T dx \quad (8.152)$$

$$L_{22} = \frac{\rho}{2} \int_0^L c^3 \left[\left(\frac{1}{4} - \frac{y_0}{c} \right) \left(\frac{3}{4} - \frac{y_0}{c} \right) \frac{dC_L}{d\theta} + \frac{\pi}{8} \right] \Phi_2 \Phi_2^T dx$$

$$M_{11} = \int_0^L m \Phi_1 \Phi_1^T dx, \quad M_{12} = \int_0^L m y_\theta \Phi_1 \Phi_2^T dx$$

$$M_{21} = \int_0^L m y_\theta \Phi_2 \Phi_1^T dx, \quad M_{22} = \int_0^L I_\theta \Phi_2 \Phi_2^T dx$$

The eigenvalue problem (8.151) can be expressed in the standard form

$$Ax = \lambda x \quad (8.153)$$

in which $x = [a^T \lambda a^T]^T$ and

$$A = \left[\begin{array}{c|c} 0 & I \\ \hline -M^{-1}(K + U^2 H) & -M^{-1}UL \end{array} \right] \quad (8.154)$$

The eigenvalue λ is a continuous function of the air speed U . When $U = 0$, the system is conservative and λ is pure imaginary. For $U \neq 0$, λ is in general complex, $\lambda = \alpha + i\omega$. It can be shown (Ref. 10) that for sufficiently small U and for $dC_L/d\theta < 2\pi$ the wing is losing energy to the surrounding air, so that the motion represents damped oscillation. This implies asymptotic stability, so that $\alpha < 0$. At some point, as U increases, α turns from negative to positive, as shown in Fig. 8.14, so that the motion turns from asymptotically stable to unstable. At the point $\alpha = 0$, at which the motion is merely stable and ready to become unstable, the air speed reaches the critical value U_{cr} . There can be more than one critical point but the lowest one is the most important, because in actual flight U increases from an initial zero value. There are two types of critical values, depending on the imaginary part ω . When $\alpha = 0$ and $\omega = 0$, so that $\lambda = 0$, the wing is said to be in critical *divergent* condition. When $\alpha = 0$ and $\omega \neq 0$ the wing is said to be in critical *flutter* condition. To compute U_{cr} , it is necessary to solve the eigenvalue problem repeatedly for increasing values of U , beginning with a small value. In the beginning all the eigenvalues will have negative real part. The first value of U for which the real part of one of the eigenvalues becomes zero is U_{cr} .

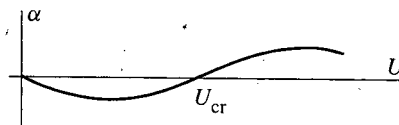


Figure 8.14 Eigenvalue real part versus the air speed

A first estimate of U_{cr} can be obtained by approximating W and Θ by means of a single term each, $n_1 = n_2 = 1$. Then, letting $\lambda = i\omega$ in Eq. (8.153) and pre-multiplying by block-diag $[I M]$, we can obtain U_{cr} from the determinantal equation

$$\det \begin{bmatrix} -i\omega & 0 & 1 & 0 \\ 0 & -i\omega & 0 & 1 \\ -k_{11} & -U_{cr}^2 h_{12} & -(i\omega m_{11} + U_{cr} l_{11}) & -(i\omega m_{12} + U_{cr} l_{12}) \\ 0 & -(k_{22} + U_{cr}^2 h_{22}) & -(i\omega m_{12} + U_{cr} l_{21}) & -(i\omega m_{22} + U_{cr} l_{22}) \end{bmatrix} \\ = \omega^4 (m_{11} m_{22} - m_{12}^2) - i\omega^3 U_{cr} [m_{11} l_{22} + m_{22} l_{11} - m_{12} (l_{12} + l_{21})] \\ - \omega^2 [U_{cr}^2 (l_{11} l_{22} - l_{12} l_{21} - h_{12} m_{12} + h_{22} m_{11}) + k_{22} m_{11} + k_{11} m_{22}] \\ - i\omega U_{cr} [U_{cr}^2 (h_{12} l_{21} + h_{22} l_{11}) + k_{22} l_{11} - k_{11} l_{22}] + k_{11} (U_{cr}^2 h_{22} + k_{22}) \\ = 0 \quad (8.155)$$

Equation (8.155) is complex, so that its satisfaction requires that both the real and imaginary part be zero, which permits a solution for both ω and U_{cr} . Indeed, equating the imaginary part to zero, we obtain

$$\omega^2 = \frac{U_{cr}^2 (h_{12} l_{21} + h_{22} l_{11}) + k_{22} l_{11} - k_{11} l_{22}}{m_{12} (l_{12} + l_{21}) - (m_{11} l_{22} + m_{22} l_{11})} \quad (8.156)$$

Then, inserting Eq. (8.156) into the real part of Eq. (8.155), we obtain the quadratic equation in U_{cr}^2

$$aU_{cr}^4 + bU_{cr}^2 + c = 0 \quad (8.157)$$

where

$$a = (h_{12} l_{21} + h_{22} l_{11}) \{ (h_{12} l_{21} + h_{22} l_{11}) - (m_{11} m_{22} - m_{12}^2) - (l_{11} l_{22} - l_{12} l_{21}) \\ - h_{12} m_{12} + h_{22} m_{11} \} [m_{12} (l_{12} + l_{21}) - (m_{11} l_{22} + m_{22} l_{11})] \\ b = 2(h_{12} l_{21} + h_{22} l_{11}) (k_{22} l_{11} - k_{11} l_{22}) (m_{11} m_{22} - m_{12}^2) \\ - [(h_{12} l_{21} + h_{22} l_{11}) (k_{22} m_{11} + k_{11} m_{22}) \\ + (k_{22} l_{11} - k_{11} l_{22}) (l_{11} l_{22} - l_{12} l_{21} - h_{12} m_{12} + h_{22} m_{11})] [m_{12} (l_{12} + l_{21}) \\ - (m_{11} l_{22} + m_{22} l_{11})] + k_{11} k_{22} [m_{12} (l_{12} + l_{21}) - (m_{11} l_{22} + m_{22} l_{11})]^2 \\ c = (k_{22} l_{11} - k_{11} l_{22})^2 (m_{11} m_{22} - m_{12}^2) \\ - (k_{22} l_{11} - k_{11} l_{22}) (k_{22} m_{11} + k_{11} m_{22}) [m_{12} (l_{12} + l_{21}) \\ - (m_{11} l_{22} + m_{22} l_{11})] + k_{11} k_{22} [m_{12} (l_{12} + l_{21}) - (m_{11} l_{22} + m_{22} l_{11})]^2 \quad (8.158)$$

The solution of Eq. (8.157) is simply

$$U_{cr}^2 = -\frac{b}{2a} \pm \frac{1}{2a} \sqrt{b^2 - 4ac} \quad (8.159)$$

so that there are four values for U_{cr} . For flutter to occur, at least one of these values must be real and positive. Then, an approximation for the critical air speed U_{cr} is given by the smallest real positive value.

Additional insights into the problem of dynamic aeroelastic instability of cantilever aircraft wings can be gained from Ref. 8.

8.10 SYSTEM RESPONSE BY APPROXIMATE METHODS

In deriving the system response, we must distinguish between conservative and non-conservative systems. Indeed, the process is quite different in the two cases, as for conservative systems the response can be obtained in the configuration space and for general nonconservative systems it is necessary to cast the problem in the state space. In this section, we discuss both cases, as follows:

i. Conservative systems

We are concerned with the solution of a boundary-value problem consisting of the differential equation

$$Lw(P, t) + m(P)\ddot{w}(P, t) = f(P, t), \quad P \text{ in } D \quad (8.160)$$

and the boundary conditions

$$B_i w(P, t) = 0, \quad i = 1, 2, \dots, p, \quad P \text{ on } S \quad (8.161)$$

The various terms are as defined in Sec. 7.16. The solution of Eqs. (8.160) and (8.161) is subject to initial conditions in the form of the initial displacement $w(P, 0) = w_0(P)$ and the initial velocity $\dot{w}(P, 0) = v_0(P)$.

We consider the case in which the boundary-value problem, Eqs. (8.160) and (8.161), does not admit a closed-form solution, so that the interest lies in an approximate one. Assuming that the operator L is self-adjoint, we propose to derive an approximate solution in conjunction with the Rayleigh-Ritz method.

To this end, we assume an approximate solution of Eq. (8.160) in the form

$$w(P, t) \cong w^{(n)}(P, t) = \boldsymbol{\phi}^T(P)\mathbf{q}(t) \quad (8.162)$$

in which $\boldsymbol{\phi} = [\phi_1 \phi_2 \dots \phi_n]^T$ is an n -vector of comparison functions and $\mathbf{q} = [q_1 q_2 \dots q_n]^T$ is an n -vector of generalized coordinates. Note that such a solution satisfies the boundary conditions (8.161) automatically. Inserting Eq. (8.162) into Eq. (8.160), premultiplying by $\boldsymbol{\phi}$ and integrating over the domain D , we obtain the spatially discretized system

$$M^{(n)}\ddot{\mathbf{q}}(t) + K^{(n)}\mathbf{q}(t) = \mathbf{Q}(t) \quad (8.163)$$

where

$$M^{(n)} = \int_D m\boldsymbol{\phi}\boldsymbol{\phi}^T dD, \quad K^{(n)} = \int_D \boldsymbol{\phi}L\boldsymbol{\phi}^T dD \quad (8.164a, b)$$

are symmetric $n \times n$ mass and stiffness matrices and

$$\mathbf{Q}(t) = \int_D \Phi f dD \quad (8.165)$$

is an n -dimensional generalized force vector.

To obtain the solution of Eq. (8.163), we first solve the eigenvalue problem

$$\mathbf{K}^{(n)} \mathbf{u} = \lambda^{(n)} \mathbf{M}^{(n)} \mathbf{u} \quad (8.166)$$

by one of the approaches discussed in Chapter 6. The solution consists of the eigenvalues $\lambda_r^{(n)} = (\omega_r^{(n)})^2$, where $\omega_r^{(n)}$ are the approximate natural frequencies, and the eigenvectors $\mathbf{u}_r^{(n)}$ ($r = 1, 2, \dots, n$). The eigenvectors are orthogonal with respect to $\mathbf{M}^{(n)}$ and $\mathbf{K}^{(n)}$ and are assumed to be normalized so as to satisfy

$$\mathbf{u}_s^{(n)T} \mathbf{M}^{(n)} \mathbf{u}_r^{(n)} = \delta_{rs}, \quad \mathbf{u}_s^{(n)T} \mathbf{K}^{(n)} \mathbf{u}_r^{(n)} = \lambda_r^{(n)} \delta_{rs}, \quad r, s = 1, 2, \dots, n \quad (8.167a, b)$$

At this point, we pause to consider questions of accuracy. We recall from Sec. 8.5 that not all eigensolutions $\lambda_r^{(n)}$, $\mathbf{u}_r^{(n)}$ ($r = 1, 2, \dots, n$) are accurate. In particular, the lower eigensolutions tend to be more accurate than the higher ones. In view of this, we assume that n is sufficiently large that the lowest m eigensolutions can be regarded as accurate. Moreover, we assume that m is sufficiently large that no mode higher than m is excited. Of course, this assumption must be verified by checking the extent of participation of the mode $m + 1$. Then, we consider a solution of Eq. (8.163) in the truncated form

$$\mathbf{q}(t) = \mathbf{U}_{tr} \boldsymbol{\eta}(t) \quad (8.168)$$

in which $\mathbf{U}_{tr} = [\mathbf{u}_1 \ \mathbf{u}_2 \ \dots \ \mathbf{u}_m]$ is an $n \times m$ truncated modal matrix and $\boldsymbol{\eta}$ is a truncated m -vector of modal coordinates. Introducing Eq. (8.168) into Eq. (8.163), premultiplying by \mathbf{U}_{tr}^T and using the orthonormality relations, Eqs. (8.167), we obtain the modal equation

$$\ddot{\boldsymbol{\eta}}(t) + \Lambda_{tr}^{(n)} \boldsymbol{\eta}(t) = \mathbf{N}(t) \quad (8.169)$$

where $\Lambda_{tr}^{(n)} = \text{diag}(\lambda_1^{(n)} \ \lambda_2^{(n)} \ \dots \ \lambda_m^{(n)}) = \text{diag}[(\omega_1^{(n)})^2 \ (\omega_2^{(n)})^2 \ \dots \ (\omega_m^{(n)})^2]$ is a truncated diagonal matrix of approximate natural frequencies squared and

$$\mathbf{N}(t) = \mathbf{U}_{tr}^T \mathbf{Q}(t) \quad (8.170)$$

is the truncated m -vector of modal forces. Equation (8.169) represents a set of independent equations of the type examined in Sec. 4.10. Hence, from Sec. 4.10, we write the response

$$\eta_r(t) = \frac{1}{\omega_r^{(n)}} \int_0^t N_r(t - \tau) \sin \omega_r^{(n)} \tau d\tau + \eta_r(0) \cos \omega_r^{(n)} t + \frac{\dot{\eta}_r(0)}{\omega_r^{(n)}} \sin \omega_r^{(n)} t, \quad r = 1, 2, \dots, m \quad (8.171)$$

in which $\eta_r(0)$ and $\dot{\eta}_r(0)$ are initial modal displacement and velocity depending on the actual initial displacement $w_0(P)$ and initial velocity $v_0(P)$, respectively. To

obtain the relation between the two types of quantities, we insert Eq. (8.168) into Eq. (8.162) and write the system response in the form

$$\begin{aligned} w(P, t) &\cong w^{(n)}(P, t) = \Phi^T(P)\mathbf{q}(t) = \Phi^T(P)U_{tr}\boldsymbol{\eta}(t) \\ &= \sum_{r=1}^m \Phi^T(P)\mathbf{u}_r\eta_r(t) = \sum_{r=1}^m w_r^{(n)}(P)\eta_r(t) \end{aligned} \quad (8.172)$$

where, by analogy with Eq. (8.91),

$$w_r^{(n)}(P) = \Phi^T(P)\mathbf{u}_r = \mathbf{u}_r^T\Phi(P) \quad (8.173)$$

are the approximate modes of the distributed system. Using Eqs. (8.164a) and (8.167a), these modes can be shown to satisfy the orthonormality relations

$$\int_D m(P)w_s^{(n)}(P)w_r^{(n)}(P) dD(P) = \delta_{rs}, \quad r, s = 1, 2, \dots, n \quad (8.174)$$

But, letting $t = 0$ in Eq. (8.172), we can write

$$w(P, 0) = w_0(P) \cong \sum_{r=1}^m w_r^{(n)}(P)\eta_r(0) \quad (8.175)$$

Hence, multiplying Eq. (8.175) by $m(P)w_s^{(n)}(P)$, integrating over the domain D and considering Eq. (8.174), we obtain

$$\eta_r(0) = \int_D m(P)w_r^{(n)}(P)w_0(P) dD(P), \quad r = 1, 2, \dots, m \quad (8.176a)$$

Similarly,

$$\dot{\eta}_r(0) = \int_D m(P)w_r^{(n)}(P)v_0(P) dD(P), \quad r = 1, 2, \dots, m \quad (8.176b)$$

The formal approximate response of the system is obtained by inserting Eq. (8.171) in conjunction with Eqs. (8.176), as well as Eqs. (8.165) and (8.170), into Eq. (8.172).

ii. Nonconservative systems

In the general case of damping, the boundary-value problem can be described by the differential equation (Sec. 7.18)

$$Lw(P, t) + C\dot{w}(P, t) + m\ddot{w}(P, t) = f(P, t), \quad P \text{ in } D \quad (8.177)$$

and the boundary conditions

$$B_i w(P, t) = 0, \quad P \text{ on } S, \quad i = 1, 2, \dots, k \quad (8.178a)$$

$$B_i w(P, t) + C_i \dot{w}(P, t) = 0, \quad P \text{ on } S, \quad i = k + 1, k + 2, \dots, p \quad (8.178b)$$

where the various terms are as defined in Sec. 7.18. Alternatively, boundary conditions (8.178b) can take the form

$$B_i w(P, t) + C_i \dot{w}(P, t) = 0, \quad P \text{ on } S, \quad i = k + 1, k + 2, \dots, p \quad (8.178c)$$

and note that Eq. (8.178b) implies the presence of a lumped mass at S and Eq. (8.178c) the presence of a damper.

In Sec. 7.18, we discussed various cases of damping in which the classical modal analysis permitted a closed-form solution of Eqs. (8.177) and (8.178), where "classical" is in the sense that the eigenfunctions corresponding to the self-adjoint system, i.e., the undamped system, are capable of diagonalizing the damped system. The most important of these is the case of proportional damping. In this section, we consider the general case in which the classical modal analysis is not able to diagonalize the system, so that no closed-form solution is possible.

Even when the operator L is self-adjoint, general damping renders the system non-self-adjoint, which implies complex eigensolutions. As demonstrated in Refs. 14 and 32, approximate solutions of differential eigenvalue problems for non-self-adjoint systems in terms of quasi-comparison functions exhibit superior convergence characteristics. In view of this, we consider a solution of Eq. (8.177) by the Galerkin method in the form of the linear combination

$$w(P, t) \cong w^{(n)}(P, t) = \Phi^T(P) \mathbf{q}(t) \quad (8.179)$$

where $\Phi(P)$ is an n -vector of quasi-comparison functions and $\mathbf{q}(t)$ an n -vector of generalized coordinates. Inserting Eq. (8.179) into Eq. (8.177), premultiplying by $\Phi(P)$ and integrating over D , we obtain a set of discretized equations having the form

$$M \ddot{\mathbf{q}}(t) + C \dot{\mathbf{q}}(t) + K \mathbf{q}(t) = \mathbf{Q}(t) \quad (8.180)$$

in which

$$M = \int_D m \Phi \Phi^T dD, \quad C = \int_D \Phi C \Phi^T dD, \quad K = \int_D \Phi L \Phi^T dD \quad (8.181)$$

are a mass matrix, damping matrix and stiffness matrix, respectively, and

$$\mathbf{Q}(t) = \int_D \Phi f dD \quad (8.182)$$

is a generalized force vector. No confusion should arise from the fact that we used the same notation for the damping matrix and the damping operator.

To obtain a solution of Eq. (8.180), we use the approach of Sec. 4.10 and cast the equation in the state form

$$\dot{\mathbf{x}}(t) = A \mathbf{x}(t) + B \mathbf{Q}(t) \quad (8.183)$$

where $\mathbf{x} = [\mathbf{q}^T \dot{\mathbf{q}}^T]^T$ is the state vector and

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} \quad (8.184)$$

are coefficient matrices. As shown in Sec. 4.10, the solution of Eq. (8.183) is

$$\mathbf{x}(t) = \Phi(t)\mathbf{x}(0) + \int_0^t \Phi(t - \tau)BQ(\tau) d\tau \quad (8.185)$$

where $\mathbf{x}(0) = [\mathbf{q}^T(0) \dot{\mathbf{q}}^T(0)]^T$ is the initial state vector and $\Phi(t - \tau) = \exp A(t - \tau)$ is the transition matrix. Note that the initial generalized displacement and velocity vectors can be obtained from the actual initial displacement $w(P, 0)$ and velocity $\dot{w}(P, 0)$ by writing

$$\mathbf{q}(0) = M^{-1} \int_D m\phi w(P, 0) dD, \quad \dot{\mathbf{q}}(0) = M^{-1} \int_D m\phi \dot{w}(P, 0) dD \quad (8.186a, b)$$

The solution of Eq. (8.183) can also be obtained by means of a state version of the modal analysis (Sec. 4.10). A modal solution of Eq. (8.183) has the advantage of permitting truncation of inaccurate higher modes, provided there is some assurance that they are not excited.

8.11 COMPONENT-MODE SYNTHESIS

In the 1950s, it became apparent that the techniques for analyzing complex structures were woefully inadequate. This led to the independent development of two techniques, the finite element method and component-mode synthesis. The first to emerge was the finite element method (Ref. 46), initially conceived as a static analysis according to which the structure is divided into small subdomains, referred to as finite elements, and the deformation over each element is described in terms of interpolation functions. Since its early beginnings, the finite element method has grown significantly in scope, finding application in a large variety of engineering areas, as well as in applied mathematics. The entire Chapter 9 is devoted to the finite element method.

Following by a few years, Hurty (Refs. 18–20) developed the component-mode synthesis as a technique for the dynamic analysis of structures consisting of assemblages of substructures. The component-mode synthesis adopts a different point of view from the finite element method, as the modeling is carried out on a much larger scale. Indeed, the general idea is to describe the motion separately over each of the substructures, referred to as components, and then constrain the components to work together as a single structure. In fact, component-mode synthesis can be regarded as an extension of the assumed-modes method to flexible multibody systems. Indeed, as in the assumed-modes method, the motion of each component is described by a linear combination of modes multiplied by time-dependent generalized coordinates. Hurty divides the component modes into three types, rigid-body modes, constraint modes and normal modes. But, because each component is modeled separately, there are redundant coordinates, as points shared by two components undergo the same motions. The removal of redundant coordinates is carried out during an assembling process in which the constituent components are constrained to act as a whole structure.

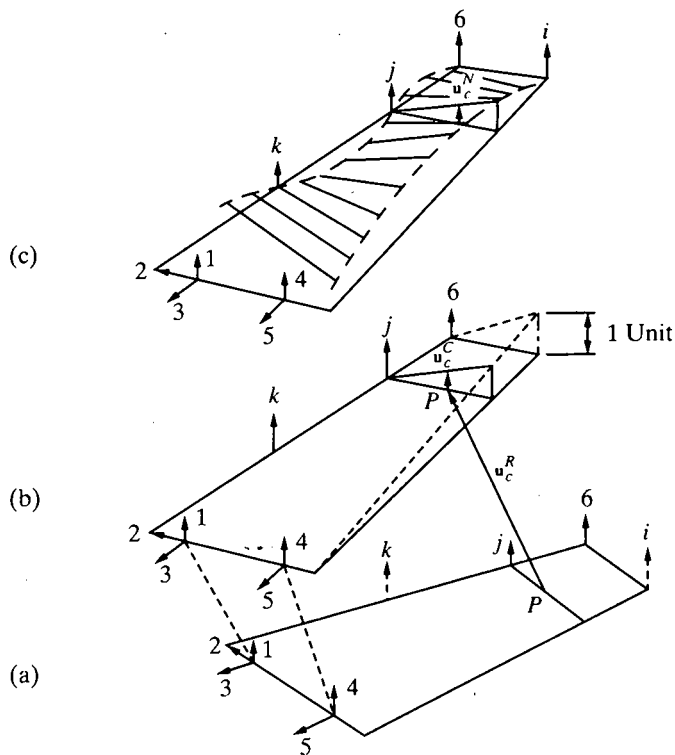


Figure 8.15 (a) Undisplaced, undeformed component (b) Component after it has undergone rigid-body displacement (c) Displacements relative to fixed-constraint motions

We consider a structure consisting of N components, concentrate on a typical component c ($c = 1, 2, \dots, N$) and express the total displacement vector of an arbitrary point $P(x, y, z)$ in the form

$$u_c(P, t) = u_c^R(P, t) + u_c^C(P, t) + u_c^N(P, t) \quad (8.187)$$

where u_c^R is a rigid-body displacement vector, u_c^C a “constraint displacement” vector and u_c^N a displacement vector relative to the fixed constraints, as shown in Fig. 8.15. Figure 8.15a displays the undisplaced, undeformed component, with a set of constraints indicated by arrows. The constraints labeled 1–6 are regarded as statically determinate and the constraints i, j and k are considered as redundant. All the constraints are movable and the result of component c being attached to adjacent components that are themselves in motion. The points shared with a given adjacent component represent the *interface* with the component in question. Figure 8.15b shows the component after it has undergone six arbitrary rigid-body displacements, defined uniquely by the displacements of the six statically determinate constraints. These displacements are represented by the rigid-body displacement vector u_c^R . The constraint displacement vector u_c^C results from the motion of the redundant coordinates relative to the rigid-body motions and it represents a linear combination of

displacements obtained by letting each of the redundant constraints undergo an arbitrary displacement, as shown in Fig. 8.15b. In addition to these displacements, there is the displacement \mathbf{u}_c^N of point P relative to the constraint motions, as depicted in Fig. 8.15c.

At this point, we begin the discretization process. To this end, we represent the three types of displacements as linear combinations of space-dependent functions multiplied by time-dependent generalized coordinates in the form

$$\begin{aligned}\mathbf{u}_c^R(P, t) &= \Phi_c^R(P)\zeta_c^R(t), \\ \mathbf{u}_c^C(P, t) &= \Phi_c^C(P)\zeta_c^C(t), \\ \mathbf{u}_c^N(P, t) &= \Phi_c^N(P)\zeta_c^N(t)\end{aligned}\quad (8.188)$$

where Φ_c^R is in general a 3×6 matrix of rigid-body modes, Φ_c^C is a matrix of constraint modes with three rows and as many columns as the number of redundant constraints, a finite number, Φ_c^N is a matrix of "fixed-constraint" normal modes with three rows and a given finite number of columns and ζ_c^R , ζ_c^C and ζ_c^N are associated generalized displacement vectors. Equations (8.187) and (8.188) can be combined into the single expression

$$\mathbf{u}_c(P, t) = \Phi_c(P)\zeta_c(t) \quad (8.189)$$

in which $\Phi_c = [\Phi_c^R \ \Phi_c^C \ \Phi_c^N]$ and $\zeta_c = [(\zeta_c^R)^T \ (\zeta_c^C)^T \ (\zeta_c^N)^T]^T$, and we observe that the net effect of Eq. (8.189) is to represent the motion of the distributed component by a finite number of degrees of freedom.

The next step is to derive the equations of motion for the discretized component. Using the analogy with the assumed-modes method (Sec. 8.7), we carry out this task by means of Lagrange's equations. To this end, we use Eq. (8.189) and write the component kinetic energy in the discretized form

$$T_c(t) = \frac{1}{2} \int_{D_c} m_c(P) \dot{\mathbf{u}}_c^T(P, t) \dot{\mathbf{u}}_c(P, t) dD_c = \frac{1}{2} \dot{\zeta}_c^T(t) M_c \dot{\zeta}_c(t) \quad (8.190)$$

where D_c is the domain of component c and

$$M_c = \int_{D_c} m_c(P) \Phi_c^T(P) \Phi_c(P) dD_c \quad (8.191)$$

is the corresponding component mass matrix. Assuming that the component is subjected to distributed viscous damping and denoting by $c_c(P)$ the distributed damping coefficient, we obtain the discretized Rayleigh's dissipation function (Sec. 4.1) for the component

$$\mathcal{F}_c = \frac{1}{2} \int_{D_c} c_c(P) \dot{\mathbf{u}}_c^T(P, t) \dot{\mathbf{u}}_c(P, t) dD_c = \frac{1}{2} \dot{\zeta}_c^T(t) C_c \dot{\zeta}_c(t) \quad (8.192)$$

in which

$$C_c = \int_{D_c} c_c(P) \Phi_c^T(P) \Phi_c(P) dD_c \quad (8.193)$$

is the associated component viscous damping matrix. Moreover, using the notation of Sec. 8.7, the component potential energy takes the discretized form

$$V_c(t) = \frac{1}{2}[\mathbf{u}_c^T, \mathbf{u}_c] = \frac{1}{2}\boldsymbol{\zeta}_c^T(t)K_c\boldsymbol{\zeta}_c(t) \quad (8.194)$$

where

$$K_c = [\Phi_c^T(P), \Phi_c(P)] \quad (8.195)$$

is the corresponding component stiffness matrix. Finally, assuming that the component is acted upon by the distributed force $\mathbf{f}_c(P, t)$, the discretized virtual work for the component can be written as

$$\delta\overline{W}_c(t) = \int_{D_c} \mathbf{f}_c^T(P, t)\delta\mathbf{u}_c(P, t) dD_c = \mathbf{Z}_c^T(t)\delta\boldsymbol{\zeta}_c(t) \quad (8.196)$$

in which

$$\mathbf{Z}_c(t) = \int_{D_c} \Phi_c^T(P)\mathbf{f}_c(P, t) dD_c \quad (8.197)$$

is the associated generalized force vector for the component, and we note that \mathbf{Z}_c excludes viscous damping forces.

By analogy with Eq. (4.14), Lagrange's equations of motion for the discretized component have the symbolic form

$$\frac{d}{dt} \left(\frac{\partial L_c}{\partial \dot{\boldsymbol{\zeta}}_c} \right) - \frac{\partial L_c}{\partial \boldsymbol{\zeta}_c} + \frac{\partial \mathcal{F}_c}{\partial \dot{\boldsymbol{\zeta}}_c} = \mathbf{Z}_c \quad (8.198)$$

where $L_c = T_c - V_c$ is the component Lagrangian. Hence, using Eqs. (8.190), (8.192) and (8.194), we obtain the component equations of motion

$$M_c\ddot{\boldsymbol{\zeta}}_c(t) + C_c\dot{\boldsymbol{\zeta}}_c(t) + K_c\boldsymbol{\zeta}_c(t) = \mathbf{Z}_c(t), \quad c = 1, 2, \dots, N \quad (8.199)$$

Next, we turn our attention to the assembling process. To this end, we first collect all component equations into the "disjoint" set of equations

$$M^{(d)}\ddot{\boldsymbol{\zeta}}(t) + C^{(d)}\dot{\boldsymbol{\zeta}}(t) + K^{(d)}\boldsymbol{\zeta}(t) = \mathbf{Z}(t) \quad (8.200)$$

where $\boldsymbol{\zeta} = [\boldsymbol{\zeta}_1^T \boldsymbol{\zeta}_2^T \dots \boldsymbol{\zeta}_N^T]^T$, $\mathbf{Z} = [\mathbf{Z}_1^T \mathbf{Z}_2^T \dots \mathbf{Z}_N^T]^T$ are disjoint displacement and force vectors and

$$M^{(d)} = \text{block-diag}[M_c], \quad C^{(d)} = \text{block-diag}[C_c], \quad K^{(d)} = \text{block-diag}[K_c] \quad (8.201)$$

are disjoint coefficient matrices. Of course, according to Eq. (8.200), the components still act independently of one another, as the vector $\boldsymbol{\zeta}$ contains all the redundant coordinates. The assembling process is designed to cause the disjoint set of components to act as a single structure, which implies elimination of the redundant coordinates. If we assume that two adjacent components r and s are joined together so that there are no relative translations and rotations between the components at the interface, then we have

$$\mathbf{u}_r = \mathbf{u}_s, \quad \boldsymbol{\theta}_r = \boldsymbol{\theta}_s \quad (8.202)$$

where θ represents a rotation vector, which implies that the rotations are small. But, the translational and rotational displacements at the interfaces are related to the generalized displacement vector ζ by means of equations of the type (8.189), in which P represents the position of the interface points. In view of this, we can combine Eqs. (8.202) corresponding to all interfaces into a single constraint equation having the general form

$$A\zeta = \mathbf{0} \quad (8.203)$$

in which A is a $c \times m$ matrix, where c is the number of constraint equations. Then, dividing the vector ζ into an n -vector \mathbf{q} of independent variables and a vector \mathbf{d} of dependent variables and partitioning the matrix A as follows:

$$A = \left[A_1 \mid A_2 \right]$$

where A_1 is a $c \times n$ matrix and A_2 is a $c \times c$ nonsingular matrix, we can rewrite Eq. (8.203) as

$$A_1\mathbf{q} + A_2\mathbf{d} = \mathbf{0} \quad (8.205)$$

which yields

$$\mathbf{d} = -A_2^{-1}A_1\mathbf{q} \quad (8.206)$$

Equation (8.206) permits us to write a relation between the n -vector $\mathbf{q}(t)$ of independent generalized coordinates for the full structure and the m -vector $\zeta(t)$, which includes redundant coordinates, in the form

$$\zeta(t) = B\mathbf{q}(t) \quad (8.207)$$

in which

$$B = \begin{bmatrix} I \\ -A_2^{-1}A_1 \end{bmatrix} \quad (8.208)$$

is an $m \times n$ matrix, where $n = m - c$ is the number of degrees of freedom of the model of the full structure. Introducing Eq. (8.207) into Eq. (8.200) and premultiplying by B^T , we obtain the *coupled* equations of motion of the full structure

$$M\ddot{\mathbf{q}}(t) + C\dot{\mathbf{q}}(t) + K\mathbf{q}(t) = \mathbf{Q}(t) \quad (8.209)$$

where

$$M = B^T M^{(d)} B, \quad C = B^T C^{(d)} B, \quad K = B^T K^{(d)} B \quad (8.210)$$

are $n \times n$ coefficient matrices and

$$\mathbf{Q}(t) = B^T \mathbf{Z}(t) \quad (8.211)$$

is an n -dimensional generalized force vector.

There are two main questions still to be addressed, namely, the choice of constraint modes and normal modes and the nature of the approximation. Hurty (Ref. 19) answers the first question, but does not address the second. In particular, the constraint modes are defined clearly in Ref. 19 as "... displacements produced by giving each redundant constraint in turn an arbitrary displacement while keeping all

other constraints fixed." As far as the normal modes are concerned, they are defined in Ref. 19 somewhat more ambiguously by stating that "... it is convenient, although not necessary, to think of these as the 'fixed-constraint' natural modes of vibration of the structure." In practice, they have been widely interpreted as fixed-fixed component modes. The nature of the approximation is significantly more involved in the component-mode synthesis than in the assumed-modes method. The question is essentially how well a combination of the constraint modes and component normal modes can approximate the behavior of the component in the context of a full structure. The question of inaccuracies introduced by the use of constraint modes arises only in problems in which the interface is a line, such as when the components are two-dimensional, rather than a point, such as when the components are one-dimensional. Clearly, if the interface is a line, then there is an infinite number of interface points, and not a finite number. As a result, the boundary conditions internal to the structure at the interface cannot be satisfied exactly. To explain this point, it is convenient to conceive of a structure with the interface consisting of a finite number of points, where the points coincide with the location of the redundant constraints. We refer to this fictitious structure as an *intermediate structure* (Ref. 27), because the model lies somewhere between the disjoint structure and the fully coupled structure, where in the latter the full infinity of interface points is considered. It is obvious that *the results of the component-modes synthesis are valid for the intermediate structure*, and not necessarily for the actual structure. The component normal modes do not require elaboration, as fixed-fixed modes are generally well defined and the nature of the approximation is the same as in the Rayleigh-Ritz method applied to a single elastic member.

Since publication of Hurty's component-mode synthesis, there have been many attempts to enhance its accuracy. The issues of component modeling and of the manner in which the various components are made to work together as a whole structure have received a great deal of attention. Hurty's method makes a sharp distinction between determinate and indeterminate constraints. In reality, no such distinction exists, and all interface constraints should really receive equal treatment. This is the essence of an idea advanced by Craig and Bampton (Ref. 6), who suggested a simplification in the treatment of the component rigid-body modes by eliminating the separation of the boundary constraints into determinate and indeterminate ones. All constraint modes are defined as the mode shapes due to successive unit displacements at each of the interface points, with all other interface points being fixed. Craig and Bampton envision an entirely discretized structure, with the constraint modes being generated by matrix operations on a computer. The normal modes and the elimination of redundant coordinates remain essentially the same as envisioned by Hurty.

The method described above is generally referred to as a "fixed constraint mode" method, because the modes used to describe the motion correspond to fixed constraints. To account for motions caused by loads at unconstrained points, in developing a computer program for the method, Bamford (Ref. 1) introduced another class of displacement modes, referred to as "attachment modes" and defined as the static deflection of the component resulting from a unit force applied at one boundary coordinate while the remaining coordinates are force free. The possibility of

using unconstrained modes has been suggested by Goldman (Ref. 12) and by Hou (Ref. 17). Some ill-conditioning problems have been experienced in using unconstrained modes. The use of unconstrained modes has also been proposed by Dowell (Refs. 7 and 21), who used Lagrange's multipliers to enforce continuity at interfaces. A somewhat different type of mode selection is advocated by MacNeal (Ref. 24) and Rubin (Ref. 42). Indeed, they use a low-frequency subset of the free-free component modes together with "residual modes," some shape functions designed to capture the contribution from the truncated normal modes. Truncation problems have been discussed by Kuhar and Stahle (Ref. 22), who present a condensation scheme similar to dynamic condensation (Ref. 28), and by Hintz (Ref. 15), who identified Hurty's elimination of redundant coordinates as a static condensation (Ref. 28).

A method developed by Benfield and Hruda (Ref. 2), and known as *component-mode substitution*, resembles both Hurty's component-mode synthesis and Gladwell's branch-mode analysis (Ref. 11). The interest in component-mode substitution lies in its connection with component-mode synthesis. The method differs from component-mode synthesis in that the component modes need not be constrained and can be free-free. More importantly, however, the component-mode substitution does not require that the generalized coordinates of the static constraint modes appear in the final formulation, thus reducing the number of degrees of freedom of the over all model. The efficiency of the method can be improved by applying stiffness and inertial loadings to the free interface coordinates of the component under consideration to account for the effect of the remaining components. It is this aspect of the method that is of interest here, as this procedure gives rise to so-called "loaded interface modes."

A comparison of the various choices of modes for Hurty's component-mode synthesis, focusing on the mode sets advocated by Craig-Bampton, MacNeal and Rubin and Benfield-Hruda is presented by Spanos and Tsuha (Ref. 43). In addition, they discuss the effect of controls on the reduction of the component order.

Finally, there is the question of the manner in which the redundant coordinates at interfaces are handled. In Hurty's component-mode synthesis, the redundant coordinates are eliminated by means of the linear transformation given by Eq. (8.207). On the other hand, Dowell (Ref. 7) and Klein and Dowell (Ref. 21) use Lagrange's multipliers for the same purpose.

The component-mode synthesis represents a sound heuristic, physically motivated approach to the dynamics of complex structures. With a proper choice of modes, the method should be capable of yielding reasonable results with a relatively small number of degrees of freedom. However, unlike the assumed-modes method, which could invoke the Rayleigh-Ritz theory to claim convergence, the component-mode synthesis cannot make such claims. Indeed, it can be argued at best that the eigenvalues of the model converge to the eigenvalues of the intermediate structure. The degree to which the intermediate structure approximates the actual structure is still an open question, and the answer depends on the extent to which the linear combination of constraint modes and normal modes is capable of satisfying the internal boundary conditions at the interfaces. This question is addressed in Sec. 8.12.

Because the finite element method and component-mode synthesis were developed with the same objective in mind, namely, to analyze complex structures, there

is the perception that the two methods compete with one another. In this regard, it should be pointed out that, whereas the finite element method is capable of producing an accurate mathematical model of a complex structure, without many of the problems associated with the selection of suitable sets of modes, the model is likely to be of extremely large order. On the other hand, a model produced by component-mode synthesis is likely to be of significantly smaller order. In view of this, under certain circumstances the two methods can be regarded as complementing one another. Indeed, it is possible to produce the component normal modes by means of the finite element method and then use them in conjunction with the component-mode synthesis to reduce the number of degrees of freedom of the model. In fact, such an approach would be consistent with the process of replacing interface lines by interface points, as discussed above. A word of caution is in order, however, as the number of interface points in a finite element model tends to be much larger than the number of interface points in a component-mode synthesis, so that the concept of an intermediate structure still applies.

8.12 SUBSTRUCTURE SYNTHESIS

Many structures, such as fixed-wing aircraft, helicopters, flexible spacecraft, flexible robots, a variety of civil structures, etc., can be modeled as assemblages of interacting flexible bodies. Hurty's component-mode synthesis discussed in Sec. 8.10 consists of representing the motion of each of the constituent substructures by means of linear combinations of rigid-body modes, constraint modes and normal modes. Various other investigators use different sets of modes in an attempt to improve the convergence of the component-mode synthesis. Note that, to generate component modes, it is generally assumed that one must solve a component eigenvalue problem (see, for example, Ref. 43).

The component-mode synthesis is basically an extension of the assumed-modes method to flexible multibodies. In essence, the various approaches discussed in Sec. 8.10, whereby different sets of component modes are used, represent special cases of the Rayleigh-Ritz method (Refs. 26 and 27). Clearly, a proper choice of component modes can produce very good results. However, in the spirit of the Rayleigh-Ritz theory, approximate solutions can be constructed from the space of admissible functions, i.e., the functions need not be modes at all. In this regard, it should be mentioned that the practice of using component modes has practical implications when the various substructures are manufactured by different companies and the structural characteristics are provided in the form of component modes. In such cases, the component-mode synthesis can be used to generate a structural model for the fully assembled structure. Still, however defined, component modes represent mere subspaces of the much larger space of admissible functions, and component-mode synthesis is part of a larger picture.

In this section, we discuss a method for the modeling of flexible multibody systems developed in Ref. 31 for systems of the type shown in Fig. 8.16. The method represents an extension of the theory developed in Sec. 8.6 for single elastic members to flexible multibody systems. Because the theory is based on expansions of the solution over the individual substructures in terms of special classes of admissible

functions, rather than modes, and the mathematical formulation of the equations of motion is entirely different from that in component-mode synthesis, the method is referred to as *substructure synthesis*.

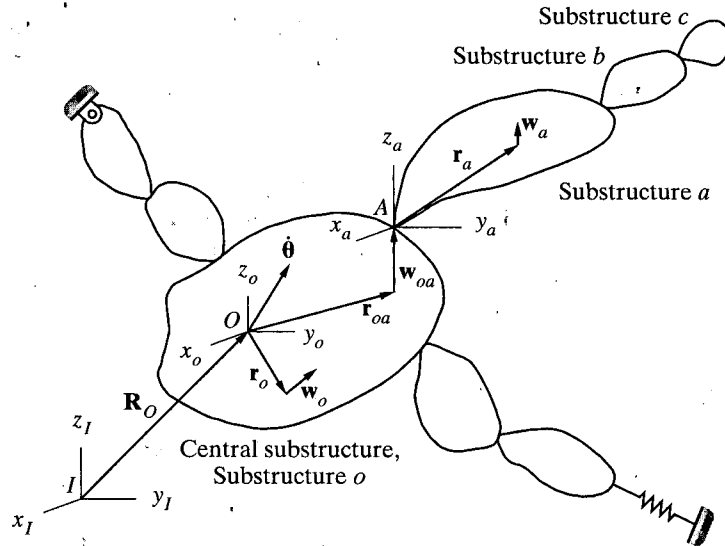


Figure 8.16 Flexible multibody system

In the Rayleigh-Ritz method, a minimizing sequence is constructed from the space of admissible functions or comparison functions, depending on the form of Rayleigh's quotient. If Rayleigh's quotient is in the form of Eq. (8.89), then admissible functions suffice. In the case of flexible multibody systems, boundary conditions cannot be defined independently of the motions of the adjacent substructures, so that comparison functions cannot be generated. Hence, the only alternative is the use of admissible functions, which include the various "substructure modes" as special cases. This, however, raises serious questions concerning the speed of convergence, as demonstrated in Sec. 8.6 for a single elastic member. This suggests the construction of an approximate solution over each of the flexible substructures from the space of quasi-comparison functions. Moreover, the geometric compatibility at interface points is ensured automatically by a kinematical procedure describing the motion of each point of the structure in a consistent manner, which obviates the question of constraints.

We propose to derive first the free-vibration equations of motion for flexible multibody systems of the type shown in Fig. 8.16. Then, the eigenvalue problem follows immediately from the free-vibration equations. Because the eigenvalue problem represents a linear problem, the interest lies in linear equations of motion, which implies that all displacements must be small, including the rotations. We derive the equations of motion by means of Lagrange's equations, which in the case of free vibration of undamped systems amounts to deriving the kinetic energy and potential

energy. The latter two are fully defined by the mass matrix and stiffness matrix, respectively.

The task of deriving the equations of motion can be simplified appreciably by adopting a consistent kinematical procedure for describing the motion. To this end, we introduce an inertial set of axes $X_I Y_I Z_I$ with the origin at I , a set of body axes $x_o y_o z_o$ with the origin at O and attached to substructure o in the undeformed state, a set of body axes $x_a y_a z_a$ with the origin at A and attached to substructure a in the undeformed state, etc. The various axes are shown in Fig. 8.16. For simplicity, we limit the formulation to substructures of the type o and a . Extension of the formulation to substructures of the type b, c , etc., is obvious, but tends to be exceedingly laborious. To carry out the extension, we observe that the motion of b relates to the motion of a in the same manner as the motion of a relates to the motion of o . The position vector of typical points in o and a can be written as

$$\mathbf{R}_o = \mathbf{R}_O + \mathbf{r}_o + \mathbf{w}_o \quad (8.212a)$$

and

$$\mathbf{R}_a = \mathbf{R}_O + \mathbf{r}_{oa} + \mathbf{w}_{oa} + \mathbf{r}_a + \mathbf{w}_a, \quad a = 1, 2, \dots, N \quad (8.212b)$$

respectively, where \mathbf{R}_O is the radius vector from I to O , \mathbf{r}_o the radius vector from O to a typical point in o , \mathbf{w}_o the elastic displacement vector of the typical point in o measured relative to axes $x_o y_o z_o$, \mathbf{r}_{oa} the radius vector from O to A , \mathbf{w}_{oa} the vector \mathbf{w}_o evaluated at A , \mathbf{r}_a the radius vector from A to a typical point in a and \mathbf{w}_a the elastic displacement of the typical point in a measured relative to axes $x_a y_a z_a$. Note that all vectors are in terms of components along local axes.

The Lagrangian formulation requires the kinetic energy, which in turn requires the velocity of typical points in the various substructures. To derive expressions for these velocities, we assume that axes $x_o y_o z_o$ rotate with the angular velocity $\dot{\boldsymbol{\theta}}$ relative to the inertial space and that axes $x_a y_a z_a$ rotate with the angular velocity $\dot{\boldsymbol{\beta}}_a$ relative to axes $x_o y_o z_o$, due to the elastic motion at A . Recalling that we are interested in linearized equations of motion, the velocity vector for a typical point in substructure o and substructure a can be written as follows:

$$\dot{\mathbf{R}}_o = \dot{\mathbf{R}}_O + \tilde{r}_o^T \dot{\boldsymbol{\theta}} + \dot{\mathbf{w}}_o \quad (8.213a)$$

$$\dot{\mathbf{R}}_a = \dot{\mathbf{R}}_O + (\tilde{r}_{oa}^T + C_a^T \tilde{r}_a^T C_a) \dot{\boldsymbol{\theta}} + \dot{\mathbf{w}}_{oa} + C_a^T \tilde{r}_a^T C_a \dot{\boldsymbol{\beta}}_a + C_a^T \dot{\mathbf{w}}_a, \quad a = 1, 2, \dots, N \quad (8.213b)$$

where $\dot{\mathbf{R}}_O$ is the velocity vector of O and C_a is a matrix of direction cosines between $x_a y_a z_a$ and $x_o y_o z_o$. Moreover, $\dot{\boldsymbol{\beta}}_a = \nabla \times \mathbf{w}_{oa}$, and we note that, in writing the angular displacements due to elastic deformations in vector form, we take into account that these deformations are small. We also note that a tilde over a symbol denotes a skew symmetric matrix formed from the corresponding vector (Sec. 2.6).

The degrees of freedom of the system are associated with the rigid-body motions of the frame $x_o y_o z_o$ and the elastic motions of the substructures. We assume that the elastic displacements can be expressed in the form

$$\mathbf{w}_s(\mathbf{r}_s, t) = \Phi_s(\mathbf{r}_s) \mathbf{q}_s(t), \quad s = o, a; a = 1, 2, \dots, N \quad (8.214)$$

where Φ_s are matrices of trial functions and \mathbf{q}_s are vectors of generalized displacements. Using Eqs. (8.213) in conjunction with Eqs. (8.214), the kinetic energy can be reduced to the form

$$\begin{aligned}
 T &= \frac{1}{2} \int_{D_o} \rho_o \dot{\mathbf{R}}_o^T \dot{\mathbf{R}}_o dD_o + \frac{1}{2} \sum_{a=1}^N \int_{D_a} \rho_a \dot{\mathbf{R}}_a^T \dot{\mathbf{R}}_a dD_a \\
 &= \frac{1}{2} m_t \mathbf{V}_O^T \mathbf{V}_O - \mathbf{V}_O^T \tilde{S}_t \boldsymbol{\omega} + \mathbf{V}_O^T \bar{\Phi}_t \dot{\mathbf{q}}_o + \mathbf{V}_O^T \sum_{a=1}^N C_a^T \bar{\Phi}_a \dot{\mathbf{q}}_o + \frac{1}{2} \boldsymbol{\omega}^T I_t \boldsymbol{\omega} \\
 &\quad + \boldsymbol{\omega}^T \tilde{\Phi}_t \dot{\mathbf{q}}_o + \boldsymbol{\omega}^T \sum_{a=1}^N H_a \dot{\mathbf{q}}_a + \frac{1}{2} \dot{\mathbf{q}}_o^T M_t \dot{\mathbf{q}}_o + \dot{\mathbf{q}}_o^T \sum_{a=1}^N J_a \dot{\mathbf{q}}_a + \frac{1}{2} \sum_{a=1}^N \dot{\mathbf{q}}_a^T M_a \dot{\mathbf{q}}_a \\
 &= \frac{1}{2} \dot{\mathbf{x}}^T M \dot{\mathbf{x}}
 \end{aligned} \tag{8.215}$$

where $\mathbf{x} = [\mathbf{R}_O^T \boldsymbol{\theta}^T \mathbf{q}_o^T \mathbf{q}_1^T \mathbf{q}_2^T \dots \mathbf{q}_N^T]^T$ is the configuration vector, in which we note that $\dot{\mathbf{R}}_o = \mathbf{V}_o$ and $\dot{\boldsymbol{\theta}} = \boldsymbol{\omega}$, and

$$M = \begin{bmatrix} m_t I & \tilde{S}_t^T & \bar{\Phi}_t & C_1^T \bar{\Phi}_1 & C_2^T \bar{\Phi}_2 & \dots & C_N^T \bar{\Phi}_N \\ \tilde{S}_t & I_t & \tilde{\Phi}_t & H_1 & H_2 & \dots & H_N \\ \bar{\Phi}_t^T & \tilde{\Phi}_t^T & M_t & J_1 & J_2 & \dots & J_N \\ \bar{\Phi}_1^T C_1 & H_1^T & J_1^T & M_1 & 0 & \dots & 0 \\ \bar{\Phi}_2^T C_2 & H_2^T & J_2^T & 0 & M_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \bar{\Phi}_N^T C_N & H_N^T & J_N^T & 0 & 0 & \dots & M_n \end{bmatrix} \tag{8.216}$$

is the mass matrix. The various quantities entering into Eq. (8.216) are as follows:

$$m_t = m_o + \sum_{a=1}^N m_a, \quad \tilde{S}_t = \tilde{S}_o + \sum_{a=1}^N (m_a \tilde{r}_{oa} + C_a^T \tilde{S}_a C_a)$$

$$\bar{\Phi}_t = \bar{\Phi}_o + \sum_{a=1}^N (m_a \Phi_{oa} - C_a^T \tilde{S}_a C_a \Upsilon_{oa})$$

$$I_t = I_o + \sum_{a=1}^N (C_a^T I_a C_a - m_a \tilde{r}_{oa}^2 - \tilde{r}_{oa} C_a^T \tilde{S}_a C_a - C_a^T \tilde{S}_a C_a \tilde{r}_{oa})$$

$$\tilde{\Phi}_t = \tilde{\Phi}_o + \sum_{a=1}^N \left[(m_a \tilde{r}_{oa} + C_a^T \tilde{S}_a C_a) \Phi_{oa} + (C_a^T I_a C_a - \tilde{r}_{oa} C_a^T \tilde{S}_a C_a) \Upsilon_{oa} \right]$$

$$\begin{aligned}
 M_t = M_o + \sum_{a=1}^N & \left(m_a \Phi_{oa}^T \Phi_{oa} - \Phi_{oa}^T C_a^T \tilde{S}_a C_a \Upsilon_{oa} \right. \\
 & \left. + \Upsilon_{oa}^T C_a^T \tilde{S}_a C_a \Phi_{oa} + \Upsilon_{oa}^T C_a^T I_a C_a \Upsilon_{oa} \right)
 \end{aligned}$$

$$\begin{aligned}
 H_s &= C_s^T \tilde{\Phi}_s + \tilde{r}_{os} C_s^T \bar{\Phi}_s, & J_s &= \Gamma_s^T C_s^T \tilde{\Phi}_s + \Phi_{os}^T C_s^T \bar{\Phi}_s, \\
 M_s &= \int_{D_s} \rho_s \Phi_s^T \Phi_s dD_s & & s = o, a; a = 1, 2, \dots, N \quad (8.217)
 \end{aligned}$$

in which

$$\begin{aligned}
 m_s &= \int_{D_s} \rho_s dD_s, & \tilde{S}_s &= \int_{D_s} \rho_s \tilde{r}_s dD_s, \\
 \Upsilon_{oa} &= \nabla \times \Phi_{oa}, & I_s &= \int_{D_s} \rho_s \tilde{r}_s \tilde{r}_s^T dD_s \quad (8.218) \\
 \bar{\Phi}_s &= \int_{D_s} \rho_s \Phi_s dD_s, & \tilde{\Phi}_s &= \int_{D_s} \rho_s \tilde{r}_s \Phi_s dD_s, \\
 \Phi_{os} &= \Phi_o(\mathbf{r}_{os}), & \Gamma_s &= \nabla \times \Phi_s(\mathbf{r}_{os})
 \end{aligned}$$

The potential energy is assumed to be due entirely to the elastic deformations and can be written as

$$V = \frac{1}{2} \mathbf{q}_o^T K_o \mathbf{q}_o + \sum_{a=1}^N \frac{1}{2} \mathbf{q}_a^T K_a \mathbf{q}_a + \sum_{b=1}^N \frac{1}{2} \mathbf{u}_b^T K_b \mathbf{u}_b = \frac{1}{2} \mathbf{x}^T K \mathbf{x} \quad (8.219)$$

where

$$K_s = [\Phi_s^T, \Phi_s], \quad s = o, a \quad (8.220)$$

are substructure stiffness matrices, K_b are boundary stiffness matrices due to the action of the springs at the boundary points B , and K is the overall stiffness matrix for the whole structure. Moreover,

$$\mathbf{u}_b \cong \mathbf{R}_O + (\tilde{r}_{oa}^T + C_a^T \tilde{r}_{ab}^T C_a) \boldsymbol{\theta} + \Phi_{oa} \mathbf{q}_o + C_a^T \Phi_{ab} \mathbf{q}_a \quad (8.221)$$

represents the displacement vector of point B , in which $\Phi_{ab} = \Phi_a(\mathbf{r}_{ab})$. The other quantities on the right side of Eq. (8.221) were defined earlier. The overall stiffness matrix can be written in the form

$$K = \begin{bmatrix}
 \kappa_{aa} & \kappa_{12} & \kappa_{13} & \kappa_{14}^1 & \kappa_{14}^2 & \dots & \kappa_{14}^N \\
 (\kappa_{12})^T & \kappa_{22} & \kappa_{23} & \kappa_{24}^1 & \kappa_{24}^2 & \dots & \kappa_{24}^N \\
 (\kappa_{13})^T & (\kappa_{23}^1)^T & K_o + \kappa_{33} & \kappa_{34}^1 & \kappa_{34}^2 & \dots & \kappa_{34}^N \\
 (\kappa_{14}^1)^T & (\kappa_{24}^1)^T & (\kappa_{34}^1)^T & K_1 + \kappa_{44}^1 & 0 & \dots & 0 \\
 (\kappa_{14}^2)^T & (\kappa_{24}^2)^T & (\kappa_{34}^2)^T & 0 & K_2 + \kappa_{44}^2 & \dots & 0 \\
 \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
 (\kappa_{14}^N)^T & (\kappa_{24}^N)^T & (\kappa_{34}^N)^T & 0 & 0 & \dots & K_N + \kappa_{44}^N
 \end{bmatrix} \quad (8.222)$$

in which

$$\begin{aligned}
 \kappa_{11} &= \sum_{a=1}^N k_a, & \kappa_{12} &= -\sum_{a=1}^N k_a (\tilde{r}_{oa} + C_a^T \tilde{r}_{ab} C_a), & \kappa_{13} &= \sum_{a=1}^N k_a \Phi_{oa} \\
 \kappa_{22} &= -\sum_{a=1}^N (\tilde{r}_{oa} + C_a^T \tilde{r}_{ab} C_a) k_a (\tilde{r}_{oa} + C_a^T \tilde{r}_{ab} C_a), \\
 \kappa_{23} &= \sum_{a=1}^N (\tilde{r}_{oa} + C_a^T \tilde{r}_{ab} C_a) k_a \Phi_{oa} & (8.223) \\
 \kappa_{33} &= \sum_{a=1}^N \Phi_{oa}^T k_a \Phi_{oa}, & \kappa_{14}^a &= k_a C_a^T \Phi_{ab}, & \kappa_{24}^a &= (\tilde{r}_{oa} + C_a^T \tilde{r}_{ab} C_a) k_a C_a^T \Phi_{ab} \\
 \kappa_{34}^a &= \Phi_{oa}^T k_a C_a^T \Phi_{ab}, & \kappa_{44}^a &= \Phi_{ab}^T C_a k_a C_a^T \Phi_{ab}
 \end{aligned}$$

As pointed out earlier in this section, Lagrange's equations for free vibrations are fully defined by the coefficient matrices in the kinetic energy and potential energy. Indeed, the equations of motion can be written simply as

$$M\ddot{\mathbf{x}}(t) + K\mathbf{x}(t) = \mathbf{0} \quad (8.224)$$

where M and K are given by Eqs. (8.216) and (8.222), respectively. Then, because free vibration of conservative systems is harmonic, $\mathbf{x}(t) = e^{i\omega t} \mathbf{x}$, Eq. (8.224) yields the eigenvalue problem

$$K\mathbf{x} = \lambda M\mathbf{x}, \quad \lambda = \omega^2 \quad (8.225)$$

From Fig. 8.16, we observe that in the case of flexible multibody systems there are no geometric boundary conditions, except when a substructure is supported externally, so that, for the most part, the only boundary conditions characterizing a substructure are natural. It follows that, according to the Rayleigh-Ritz theory, the admissible functions need satisfy nothing. To be sure, they must be p times differentiable, but this requirement is satisfied routinely by virtually all choices. Hence, in theory, admissible functions corresponding to the modes of free-free substructures should be a suitable choice. However, this turns out not to be the case in general.

In the case of substructures in the form of beams in bending, there are four quantities entering into the boundary conditions, namely, displacement, slope, bending moment and shearing force, where the latter two involve second and third derivatives with respect to the spatial variable, respectively. In component-mode synthesis, geometric compatibility is enforced by means of constraint equations. The kinematical procedure used here ensures geometric compatibility at boundary points automatically, which is accomplished by defining the various sets of body axes so as to guarantee displacement and slope compatibility at boundary points common to any two substructures. This obviates the need for constraint equations enforcing such geometric compatibility. In addition, rigid-body motions are included in the displacement vector of the central substructure o . But, this substructure synthesis

goes beyond ensuring geometric compatibility. Indeed, through the use of quasi-comparison functions, provisions are also made for balancing to a large degree the bending moment and shearing force at boundary points between any two substructures, thus satisfying the natural boundary conditions approximately at these points. In this regard, a distinction must be made between substructure o and substructures a ($a = 1, 2, \dots, N$). Indeed, for substructure o , it is necessary to make provisions for nonzero displacement, slope, bending moment and shearing force at the boundary points A . On the other hand, because displacement and slope compatibility are guaranteed automatically by the kinematical procedure, for substructures of type a it is only necessary to make provisions for nonzero bending moment and shearing force at points A . All this is done through a judicious choice of admissible functions. For example, linear combinations of free-free functions alone do not qualify as quasi-comparison functions for substructure o , because they are characterized by zero bending moment and shearing force at boundary points. However, a set including free-free functions and clamped-clamped functions can qualify as quasi-comparison functions if the set provides for nonzero displacement, slope, bending moment and shearing force of arbitrary magnitude at all boundary points. This implies that there must be a minimum number of functions in the set. As an example, we consider substructure o as a typical beam in bending and confine ourselves to transverse displacements only, so that there are four arbitrary quantities at each end, for a total of eight. Hence, in addition to one rigid-body translation and one rigid-body rotation, it is necessary to include six shape functions in the set, perhaps three free-free functions and three clamped-clamped functions. This choice of shape functions is far from being unique. Another choice that can prove quite suitable, and one likely to cause initial skepticism, is a set of clamped-free and free-clamped functions. It should be reiterated that the preceding functions are not modes at all, as there is no conceivable substructure that can be free-free and clamped-clamped, or clamped-free and free-clamped at the same time. Going one step further, in the numerical example to follow, we demonstrate that sine and cosine functions can constitute a suitable set from which to construct quasi-comparison functions.

The fact that a given choice of quasi-comparison functions for the substructures is capable of providing both for the satisfaction of geometric compatibility and for the matching of bending moment and shearing force at boundary points does not mean that the natural boundary conditions will actually be satisfied exactly. Indeed, in rendering the Rayleigh quotient in terms of the energy inner product stationary, the natural boundary conditions will be satisfied only approximately, and so will the differential equations. The substructure synthesis process tends to reduce the error at all points of the structure, regardless of whether they are boundary points or points in the interior of the substructures. In using quasi-comparison functions, instead of mere admissible functions, the process is given the chance to reduce errors at all points of the structure, resulting in superior convergence characteristics.

As mentioned on several occasions, the substructure synthesis is in fact a Rayleigh-Ritz method. The main difference between the substructure synthesis presented here and the classical Rayleigh-Ritz method is that here the admissible functions are local in the sense that they are defined over the domain of a given substructure, whereas in the classical Rayleigh-Ritz method they are global, in the sense

that they are defined over the entire structure. Of course, the fact that this substructure synthesis is a Rayleigh-Ritz method has many implications. The most important of these is that most of the theory developed in conjunction with the Rayleigh-Ritz method is valid for the present substructure synthesis as well. Having the backing of the Rayleigh-Ritz theory permits us to draw some immediate conclusions concerning the convergence of substructure synthesis. To this end, we assume that the number of degrees of freedom of the system is n . Then, as the number of admissible functions entering into the comparison functions increases, we can state that

$$\lim_{n \rightarrow \infty} \lambda_r^n = \lambda_r \quad (8.226)$$

or the computed eigenvalues converge to the actual eigenvalues. Moreover, they approach the actual eigenvalues from above. Although the Rayleigh-Ritz method does not permit a similar statement concerning the computed eigenvectors, and hence concerning the approximate eigenfunctions, it is safe to say that, as the number of admissible functions entering into the quasi-comparison functions increases, the error in satisfying the differential equation and the boundary conditions tends to decrease. The rate of convergence depends largely on the choice of quasi-comparison functions, and tends to be faster for eigenvalues than for eigenvectors. This can be attributed to the stationarity of Rayleigh's quotient, which implies that, if an approximate eigenvector differs from the corresponding actual eigenvector by a small quantity of first order, the approximate eigenvalue differs from the corresponding actual eigenvalue by a small quantity of second order.

From the preceding discussion, we conclude that the substructure synthesis presented here is different philosophically from the component-mode synthesis. In fact, it is closer in nature to the hierarchical finite element method (Ref. 29). Indeed, both this substructure synthesis and the hierarchical finite element method describe the motion in terms of local admissible functions. In the first, the local functions are admissible functions capable of yielding quasi-comparison functions defined over entire substructures, and in the second, the local functions are polynomials defined over finite elements. Moreover, in substructure synthesis convergence is achieved by increasing the number of admissible functions entering into the quasi-comparison function for a given substructure, and in the hierarchical finite element method convergence is achieved by increasing the number and degree of polynomials for a given finite element. The latter is in contrast with the ordinary finite element method, in which convergence is achieved by keeping the number of polynomials constant and refining the finite element mesh.

The substructure synthesis method described in this section is suitable for structures for which the substructures represent one-dimensional elastic members.

Example 8.4

The theory just developed is applied to the structure shown in Fig. 8.17 (Ref. 31). The structure consists of three substructures, the central substructure o and two substructures of type a , with the supports being mounted on springs. The central substructure is a uniform beam, and the other two substructures are tapered beams, as shown in Fig. 8.17.

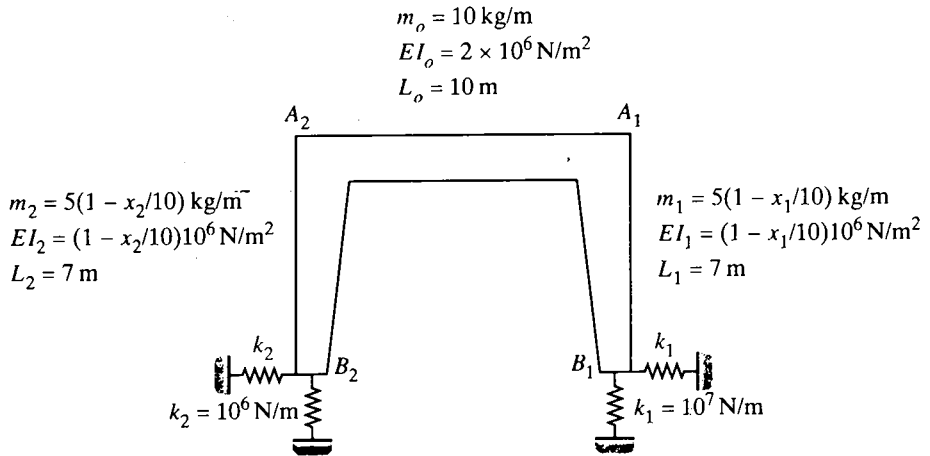


Figure 8.17 Structure consisting of three substructures

We propose to describe the motion of the substructures in five different ways. In case 1, we represent the motion of all three substructures by means of mere admissible functions. To this end, we use free-free functions for the central substructure and clamped-free functions for the remaining two substructures, all functions corresponding to modes for the associated uniform beams. As pointed out earlier, the boundary conditions for the central substructure cannot be satisfied with a finite number of free-free functions. Similarly, the boundary conditions at points B_1 and B_2 in the case of substructures of type a cannot be satisfied with a finite number of clamped-free functions. Note that the geometric boundary conditions at points A_1 and A_2 are satisfied automatically by the kinematical procedure implied by Eqs. (8.212)–(8.214). In case 2, the motion of the central substructure is represented by means of an improved set of admissible functions and the motion of the remaining two substructures is represented by means of quasi-comparison functions. In particular, for the central substructure, we use a combination of free-free and pinned-pinned functions. Although this combination of functions represents an improvement over case 1, it still falls in the class of mere admissible functions, as the bending moment at the boundaries remains zero with a finite number of terms. For the other two substructures, we use combinations of clamped-free and clamped-pinned functions, so that these combinations of functions qualify as quasi-comparison functions. In cases 3–5, the motion of all three substructures is represented by quasi-comparison functions. In case 3, we use combinations of free-free and clamped-clamped functions for the central substructure, so that both boundary conditions at each end can be satisfied with a finite number of functions; the quasi-comparison functions for the other two substructures remain as in case 2. Cases 4 and 5 differ from case 3 in that in case 4 the motion of the central substructure is represented by clamped-free and free-clamped functions, and in case 5 by pinned-pinned and cosine shape functions. The motion of the other two substructures is represented in cases 3, 4 and 5 as in case 2. Figures 8.18a–e show the various functions for cases 1–5.

The various types of functions used have closed-form expressions. The free-free functions have the expression

$$\phi_i(x) = \cosh \beta_i x + \cos \beta_i x - \sigma_i (\sinh \beta_i x + \sin \beta_i x) \tag{a}$$

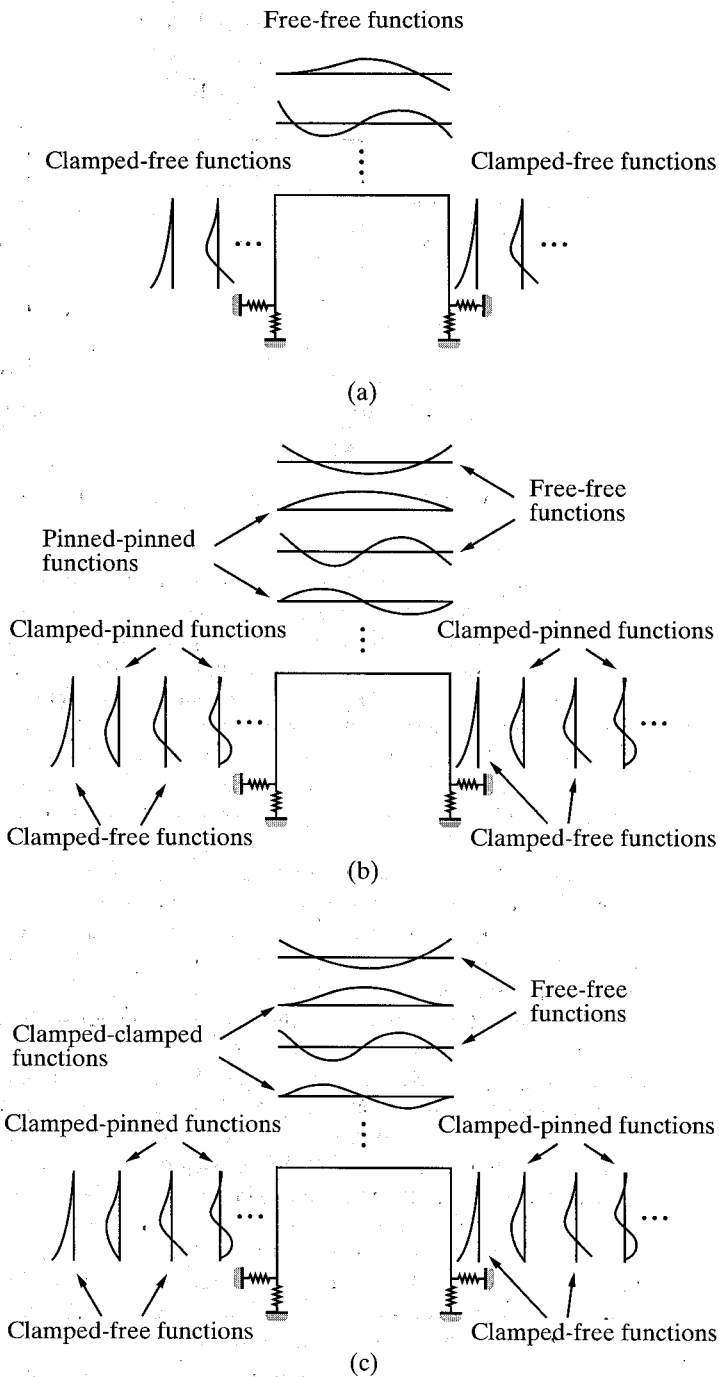


Figure 8.18 (a) Admissible functions for all three substructures (b) Improved admissible functions for beam and quasi-comparison functions for columns (c) Quasi-comparison functions for all three substructures

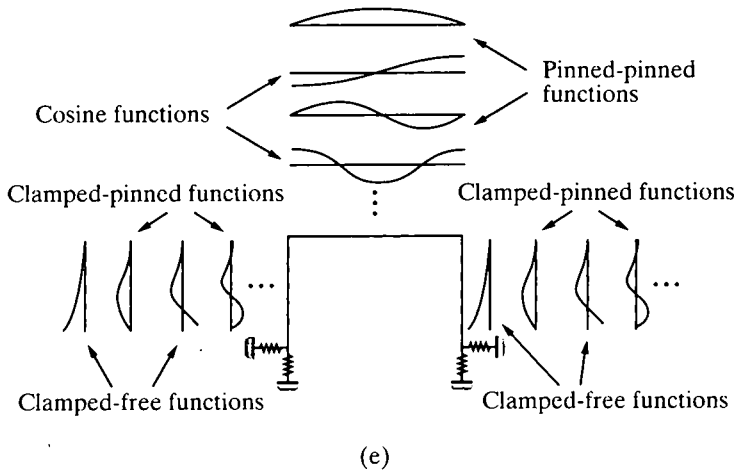
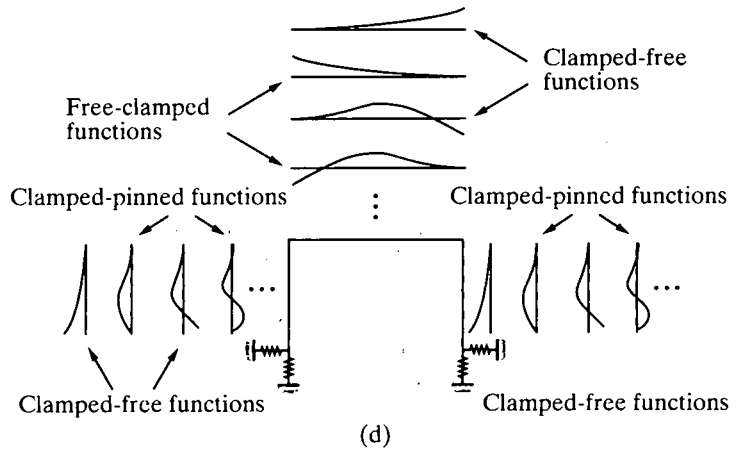


Figure 8.18 (Continued) **(d)** Quasi-comparison functions for all three substructures **(e)** Quasi-comparison functions for all three substructures

The clamped-free, clamped-clamped and clamped-pinned functions are given by

$$\phi_i = \cosh \beta_i x - \cos \beta_i x - \sigma_i (\sinh \beta_i x - \sin \beta_i x) \tag{b}$$

where the constants β_i and σ_i vary from case to case; their numerical values are given by Blevins.² On the other hand, the pinned-pinned functions have the form

$$\phi_i(x) = \sin \frac{i\pi x}{L} \tag{c}$$

and the cosine functions are

$$\phi_i(x) = \cos \frac{i\pi x}{L} \tag{d}$$

² Blevins, R.D., *Formulas for Natural Frequency and Mode Shape*, Van Nostrand Reinhold, New York, 1979.

Finally, the stiffness matrices for beams in bending have the entries

$$k_{oij} = \int_0^{l_o} EI_0(x) \phi_i''(x) \phi_j''(x) dx, \quad k_{aij} = \int_0^{l_a} EI_a(x) \phi_i''(x) \phi_j''(x) dx, \quad a = 1, 2 \quad (e)$$

where primes denote differentiations with respect to x . The numerical values for the various parameters are shown in Fig. 8.17.

The eigenvalue problem for the system shown in Fig. 8.17 was solved for the five cases just described and the three lowest natural frequencies are listed in Tables 8.4–8.6. The results are in agreement with the expectations. In case 1, in which the motion is expressed in terms of one type of admissible functions only for each substructure, the convergence is unsatisfactory. Using a 27-degree-of-freedom model, the computed natural frequencies are relatively far from the actual ones, and improvement with the addition of degrees of freedom is very slow. At this point, there is no indication how many degrees of freedom will be necessary for convergence. In case 2, in which the motion of substructure o is described by means of more suitable admissible functions than in case 1 and the motion of substructures a is represented by means of quasi-comparison functions, the results are significantly better than in case 1, although convergence is still elusive. In cases 3–5, in which the motion of all substructures is represented by means of quasi-comparison functions, convergence is relatively rapid, with the results of case 4 being better than those of cases 3 and 5. Clearly, the results are far superior to those obtained in the first two cases. As pointed out earlier, there is a minimum number of terms necessary before the linear combinations of admissible functions become quasi-comparison functions. Hence, in cases 3–5, the results for small numbers of degrees of freedom are not meaningful.

TABLE 8.4 First Natural Frequency

DoF	Case 1	Case 2	Case 3	Case 4	Case 5
6	1.75746	1.75746	1.75746	1.69875	1.75745
9	1.57714	1.65474	1.65474	1.47335	1.47530
12	1.56932	1.56538	1.56538	1.47102	1.47084
15	1.53536	1.49632	1.47391	1.47073	1.47077
18	1.53462	1.49632	1.47391	1.47073	1.47077
21	1.52395	1.49632	1.47391	1.47073	1.47073
24	1.52339	1.48726	1.47157	1.47073	1.47073
27	1.51702	1.48138	1.47073	1.47073	1.47073

To verify how well the natural boundary condition at points A is satisfied, we define the error in the bending moment as

$$\epsilon_M^n = M(A-) - M(A+) \quad (f)$$

where $M(A-)$ denotes the bending moment at point A_2 computed from the solution for substructure o and $M(A+)$ is the same quantity corresponding to substructure $a = 2$. Figure 8.19 shows plots of ϵ_M^n versus n for all five cases just discussed. It is clear that the solutions in terms of quasi-comparison functions are far superior to those in terms of mere admissible functions.

TABLE 8.5 Second Natural Frequency

DoF	Case 1	Case 2	Case 3	Case 4	Case 5
6	8.59485	8.59485	8.59485	11.62478	8.71827
9	8.29244	8.25873	8.23772	9.76602	8.33865
12	8.26215	8.24909	8.22873	8.22884	8.32848
15	8.25375	8.24867	8.22834	8.21273	8.21143
18	8.24792	8.23106	8.20832	8.21197	8.20475
21	8.24659	8.22109	8.20449	8.20455	8.20475
24	8.24418	8.22108	8.20447	8.20443	8.20475
27	8.24369	8.22107	8.20446	8.20443	8.20444

TABLE 8.6 Third Natural Frequency

DoF	Case 1	Case 2	Case 3	Case 4	Case 5
6	26.86095	26.86095	26.86095	28.98808	26.92279
9	21.78531	21.91468	21.35960	20.37671	20.24966
12	21.30582	21.11908	20.98775	20.18728	19.93381
15	20.66517	20.11777	20.03998	19.94335	19.93298
18	20.61882	20.11582	20.03894	19.93163	19.93137
21	20.46641	20.11493	20.03876	19.93120	19.93120
24	20.45673	20.05149	19.95669	19.93111	19.93111
27	20.37866	20.00927	19.93139	19.93102	19.93102

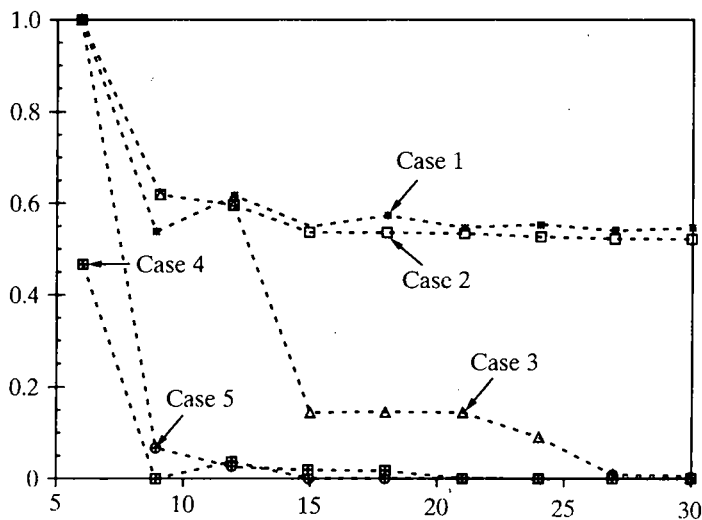


Figure 8.19 Bending moment error at point A₂ versus the number of degrees of freedom of the model

8.13 SYNOPSIS

The spatial discretization methods can be divided broadly into lumping procedures and series discretization methods. Lumping techniques appeal to physical intuition and tend to be easy to understand. Indeed, the system parameters, i.e., the mass distribution, or the stiffness distribution, or both, are lumped at given points of the system. For the most part this is a heuristic process, with no analytical guidelines to call upon. The best known methods are the lumped-parameter method using influence coefficients, Holzer's method for torsional vibration of shafts (and hence for the transverse vibration of strings and axial vibration of rods) and Myklestad's method for bending vibration of beams. In the first, the mass is lumped at discrete points, thus yielding a diagonal mass matrix. On the other hand, the stiffness properties are accounted for in an "exact" manner through flexibility influence coefficients. This being a discretized version of the integral form of the eigenvalue problem, superior results can be expected for a sufficiently large number of discrete points. Moreover, at least in theory, the method is applicable to all types of vibrating systems. In practice, the determination of the influence coefficients can cause serious difficulties, particularly for systems with nonuniform stiffness distribution and/or complex boundary conditions, or for two-dimensional problems. In Holzer's method, both the mass and stiffness are lumped. The mass is lumped into rigid disks at discrete points and the shaft segments between these points are assumed to be massless and to have uniform torsional stiffness. The computation of the natural frequencies and modes of vibration can be carried out in a systematic manner by means of transfer matrices. Myklestad's method extends the ideas to the bending vibration of beams.

Lumped-parameter methods lack mathematical rigor and their convergence is not easy to judge. By contrast, series discretization methods do not suffer from these drawbacks. They tend to be more abstract, however. In the case of conservative systems, the discretization process is embedded into a variational approach with its origin in Rayleigh's principle, which states that the frequency of vibration has a minimum in the neighborhood of the fundamental mode. The fundamental mode is actually not known, and any guess for the fundamental mode yields a frequency of vibration larger than the lowest natural frequency. It is therefore natural to attempt to improve the guess of the fundamental mode, thus lowering the estimate of the lowest natural frequency. This improved guess is in the form of a series of admissible functions with undetermined coefficients and the coefficients are determined so as to render Rayleigh's quotient stationary. The procedure is commonly known as the Rayleigh-Ritz method. The main drawback of the method lies in the difficulty of coming up with suitable admissible functions, making the approach more of an art than a method. The weighted residuals method is really a family of series discretization procedures based on the idea of reducing the approximation error. It is not a variational approach, so that it is applicable to both self-adjoint and non-self-adjoint systems. By far the best known of the weighted residuals methods is Galerkin's method, which is equivalent to the Rayleigh-Ritz method for self-adjoint systems. The convergence of approximations for both self-adjoint and non-self-adjoint systems can be improved through the use of quasi-comparison functions.

As originally envisioned by Hurty, the component-mode synthesis is an extension of the Rayleigh-Ritz method to complex structures with identifiable substructures, referred to as "components." The term "mode" can be traced to the wide assumption that the admissible functions used to represent the motion of the individual components must be some loosely defined modes of vibration. Substructure synthesis represents an extension of the enhanced Rayleigh-Ritz method of Sec. 8.6 to structures consisting of chains of substructures. A consistent kinematical procedure obviates the problem of using constraints to force substructures to work together as a single structure. The method is able to accommodate substructures rotating relative to one another. Here too, the use of quasi-comparison functions can improve convergence.

Whereas the Rayleigh-Ritz and the Galerkin methods have many advantages over other approximate techniques, they also have serious shortcomings. In particular, the applicability of the methods is confined to relatively simple systems, such as one-dimensional ones and two-dimensional ones with rectangular and circular boundaries, although the formulation can be modified so as to accommodate systems with trapezoidal boundaries. The component-mode synthesis and substructure synthesis can extend the usefulness of the series discretization approach, but the above limitations still apply to the individual substructures. Moreover, the question of generating suitable admissible functions is not entirely settled. In addition, the computation of the mass and stiffness matrices generally requires extensive numerical integrations. Many of these shortcomings can be attributed to the fact that the admissible functions are global functions, in the sense that they extend over the entire domain of the elastic member, or of the substructure. Another series discretization method, the finite element method, does not have these drawbacks by virtue of the fact that it uses local admissible functions, defined over small subdomains of the structure. Because these subdomains are small, the local admissible functions can be chosen in the form of low-degree polynomials, and the finite element mesh can be constructed so as to accommodate boundaries with very complex geometry. Finally, the process of computing the mass and stiffness matrices can be automated, relieving the analyst from many computer coding chores. All these attributes make the finite element method a very versatile one.

It should be pointed out here that the Rayleigh-Ritz theory does not actually require that the admissible functions be global, although this has generally been the practice, and local functions are indeed admissible, provided they satisfy the differentiability requirements. Hence, although not conceived originally as such, the finite element method does represent another version of the Rayleigh-Ritz method differing from the classical one presented in this chapter in the nature of the admissible functions. The identification of the finite element method as a Rayleigh-Ritz method was very fortunate indeed, as the mathematical foundation of the Rayleigh-Ritz method could be extended instantly to the finite element method, a foundation lacking originally in the heuristically developed finite element method. Due to its extreme versatility, the finite element method has become the method of choice in many areas of engineering analysis, reaching far beyond the original structural applications. In recognition of its dominant role in vibrations, the entire Chapter 9 is devoted to the finite element method.

PROBLEMS

- 8.1 Formulate the eigenvalue problem for a uniform string fixed at both ends by means of the lumped-parameter method using flexibility influence coefficients. Solve the eigenvalue problem for $n = 20$, compare the results with the exact solution obtained in Sec. 7.6 and draw conclusions concerning the accuracy of the approximate solution.
- 8.2 A shaft in torsional vibration fixed at both ends has the mass polar moment of inertia and torsional stiffness distributions

$$I(x) = I \left[\frac{3}{4} + \frac{x}{L} - \left(\frac{x}{L} \right)^2 \right], \quad GJ(x) = GJ \left[\frac{3}{4} + \frac{x}{L} - \left(\frac{x}{L} \right)^2 \right]$$

Formulate and solve the eigenvalue problem by means of the lumped-parameter method using flexibility influence coefficients for the case in which $n = 20$.

- 8.3 A shaft in torsional vibration fixed at $x = 0$ and with a torsional spring of stiffness $k = GJ/L$ at $x = L$ has the mass polar moment of inertia and torsional stiffness distributions

$$I(x) = \left[1 - \frac{1}{2} \left(\frac{x}{L} \right)^2 \right], \quad GJ(x) = GJ \left[1 - \frac{1}{2} \left(\frac{x}{L} \right)^2 \right]$$

Formulate and solve the eigenvalue problem by means of the lumped-parameter method using flexibility influence coefficients for the case in which $n = 20$.

- 8.4 A cantilever beam clamped at $x = 0$ has rectangular cross section of unit width and height varying according to $h(x) = h(1 - 2x/3L)$. Formulate and solve the eigenvalue problem for the bending vibration of the beam by means of the lumped-parameter method using flexibility influence coefficients for the case in which $n = 20$.
- 8.5 A simply supported beam has the mass and stiffness distributions

$$m(x) = m \left[1 + 12 \frac{x}{L} - 12 \left(\frac{x}{L} \right)^2 \right], \quad EI(x) = EI \left[1 + 12 \frac{x}{L} - 12 \left(\frac{x}{L} \right)^2 \right]$$

Formulate and solve the eigenvalue problem by means of the lumped-parameter method using flexibility influence coefficients for the case in which $n = 20$.

- 8.6 Solve Problem 8.2 by means of Holzer's method, compare results and draw conclusions concerning the relative accuracy; provide arguments in support of your conclusions.
- 8.7 Solve Problem 8.3 by means of Holzer's method, compare results, and draw conclusions concerning the relative accuracy; provide arguments in support of your conclusions.
- 8.8 Solve Problem 8.4 by means of Myklestad's method, compare results and draw conclusions concerning the relative accuracy; provide arguments in support of your conclusions.
- 8.9 Solve Problem 8.5 by means of Myklestad's method, compare results and draw conclusions concerning the relative accuracy; provide arguments in support of your conclusions.
- 8.10 Estimate the lowest natural frequency of the string of Problem 8.1 by means of Rayleigh's energy method using the static displacement curve as a trial function. Compare the estimate with the exact solution obtained in Sec. 7.6 and draw conclusions as to the accuracy of the estimate.
- 8.11 Estimate the lowest natural frequency of the shaft of Problem 8.2 by means of Rayleigh's energy method using as a trial function the static displacement curve due to a distributed torque proportional to the mass polar moment of inertia. Compare the estimate with results obtained in Problems 8.2 and 8.6 and draw conclusions concerning the relative accuracy of the estimate.

- 8.12** Solve Problem 8.11 with Problems 8.3 and 8.7 replacing Problems 8.2 and 8.6, respectively.
- 8.13** Solve Problem 8.11 with Problems 8.4 and 8.8 replacing Problems 8.2 and 8.6, respectively.
- 8.14** Solve Problem 8.11 with Problems 8.5 and 8.9 replacing Problems 8.2 and 8.6, respectively.
- 8.15** Estimate the lowest natural frequency of the beam of Problem 7.32 by means of Rayleigh's energy method using the static displacement curve as a trial function. Compare the estimate with the exact solution obtained in Problem 7.32 and draw conclusions concerning the accuracy of the estimate.
- 8.16** Solve Problem 8.2 by the Rayleigh-Ritz method using the eigenfunctions of a uniform shaft fixed at both ends as comparison functions for $n = 1, 2, \dots, 6$. Construct an array as in Eq. (8.103) and draw conclusions.
- 8.17** Solve Problem 8.3 by the Rayleigh-Ritz method using the eigenfunctions of a uniform fixed-free shaft as admissible functions for $n = 1, 2, \dots, 6$. Construct an array as in Eq. (8.103) and draw conclusions.
- 8.18** Solve Problem 8.5 by the Rayleigh-Ritz method using the eigenfunctions of a uniform simply supported beam as admissible functions for $n = 1, 2, \dots, 6$. Construct an array as in Eq. (8.103) and draw conclusions.
- 8.19** Solve Problem 7.32 by the Rayleigh-Ritz method using the eigenfunctions of a clamped-pinned beam (without the spring) as admissible functions for $n = 1, 2, \dots, 6$. Construct an array as in Eq. (8.103) and draw conclusions.
- 8.20** Solve Problem 7.40 by the Rayleigh-Ritz method using the eigenfunctions of a uniform membrane free at $x = 0, a$ and fixed at $y = 0, b$ as admissible functions for $m = 1, 2$; $n = 1, 2, 3$. Compare the results with those obtained in Problem 7.40 and draw conclusions.
- 8.21** Solve Problem 7.41 by means of the Rayleigh-Ritz method using admissible functions in the form of the products $f_i(r)g_j(\theta)$, where $f_i(r) = (r/a)^i$ and $g_j(\theta)$ are trigonometric functions, for $i = 0, 1, \dots, 8$; $j = 0, 1, 2, 3$. Compare the natural frequencies with those obtained in Problem 7.41 and draw conclusions.
- 8.22** Solve Problem 7.47 by the Rayleigh-Ritz method using products of beam eigenfunctions as admissible functions.
- 8.23** A rectangular plate simply supported at the boundaries $x = 0, a$ and $y = 0, b$ has the mass density and flexural rigidity

$$m(x, y) = m \left[1 + 0.25 \frac{x}{a} \left(1 - \frac{x}{a} \right) \frac{y}{b} \left(1 - \frac{y}{b} \right) \right]$$

$$D_E(x, y) = D_E \left[1 + 0.75 \frac{x}{a} \left(1 - \frac{x}{a} \right) \frac{y}{b} \left(1 - \frac{y}{b} \right) \right].$$

Solve the eigenvalue problem by the Rayleigh-Ritz method and give the values of the four lowest natural frequencies and the expressions of the four associated natural modes.

- 8.24** Solve Problem 8.17 by the enhanced Rayleigh-Ritz method using quasi-comparison functions of your own choice. Compare convergence of the three lowest natural frequencies obtained here with the convergence of the three lowest natural frequencies obtained in Problem 8.17 and draw conclusions.

- 8.25** Solve Problem 8.19 by the enhanced Rayleigh-Ritz method using quasi-comparison functions from two families, clamped-pinned and clamped-clamped functions. Compare convergence of the three lowest natural frequencies obtained here with the convergence of the three lowest natural frequencies obtained in Problem 8.19 and draw conclusions.
- 8.26** Solve Problem 8.25 by the enhanced Rayleigh-Ritz method using quasi-comparison functions from two families, clamped-pinned functions and a second family of functions of your own choice.
- 8.27** Solve Problem 7.40 by the enhanced Rayleigh-Ritz method using quasi-comparison functions of your own choice. Compare the results with the results obtained in Problem 8.20 and draw conclusions.
- 8.28** The boundary-value problem for a given distributed-parameter system is defined by the differential equation

$$Lw + C\dot{w} + m\ddot{w} = 0$$

and suitable boundary conditions, where L is a stiffness operator and C a damping operator. Use Galerkin's method to derive an algebraic eigenvalue problem approximating the associated differential eigenvalue problem.

- 8.29** Assume that the system of Problem 8.3 is subjected to damping of the Kelvin-Voigt type with $c = 0.1$ (Sec. 7.18). Then, use the formulation of Problem 8.28, let the system parameters be uniformly distributed, solve the algebraic eigenvalue problem for $n = 3, 4, 5, 6$ and discuss the behavior of the eigenvalues.
- 8.30** Solve Problem 8.29 with Problem 7.32 replacing Problem 8.3.
- 8.31** Solve Problem 8.16 by the collocation method. Determine the required number n of terms in the approximation to match the accuracy of the lowest eigenvalue obtained in Problem 8.16 for $n = 6$.
- 8.32** Solve Problem 8.31 with Problem 8.17 replacing Problem 8.16.
- 8.33** Solve Problem 8.31 with Problem 8.19 replacing Problem 8.16.
- 8.34** Derive the response of the system of Problem 8.16 with $n = 6$ to the concentrated torque $M_0 u(t)$ applied at $x = L/2$, where $u(t)$ is the unit step function. Discuss the mode participation in the response.
- 8.35** Derive the response of the system of Problem 8.19 with $n = 6$ to the distributed force $f(x, t) = f_0(1 - x/L)\delta(t)$, where $\delta(t)$ is the unit impulse.
- 8.36** Derive the response of the system of Problem 8.29 with $n = 6$ to the distributed torque $m(x, t) = m_0 r(t)$, where $r(t)$ is the unit ramp function (Sec. 1.7).
- 8.37** Derive the response of the system of Problem 8.30 with $n = 6$ to the concentrated force $F_0[u(t) - u(t - T)]$ applied at $x = L/2$, where $u(t)$ is the unit step function.
- 8.38** Derive the response of the system of Problem 8.31 with $n = 6$ to the distributed torque $m(x, t) = m_0[r(t) - r(t - T)]$, where $r(t)$ is the unit ramp function (Sec. 1.7). Discuss the mode participation in the response.
- 8.39** Derive the response of the system of Problem 8.33 with $n = 6$ to the distributed torque $m(x, t) = m_0(1 - x/2L)[r(t) - r(t - T) - T u(t - 2T)]$, where $r(t)$ is the unit ramp function (Sec. 1.7) and $u(t)$ the unit step function.
- 8.40** Derive the response of the plate of Problem 8.23 to the force $f(x, y, t) = f_0[r(t) - r(t - T)]$ distributed uniformly over the rectangular area defined by $a/2 < x < 3a/4$, $b/4 < y < 3b/4$, where $r(t)$ is the unit ramp function (Sec. 1.7).

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THE FINITE ELEMENT METHOD

The finite element method must be regarded as the most successful technique of structural analysis. Originally conceived by Turner, Clough, Martin and Topp in the mid 1950s (Ref. 26) as a procedure for static stress analysis of complex structures, the method has been expanding rapidly into many engineering areas. The phenomenal success of the finite element method can be attributed to a large extent to timing, as at the same time the finite element method was being developed so were increasingly powerful digital computers. In fact, in many ways the computer has helped greatly with the development of the method.

Although developed independently of the Rayleigh-Ritz method, the finite element method was demonstrated later to be a Rayleigh-Ritz method. To distinguish between the two, we refer to the original one as the *classical Rayleigh-Ritz method*. We recall from Sec. 8.5 that the classical Rayleigh-Ritz method represents a variational approach whereby a distributed system is approximated by a discrete one by assuming a solution of the differential eigenvalue problem as a finite series of admissible functions. The wide use of the classical Rayleigh-Ritz method has been limited by the inability to generate suitable admissible functions for a large number of problems. This inability can be attributed to the traditional manner in which the method has been applied rather than to inherent flaws in the method itself. Indeed, a major source of difficulties is due to the insistence on using global admissible functions, and there is nothing in the theory requiring that the functions be global. In this regard, it should be noted that systems with complex boundary conditions, or complex geometry, cannot be accommodated easily by global admissible functions. Such cases are quite common in two- and three-dimensional structures. Moreover,

global admissible functions tend to have complicated expressions, difficult to handle on a routine basis. The basic difference between the classical Rayleigh-Ritz method and the finite element method lies in the fact that in the latter an approximate solution is constructed using local admissible functions defined over small subdomains of the structure. In this regard it should perhaps be mentioned that, in a paper generally regarded as the forerunner of the finite element method, Courant (Ref. 8) used a variational approach in conjunction with linear admissible functions defined over small triangular subdomains to produce an approximate solution to St. Venant's torsion problem, thus preceding the development of the finite element method by over a decade. The reason why Courant's work did not attract more attention can be attributed to poor timing. Indeed, in the early 1940s, computers capable of solving large sets of equations of equilibrium, or equations of motion, did not exist, so that the method was not practical then.

The concept of local functions defined over small subdomains carries enormous implications, and is the key to the success of the finite element method. In the first place, because the subdomains are small, good approximations can be realized with local admissible functions in the form of low-degree polynomials. These low-degree polynomials, often referred to as interpolation functions, not only make the computation of the stiffness and mass matrices appreciably easier, but also eliminate the troublesome task of choosing suitable admissible functions, as given classes of problems call for certain choices of polynomials. Perhaps more important is the fact that the computations lend themselves to automation. Indeed, the computer is not only able to solve the discretized equations of equilibrium, or equations of motion, but also to carry out such diverse tasks as the formulation of the equations by making decisions concerning the finite element mesh and the assembly of the stiffness and mass matrices. Finally, the finite element method has no equal in its ability to accommodate systems with complicated geometries and parameter distributions. Of course, the geometry can be a serious concern in two- and three-dimensional problems. To match a given irregular boundary, or to accommodate parameter nonuniformities, not only the size of the finite elements can be changed but also their shape. This extreme versatility, coupled with the fact that many powerful computer codes based on the method have become available, has made the finite element method the method of choice for static and dynamic analysis of structures.

In this chapter, we begin with the presentation of the finite element method as a Rayleigh-Ritz method. Then, the procedure for determining element stiffness and mass matrices and for assembling them into global stiffness and mass matrices is demonstrated by means of second-order systems, such as strings in transverse vibration, first using linear and then higher-degree interpolation functions. The approach is subsequently extended to fourth-order systems, such as beams in bending. The real power of the finite element method becomes evident in two-dimensional systems, such as membranes and plates. Here we encounter finite elements of various shapes, such as triangular, rectangular and quadrilateral elements, as well as elements with curved boundaries. Another version of the method, known as the hierarchical finite element method, combines some of the best features of the finite element and classical Rayleigh-Ritz methods. The chapter concludes with a discussion of the system response.

9.1 THE FINITE ELEMENT METHOD AS A RAYLEIGH-RITZ METHOD

The *finite element method* is a technique for the spatial discretization of distributed-parameter systems. It consists of dividing the domain D of the system into a set of subdomains and describing the motion over each of these subdomains by means of a linear combination of trial functions. The subdomains are called *finite elements*, the set of finite elements is known as the *mesh* and the trial functions are referred to as *interpolation functions*. To introduce the ideas, we consider first a one-dimensional domain, such as that shown in Fig. 9.1, and denote the number of elements by n and the length of the elements by h , so that $nh = L$, where L is the length of the domain. For the sake of this discussion, we assumed that the elements are equal in length, although in general their length can vary, depending on the nature of the problem. The boundary points between two elements are known as *nodes* and their displacements as *nodal displacements*.¹ For example, node j lies between the elements $j - 1$ and j at a distance jh from the left end and has the displacement a_j . Figure 9.1 shows a displacement profile $w^{(n)}(x)$ approximating the actual displacement curve $w(x)$, in which the displacement $w^{(n)}(x)$ at any point x between two typical nodes $(j - 1)h$ and jh varies linearly from a_{j-1} at $(j - 1)h$ to a_j at jh . But, observing that the displacement profile can be generated by a superposition of triangles of width $2h$ and height a_j , except for the last triangle which is of width h , we can express $w^{(n)}(x)$ as the linear combination

$$w^{(n)}(x) = \sum_{j=1}^n a_j \phi_j(x) \quad (9.1)$$

where $\phi_j(x)$ are the *roof functions* shown in Fig. 9.2. All roof functions have *unit amplitude* and extend over two elements, $(j - 1)h \leq x \leq (j + 1)h$, with the exception of the function $\phi_n(x)$, which extends over the single element $(n - 1)h \leq x \leq nh = L$. We note with interest that the roof functions are *nearly orthogonal*, as ϕ_j is orthogonal to all other functions, except ϕ_{j-1} and ϕ_{j+1} , which are the only two functions overlapping ϕ_j . The near orthogonality has important computational implications.

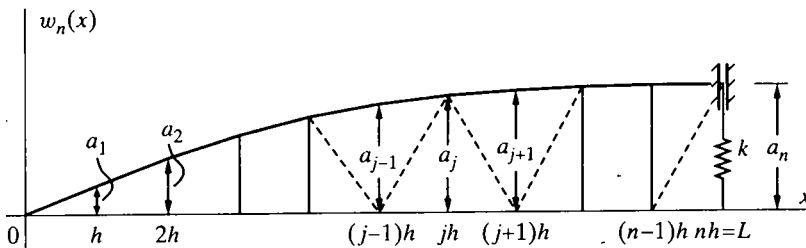


Figure 9.1 Displacement profile approximated by the finite element method

¹ This is an unfortunate term, in view of the fact that in vibrations nodes are defined as points of zero displacement. Nevertheless, the term is entrenched in finite element literature, so that we adopt it.

Comparing Eq. (9.1) with Eq. (8.76), we conclude that the finite element approximation has the same form as the Rayleigh-Ritz approximation. In fact, for second-order systems, such as strings, rods and shafts, not only that *the finite element solution* has the same form, but it is a *Rayleigh-Ritz solution*, provided the eigenvalue problem is formulated in variational form, as root functions are admissible for such systems. Moreover, observing that $w^{(n)}(x)$ is zero and $dw^{(n)}(x)/dx$ is different from zero at $x = 0$ and both $w^{(n)}(x)$ and $dw^{(n)}(x)/dx$ are different from zero at $x = L$, we conclude that $w^{(n)}(x)$ represents not a mere linear combination of admissible functions but a quasi-comparison function for second-order systems fixed at $x = 0$ and supported by a spring at $x = L$. Hence, Eq. (9.1) represents a solution for the enhanced version of the Rayleigh-Ritz method presented in Sec. 8.6. It follows that the finite element method can be based on the same mathematical foundation as the Rayleigh-Ritz method, although significant procedural differences remain. In fact, the enormous success of the finite element method can be attributed to these procedural differences. In recognition of the fact that the finite element method is a Rayleigh-Ritz method, we refer to the approach presented in Secs. 8.5 and 8.6 as the *classical Rayleigh-Ritz method*. Before we proceed with the details of the finite element method, a comparison with the classical Rayleigh-Ritz method should prove quite rewarding.

For second-order systems, the variational approach requires that the trial functions $\phi_j(x)$ be from the energy space \mathcal{K}_G^1 , i.e., they must be mere admissible functions. In the case at hand, the functions ϕ_j must be merely continuous. This rules out piecewise constant functions, but piecewise linear functions are admissible, provided they contain no discontinuities. Moreover, these piecewise linear functions need not be defined over the entire domain $D : 0 \leq x \leq L$, but only over certain subdomains $D_j : (j-1)h \leq x \leq jh$, and can be identically zero everywhere else. We refer to such a basis as a *local basis*, in contrast with the classical Rayleigh-Ritz method, which uses *global bases*. Clearly, the roof functions of Fig. 9.2 represent a local basis from the energy space \mathcal{K}_G^1 . In fact, they represent the simplest set of admissible functions. Hence, the appeal of the finite element method can be attributed to the fact that the admissible functions constitute a local basis of the simplest form permitted by the Rayleigh-Ritz theory.

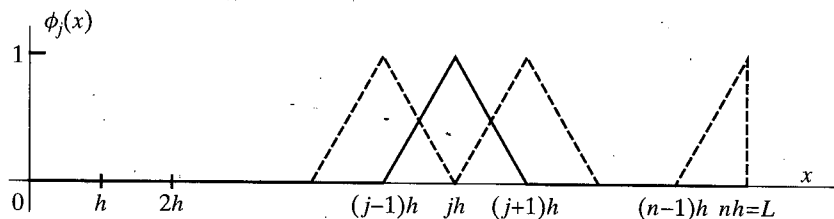


Figure 9.2 Roof functions as admissible functions

It was mentioned earlier that the roof functions are nearly orthogonal. To this should be added that *the near orthogonality holds regardless of any weighting functions*, as orthogonality is simply the result of absence of overlap between any two roof functions. The fact that all roof functions have unit amplitude has significant

physical implications. Indeed, *this gives the coefficients a_j in series (9.1) a great deal of physical significance, as it renders a_j equal to the approximate displacement $w^{(n)}(jh)$ of the node $x = x_j = jh$.* By contrast, in the classical Rayleigh-Ritz method, the coefficients a_j represent abstract quantities, not unlike the coefficients in a Fourier series expansion. Another difference between the finite element method and the classical Rayleigh-Ritz method lies in the nature of the convergence. In the classical Rayleigh-Ritz method, the addition of another admissible function ϕ_{n+1} to series (9.1) enlarges the Ritz space from \mathcal{R}_n to \mathcal{R}_{n+1} without affecting \mathcal{R}_n , where \mathcal{R}_n is a subspace of \mathcal{R}_{n+1} . The implication is that the mass matrices $M^{(n)}$ and $M^{(n+1)}$ on the one hand and the stiffness matrices $K^{(n)}$ and $K^{(n+1)}$ on the other hand possess the embedding property, Eqs. (8.100). As a result, the two sets of eigenvalues computed by means of the classical Rayleigh-Ritz method satisfy the separation theorem; inequalities (8.101), which guarantee monotonic convergence from above. By contrast, in the finite element method, the addition of another function ϕ_{n+1} can be done only by refining the mesh, which amounts to dividing the domain into $n + 1$ elements. As a result, *the entire set of n admissible functions ϕ_j ($j = 1, 2, \dots, n$) changes*, in the sense that the roof functions are now defined over smaller subdomains, even though the amplitudes remain unity. In view of this, the matrices $M^{(n)}$, $M^{(n+1)}$, $K^{(n)}$ and $K^{(n+1)}$ do not possess the embedding property, and there is no mathematical proof that the separation theorem holds (Ref. 14). This does not mean that it does not hold, or that the finite element method does not converge. Indeed, the method does converge (Ref. 22), provided the elements satisfy certain conditions, but proof of convergence is not an easy matter, particularly for two-dimensional domains, where mesh refinement presents many options.

The above considerations may seem trivial when we consider the real reasons why the finite element method has gained such universal acceptance. Among these reasons we cite the virtually routine choice of admissible functions, the minimal effort in producing the mass and stiffness matrices and versatility. Although the classical Rayleigh-Ritz method has many attributes from a mathematical point of view, the method has weaknesses from a practical point of view. In particular, the selection of admissible functions is a constant source of consternation, although the development of the class of quasi-comparison functions can at times mitigate the situation. Still, the selection must be regarded as an art rather than a well-established process. By contrast, in the finite element method the selection process is relatively routine, as there is by now an established inventory of interpolation functions ready to be used for most systems of interest. The finite element method has also a clear edge in the computation of the mass and stiffness matrices. In the classical Rayleigh-Ritz method, the computation involves integrations of relatively complicated functions over the entire domain. By contrast, the generation of the mass and stiffness matrices is relatively routine in the finite element method, as it consists of assembling predetermined element matrices. To put it in simple terms, the comparison is between a custom-made process requiring a great deal of experience and physical insight and an automated, mass-production process, whereby full advantage is taken of the power of the digital computer. In fact, the wide acceptance of the finite element method can be attributed in large part to the development of increasingly powerful computers permitting numerical solutions to very complex problems. Finally, whereas the

classical Rayleigh-Ritz method is essentially a structural method capable of treating linear systems with relatively uncomplicated geometry and parameter distributions, the finite element method, developed originally as a method for analyzing stresses in complicated aircraft structures, has evolved into a technique of great versatility, as it can be applied to a large variety of linear and nonlinear engineering problems.

The comparison is not as one-sided as it may seem, however. In the first place, the finite element method has a large drawback in that it requires a significantly larger number of degrees of freedom than the classical Rayleigh-Ritz method to achieve comparable accuracy (see, for example, Ref. 15). Moreover, the development of the component-mode synthesis (Sec. 8.11) and substructure synthesis (Sec. 8.12) has extended the usefulness of the concepts underlying the classical Rayleigh-Ritz method. Perhaps the single most important argument for an in-depth study of the classical Rayleigh-Ritz method is that so much of its mathematical theory extends to the finite element method.

9.2 SECOND-ORDER PROBLEMS. LINEAR INTERPOLATION FUNCTIONS.

In Sec. 9.1, we introduced the general ideas behind the finite element method as applied to eigenvalue problems without entering into any procedural details. In particular, a question of interest is how to generate the mass and stiffness matrices for a given set of trial functions. Of course, in the Rayleigh-Ritz method, the mass matrix involves the evaluation of weighted inner products, Eq. (8.88b), and the stiffness matrix requires energy inner products, Eq. (8.90). In the classical Rayleigh-Ritz method, the process is carried out in one step. By contrast, in the finite element method, the process is carried out in two steps, namely, the evaluation of element matrices and the assembly of these matrices to obtain global matrices. In this section, we demonstrate the process for the second-order system shown in Fig. 9.1.

From Eq. (8.89), Rayleigh's quotient can be written the form

$$R = \frac{[w, w]}{(\sqrt{mw}, \sqrt{mw})} = \frac{N}{D} = \frac{\sum_{j=1}^n N_j}{\sum_{j=1}^n D_j} \quad (9.2)$$

where N and D simply denote the numerator and denominator, respectively, and N_j and D_j represent the contributions from element j . We assume that the system of Fig. 9.1 represents a string in transverse vibration, so that, using the analogy with Eq. (8.104) for a rod in axial vibration, the element numerator and denominator can be written as

$$N_j = [w, w]_j = \int_{(j-1)h}^{jh} T(x) \left[\frac{dw(x)}{dx} \right]^2 dx + \delta_{nj} k w^2(L), \quad j = 1, 2, \dots, n \quad (9.3a)$$

$$D_j = (\sqrt{mw}, \sqrt{mw})_j = \int_{(j-1)h}^{jh} \rho(x) w^2(x) dx, \quad j = 1, 2, \dots, n \quad (9.3b)$$

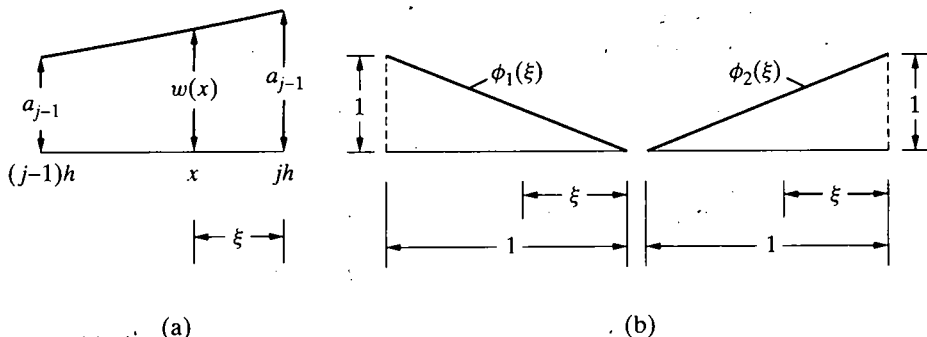


Figure 9.3 (a) String displacement over element j showing local coordinate
 (b) Linear interpolation functions

in which $T(x)$ is the tension in the string, k the end spring and $\rho(x)$ the mass density. At this point, we propose to simplify the evaluation of N_j and D_j greatly. To this end, we refer to Fig. 9.3a and introduce the local nondimensional coordinate²

$$\xi = \frac{jh - x}{h} = j - \frac{x}{h} \quad (9.4)$$

Then, we express the displacement w inside element j in terms of the nodal coordinates a_{j-1} and a_j in the form

$$w(\xi) = \phi_1(\xi)a_{j-1} + \phi_2(\xi)a_j = \Phi^T \mathbf{a}_j \quad (9.5)$$

where ϕ_1 and ϕ_2 are *trial functions*, also known as *shape functions*, or *interpolation functions, elements*, and $\Phi = [\phi_1 \ \phi_2]^T$ is an associated vector. Moreover, $\mathbf{a}_j = [a_{j-1} \ a_j]^T$ is known as a *nodal vector*. In the case at hand, *the elements are linear* and can be expressed as

$$\phi_i(\xi) = c_{i1} + c_{i2}\xi, \quad i = 1, 2 \quad (9.6)$$

in which c_{i1} and c_{i2} are constants yet to be determined. They can be determined by considering the fact that the displacement $w(\xi)$ at a given node must be equal to the corresponding nodal displacement. To this end, we use Eq. (9.5) and write

$$\begin{aligned} w(1) &= \phi_1(1)a_{j-1} + \phi_2(1)a_j = a_{j-1} \\ w(0) &= \phi_1(0)a_{j-1} + \phi_2(0)a_j = a_j \end{aligned} \quad (9.7)$$

from which we conclude that ϕ_1 and ϕ_2 must satisfy the *end conditions*

$$\phi_1(1) = 1, \quad \phi_1(0) = 0 \quad (9.8a)$$

and

$$\phi_2(1) = 0, \quad \phi_2(0) = 1 \quad (9.8b)$$

² Referred to in finite element terminology as a *natural*, or *normal coordinate*, another unfortunate term in view of the fact that in vibrations the term is used for decoupled modal coordinates.

respectively. Inserting Eqs. (9.8a) and (9.8b) into Eqs. (9.6), in sequence, we obtain two pairs of algebraic equations having the solution

$$c_{11} = 0, \quad c_{12} = 1 \quad (9.9a)$$

$$c_{21} = 1, \quad c_{22} = -1 \quad (9.9b)$$

so that the interpolation functions have the form

$$\phi_1 = \xi, \quad \phi_2 = 1 - \xi \quad (9.10)$$

The interpolation functions ϕ_1 and ϕ_2 are displayed in Fig. 9.3b. Note that, although we could have written the expressions for ϕ_1 and ϕ_2 directly from Fig. 9.3a, we chose to go through the process defined by Eqs. (9.6)–(9.10) in order to introduce the general procedure for generating interpolation functions.

The preceding derivation of the interpolation functions can be cast in matrix form. This may seem as a frivolous exercise, and for the simple task of generating two linear interpolation functions it is. However, the same procedure can be used for significantly more involved cases, and this simple example permits us to illustrate the ideas in an effective manner. Inserting Eqs. (9.8) into Eqs. (9.6), we obtain two nonhomogeneous algebraic equations for the constants c_{i1} and c_{i2} , which can be written in the matrix form

$$A\mathbf{c}_i = \mathbf{e}_i, \quad i = 1, 2 \quad (9.11)$$

where

$$A = \begin{bmatrix} 1 & \xi_1 \\ 1 & \xi_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad (9.12)$$

in which ξ_1 and ξ_2 are the values of ξ at the first and second node, respectively, i.e., $\xi = \xi_1 = 1$ and $\xi = \xi_2 = 0$. Moreover, $\mathbf{c}_i = [c_{i1} \ c_{i2}]^T$ are two-dimensional vectors of constants and \mathbf{e}_i are the two-dimensional standard unit vectors, $\mathbf{e}_1 = [1 \ 0]^T$ and $\mathbf{e}_2 = [0 \ 1]^T$. The solution of Eq. (9.11) is simply

$$\mathbf{c}_i = A^{-1}\mathbf{e}_i, \quad i = 1, 2 \quad (9.13)$$

from which we conclude that \mathbf{c}_1 and \mathbf{c}_2 are the first and second column of A^{-1} , respectively. Hence, we can write

$$[\mathbf{c}_1 \ \mathbf{c}_2] = \begin{bmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{bmatrix} = A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \quad (9.14)$$

which agrees with Eqs. (9.9).

At this point, we are ready to evaluate N_j and D_j . In the process, we derive the stiffness and mass matrices defining the eigenvalue problem. To this end, we must perform the integrations indicated in Eqs. (9.3). But, in view of the fact that the displacement w , Eq. (9.5), is in terms of the local coordinate ξ , we must first carry out the transformation from x to ξ . Hence, from Eq. (9.4), we transform the

differential element, the derivative with respect to x and the limits of integration, as follows:

$$dx = -h d\xi, \quad \frac{d}{dx} = \frac{d}{d\xi} \frac{d\xi}{dx} = -\frac{1}{h} \frac{d}{d\xi} \quad (9.15a, b)$$

$$x = jh \rightarrow \xi = 0 \quad (9.15c)$$

$$x = (j-1)h \rightarrow \xi = 1 \quad (9.15d)$$

Introducing Eqs. (9.5) and (9.15) into Eqs. (9.3), we obtain

$$\begin{aligned} N_j &= -\frac{1}{h} \int_1^0 T_j(\xi) \mathbf{a}_j^T \frac{d\Phi(\xi)}{d\xi} \frac{d\Phi^T(\xi)}{d\xi} \mathbf{a}_j d\xi + \delta_{jn} k \mathbf{a}_j^T \Phi(0) \Phi^T(0) \mathbf{a}_j \\ &= \mathbf{a}_j^T K_j \mathbf{a}_j, \quad j = 1, 2, \dots, n \end{aligned} \quad (9.16a)$$

$$D_j = -h \int_1^0 \rho_j(\xi) \mathbf{a}_j^T \Phi(\xi) \Phi^T(\xi) \mathbf{a}_j d\xi = \mathbf{a}_j^T M_j \mathbf{a}_j, \quad j = 1, 2, \dots, n \quad (9.16b)$$

where

$$K_j = \frac{1}{h} \int_0^1 T_j(\xi) \Phi'(\xi) \Phi'^T(\xi) d\xi + \delta_{jn} k \Phi(0) \Phi^T(0), \quad j = 1, 2, \dots, n \quad (9.17a)$$

are *element stiffness matrices*, in which $T_j(\xi)$ is the tension over element j and primes denote derivatives with respect to ξ , and

$$M_j = h \int_0^1 \rho_j(\xi) \Phi(\xi) \Phi^T(\xi) d\xi, \quad j = 1, 2, \dots, n \quad (9.17b)$$

are *element mass matrices*, in which $\rho_j(\xi)$ is the mass density over element j . Hence, inserting Eqs. (9.10) into Eqs. (9.17), the element stiffness and mass matrices become

$$K_j = \frac{1}{h} \int_0^1 T_j(\xi) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\xi + \delta_{jn} k \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad j = 1, 2, \dots, n \quad (9.18a)$$

and

$$M_j = h \int_0^1 \rho_j(\xi) \begin{bmatrix} \xi^2 & \xi(1-\xi) \\ \xi(1-\xi) & (1-\xi)^2 \end{bmatrix} d\xi, \quad j = 1, 2, \dots, n \quad (9.18b)$$

respectively. It should be pointed out that, due to the fact that $a_0 = 0$, we must remove the first row and column from matrices K_1 and M_1 . For computer programming purposes, it is perhaps simpler if the first row and column are removed after the assembly process.

The assembly process consists of inserting Eqs. (9.16) into Eq. (9.2) and writing the Rayleigh quotient in the discrete form

$$R = \frac{\sum_{j=1}^n \mathbf{a}_j^T K_j \mathbf{a}_j}{\sum_{j=1}^n \mathbf{a}_j^T M_j \mathbf{a}_j} = \frac{\mathbf{a}^T K \mathbf{a}}{\mathbf{a}^T M \mathbf{a}} \quad (9.19)$$

where $\mathbf{a} = [a_1 a_2 \dots a_n]^T$ is the *nodal vector*, from which the component $a_0 = 0$ was excluded, and K and M are the *global stiffness* and *mass matrices*, from which the first row and column were deleted. In carrying out the assembly of the element matrices, we observe that the nodal displacement a_j appears as the bottom component in \mathbf{a}_j and as the top component in \mathbf{a}_{j+1} . Consistent with this, there are two element matrix entries corresponding to a_j , the entry (2,2) of K_j and M_j and the entry (1, 1) of K_{j+1} and M_{j+1} . The assembly consists of elimination of the duplication of a_j from the nodal vector \mathbf{a} and adding correspondingly the (2,2) entries in K_j and M_j to the (1,1) entries in K_{j+1} and M_{j+1} ($j = 1, 2, \dots, n - 1$). The resulting global matrices K and M are real symmetric and positive definite; they are displayed schematically in the form

$$K = \left[\begin{array}{c} K_1 \\ K_2 \\ K_3 \\ \vdots \\ K_{n-1} \\ K_n \end{array} \right], \quad M = \left[\begin{array}{c} M_1 \\ M_2 \\ M_3 \\ \vdots \\ M_{n-1} \\ M_n \end{array} \right] \quad (9.20)$$

where we note that the shaded areas denote entries representing the sum of (1, 1) and (2, 2) entries as described above. We note that the near orthogonality of the roof functions is responsible for the *banded* nature of the stiffness and mass matrices, where the *half-bandwidth* is equal to one. A matrix is said to have half-bandwidth t if the entries (r, s) are zero for all r and s satisfying the inequality $s > r + t$.

By analogy with the classical Rayleigh-Ritz method (Sec. 8.5), the requirement that Rayleigh's quotient, Eq. (9.19), be stationary yields the algebraic eigenvalue problem

$$K\mathbf{a} = \lambda M\mathbf{a} \quad (9.21)$$

Moreover, because both K and M are real symmetric positive definite matrices, the eigenvalue problem can be reduced to one in terms of a single real symmetric positive definite matrix A (Sec. 4.6), which can be solved with ease by any of the methods discussed in Chapter 6.

For sufficiently small h , the parameters $T_j(\xi)$ and $\rho_j(\xi)$ can be regarded as being constant over the width of the element, although they can still vary from element to element, so that $T_j(\xi) \cong T_j = \text{constant}$, $\rho_j(\xi) \cong \rho_j = \text{constant}$ ($j = 1, 2, \dots, n$). Under these circumstances, following the integrations indicated in Eqs. (9.18), the global stiffness and mass matrices, Eqs. (9.20), can be shown to have the form

$$K = \frac{1}{h} \times$$

$$\begin{bmatrix}
 T_1 + T_2 & -T_2 & 0 & 0 & \dots & 0 & 0 & 0 \\
 & T_2 + T_3 & -T_3 & 0 & \dots & 0 & 0 & 0 \\
 & & T_3 + T_4 & -T_4 & \dots & 0 & 0 & 0 \\
 & & & T_4 + T_5 & \dots & 0 & 0 & 0 \\
 & & & & \dots & & & \\
 & & \text{symm} & & & T_{n-2} + T_{n-1} & -T_{n-1} & 0 \\
 & & & & & & T_{n-1} + T_n & -T_n \\
 & & & & & & & T_n + kh
 \end{bmatrix}
 \tag{9.22a}$$

and

$$M = \frac{h}{6} \times
 \begin{bmatrix}
 2(\rho_1 + \rho_2) & \rho_2 & 0 & 0 & \dots & 0 & 0 & 0 \\
 & 2(\rho_2 + \rho_3) & \rho_3 & 0 & \dots & 0 & 0 & 0 \\
 & & 2(\rho_3 + \rho_4) & \rho_4 & \dots & 0 & 0 & 0 \\
 & & & 2(\rho_4 + \rho_5) & \dots & 0 & 0 & 0 \\
 & & & & \dots & & & \\
 & & \text{symm} & & & 2(\rho_{n-2} + \rho_{n-1}) & \rho_{n-1} & 0 \\
 & & & & & & 2(\rho_{n-1} + \rho_n) & \rho_n \\
 & & & & & & & 2\rho_n
 \end{bmatrix}
 \tag{9.22b}$$

respectively. Of course, for a uniform string under constant tension, the stiffness and mass matrices assume the familiar form

$$K = \frac{T}{h} \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ & 2 & -1 & \dots & 0 & 0 \\ & & 2 & \dots & 0 & 0 \\ & & & \dots & & \\ & & \text{symm} & & 2 & -1 \\ & & & & & 1 + kh/T \end{bmatrix}, \quad M = \frac{\rho h}{6} \begin{bmatrix} 4 & 1 & 0 & \dots & 0 & 0 \\ & 4 & 1 & \dots & 0 & 0 \\ & & 4 & \dots & 0 & 0 \\ & & & \dots & & \\ & & \text{symm} & & 4 & 1 \\ & & & & & 2 \end{bmatrix}
 \tag{9.23a,b}$$

The formulation presented in this section applies to all second-order systems, including not only strings in transverse vibration but also rods in axial vibration and shafts in torsional vibration. The formulation is in terms of linear elements, which are the simplest elements satisfying the differentiability conditions required of admissible functions for second-order systems. Indeed, the displacement profile consists of a concatenation of linear segments and the transverse force profile is sectionally constant. This implies that the transverse force density, which is equal to $\partial(T\partial w/\partial x)/\partial x$, is a collection of spatial Dirac delta functions. Clearly, this may be mathematically acceptable, but is hard to reconcile with the physics of the problem, which dictates that the transverse force density be continuous throughout the domain. Of course, the problem disappears as $n \rightarrow \infty$. Not surprisingly, convergence of a solution in terms of linear elements is relatively slow, as shown in Example 9.1.

Still, such solutions are almost universally used. Convergence can be expedited by means of higher-order interpolation functions, as shown in Sec. 9.3. The question of convergence of the finite element method is examined later in this chapter.

Example 9.1

Solve the eigenvalue problem for the rod in axial vibration discussed in Sec. 8.6 by means of the finite element method in terms of linear interpolation functions. Compare the results with those obtained in Sec. 8.6 by means of the classical Rayleigh-Ritz method using mere admissible functions and the enhanced classical Rayleigh-Ritz method using quasi-comparison functions and draw conclusions.

The rod in axial vibration of Sec. 8.6 is entirely analogous to the string in transverse vibration discussed in this section. Hence, the element stiffness and mass matrices remain as given by Eqs. (9.18), except that we must replace the tension by the axial stiffness and change the notation for the mass density. Inserting Eq. (9.4) into Eqs. (8.105), the system parameters transform into

$$EA(\xi) = \frac{6EA}{5} \left[1 - \frac{1}{2} \left(\frac{h}{L} \right)^2 (j - \xi)^2 \right],$$

$$m(\xi) = \frac{6m}{5} \left[1 - \frac{1}{2} \left(\frac{h}{L} \right)^2 (j - \xi)^2 \right], \quad k = \frac{EA}{L}$$
(a)

so that, introducing Eqs. (a), into Eqs. (9.18) and carrying out the indicated integrations, we obtain the element stiffness matrices

$$K_j = \frac{6EA}{5L} \left[n - \frac{1}{6n} (1 - 3j + 3j^2) \right] \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \delta_{jn} \frac{EA}{L} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$j = 1, 2, \dots, n$$
(b)

and the element mass matrices

$$M_j = \frac{mL}{5n} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} - \frac{mL}{100n} \begin{bmatrix} 2(6 - 15j + 10j^2) & 3 - 10j + 10j^2 \\ 3 - 10j + 10j^2 & 2(1 - 5j + 10j^2) \end{bmatrix},$$

$$j = 1, 2, \dots, n$$
(c)

in which we set $L/h = n$. It should be pointed out here that the element stiffness and mass matrices, Eqs. (b) and (c), are "exact," in the sense that their entries were computed using the actual expressions for the system parameters, Eqs. (a). This was done because this example involves such small numbers n of elements that the assumption of constant parameter values over the elements would cause gross errors. In ordinary finite element practice, the parameters are taken as constant over the elements as a rule, because the number n of elements tends to be large and the width of the elements tends to be small.

Recalling that the first row and column of K_1 and M_1 must be omitted and using the scheme (9.20), we obtain the global stiffness matrix

$$K = \frac{6EA n}{5L} \begin{bmatrix} 2 & -1 & 0 & \dots & 0 & 0 \\ & 2 & -1 & \dots & 0 & 0 \\ & & 2 & \dots & 0 & 0 \\ \text{symm} & & & \dots & & \\ & & & & 2 & -1 \\ & & & & & 1 + 1/6n \end{bmatrix}.$$

$$-\frac{EA}{5Ln} \begin{bmatrix} 8 & -7 & 0 & \dots & 0 & 0 \\ & 26 & -19 & \dots & 0 & 0 \\ & & 56 & \dots & 0 & 0 \\ \text{symm} & & & \dots & & \\ & & & & 2(4 - 6n + 3n^2) & -(1 - 3n + 3n^2) \\ & & & & & 1 - 3n + 3n^2 \end{bmatrix} \quad (d)$$

and the global mass matrix

$$M = \frac{mL}{5n} \begin{bmatrix} 4 & 1 & 0 & \dots & 0 & 0 \\ & 4 & 1 & \dots & 0 & 0 \\ & & 4 & \dots & 0 & 0 \\ \text{symm} & & & \dots & & \\ & & & & 4 & 1 \\ & & & & & 2 \end{bmatrix}$$

$$-\frac{mL}{100n^3} \begin{bmatrix} 44 & 23 & 0 & \dots & 0 & 0 \\ & 164 & 63 & \dots & 0 & 0 \\ & & 364 & \dots & 0 & 0 \\ \text{symm} & & & \dots & & \\ & & & & 2(37 - 50n + 20n^2) & 3 - 10n + 10n^2 \\ & & & & & 2(1 - 5n + 10n^2) \end{bmatrix} \quad (e)$$

The eigenvalue problem associated with the matrices K and M has been solved in Ref. 15 for $n = 1, 2, \dots, 30$, and the first three approximate natural frequencies are listed in Table 9.1. Contrasting the results of Table 9.1 with the results of Table 8.1, we arrive to mixed conclusions. Indeed, the finite element method using linear elements shows faster convergence to ω_1 and slower convergence to ω_2 and ω_3 than the classical Rayleigh-Ritz method using mere admissible functions. On the other hand, from Tables 8.3 and 9.1, we conclude that convergence of the finite element method using linear

TABLE 9.1 First, Second, and Third Natural Frequencies Computed by the Finite Element Method Using Linear Elements

n	$\omega_1^{(n)} \sqrt{mL^2/EA}$	$\omega_2^{(n)} \sqrt{mL^2/EA}$	$\omega_3^{(n)} \sqrt{mL^2/EA}$
1	2.67261	—	—
2	2.32551	6.27163	—
3	2.26469	5.68345	9.88405
4	2.24326	5.43128	9.41491
5	2.23330	5.31181	8.98398
⋮	⋮	⋮	⋮
28	2.21609	5.10625	8.14407
29	2.21605	5.10580	8.14218
30	2.21602	5.10539	8.14049

elements is very slow compared to the enhanced Rayleigh-Ritz method using quasi-comparison functions. Indeed, in the enhanced Rayleigh-Ritz method using quasi-comparison functions $\omega_1^{(n)}$ and $\omega_2^{(n)}$ reach convergence to ω_1 and ω_2 , respectively, with $n = 6$ and $\omega_3^{(n)}$ achieves convergence to ω_3 with $n = 13$. By contrast, the finite element method is still not close to convergence with $n = 30$.

9.3 HIGHER-DEGREE INTERPOLATION FUNCTIONS

As pointed out on several occasions, one of the reasons for the wide appeal of the finite element method is the simplicity of the admissible functions. Indeed, as a rule, they are low-degree polynomials. For second-order systems, the lowest degree admissible is the first, i.e., the interpolation functions are linear. But, as discovered in Example 9.1, there is a price to be paid for this simplicity in that convergence tends to be slow. Hence, the question arises as to whether convergence can be accelerated by using higher-degree polynomials. In this section, we address the question by considering quadratic and cubic interpolation functions. Higher-degree polynomials can be generated by means of Lagrange's interpolation formula (Ref. 11). Quadratic and cubic polynomials can be generated just as easily by means of the approach of Sec. 9.2.

Quadratic interpolation functions, or quadratic elements, can be generated by means of the quadratic polynomials

$$\phi_i = c_{i1} + c_{i2}\xi + c_{i3}\xi^2, \quad i = 1, 2, 3 \quad (9.24)$$

but we run immediately into a problem when we try to determine the coefficients c_{i1} , c_{i2} and c_{i3} ($i = 1, 2, 3$). Indeed, there are three coefficients to be determined for every element and only two nodes available for producing the three necessary conditions. It follows that another node must be created. For simplicity, we choose the location of the third node at $\xi = 1/2$ and denote the corresponding nodal displacement by $a_{j-1/2}$. Clearly, the point $\xi = 1/2$ represents an *internal node*, which makes the points $\xi = 0$ and $\xi = 1$ *external nodes*. Following the pattern established in Sec. 9.2, we express the approximate displacement in the form

$$w(\xi) = \phi_1(\xi)a_{j-1} + \phi_2(\xi)a_{j-1/2} + \phi_3(\xi)a_j = \boldsymbol{\phi}^T \mathbf{a}_j \quad (9.25)$$

where the vector $\boldsymbol{\phi} = [\phi_1 \ \phi_2 \ \phi_3]^T$ of interpolation functions and the nodal vector $\mathbf{a}_j = [a_{j-1} \ a_{j-1/2} \ a_j]^T$ are now three-dimensional.

The determination of the coefficients c_{i1} , c_{i2} , c_{i3} ($i = 1, 2, 3$) to be used in Eqs. (9.24) follows the pattern established in Sec. 9.2, Eqs. (9.11)–(9.14). To this end, we observe that the values of ξ at the nodes, taken in sequence, are $\xi_1 = 1$, $\xi_2 = 1/2$ and $\xi_3 = 0$ and write

$$A = \begin{bmatrix} 1 & \xi_1 & \xi_1^2 \\ 1 & \xi_2 & \xi_2^2 \\ 1 & \xi_3 & \xi_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1/2 & 1/4 \\ 1 & 0 & 0 \end{bmatrix} \quad (9.26)$$

so that

$$[c_1 \ c_2 \ c_3] = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} = A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 4 & -3 \\ 2 & -4 & 2 \end{bmatrix} \quad (9.27)$$

Inserting the values of the constants c_{i1} , c_{i2} and c_{i3} ($i = 1, 2, 3$) from Eq. (9.27) into Eqs. (9.24), we obtain the *quadratic interpolation functions*, or *quadratic elements*

$$\phi_1 = \xi(2\xi - 1), \quad \phi_2 = 4\xi(1 - \xi), \quad \phi_3 = (1 - \xi)(1 - 2\xi) \quad (9.28)$$

The functions ϕ_i ($i = 1, 2, 3$) are displayed in Fig. 9.4.

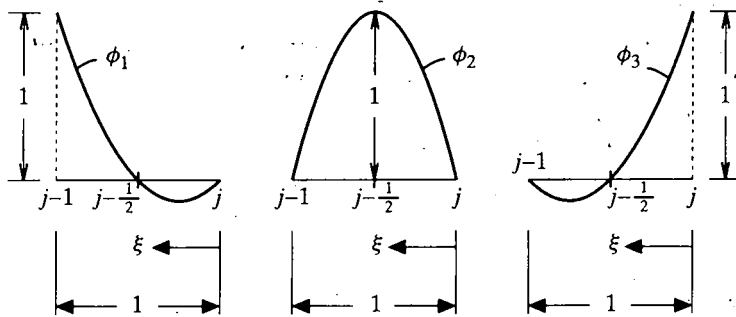


Figure 9.4 Quadratic interpolation functions

The stiffness and mass matrices remain in the general form of Eqs. (9.17a) and (9.17b), respectively. Inserting Eqs. (9.28) into Eqs. (9.17), we obtain the element stiffness and mass matrices in the more explicit form

$$K_j = \frac{1}{h} \int_0^1 T_j(\xi) \begin{bmatrix} (4\xi - 1)^2 & 4(4\xi - 1)(1 - 2\xi) & (4\xi - 1)(4\xi - 3) \\ & 16(1 - 2\xi)^2 & 4(1 - 2\xi)(4\xi - 3) \\ \text{symm} & & (4\xi - 3)^2 \end{bmatrix} d\xi + \delta_{jn} k \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad j = 1, 2, \dots, n \quad (9.29a)$$

and

$$M_j = h \int_0^1 \rho_j(\xi) \begin{bmatrix} \xi^2(2\xi - 1)^2 & 4\xi^2(2\xi - 1)(1 - \xi) & -\xi(1 - \xi)(1 - 2\xi)^2 \\ & 16\xi^2(1 - \xi)^2 & 4\xi(1 - \xi)^2(1 - 2\xi) \\ \text{symm} & & (1 - \xi)^2(1 - 2\xi)^2 \end{bmatrix} d\xi, \quad j = 1, 2, \dots, n \quad (9.29b)$$

respectively. Using the assembly process described in Sec. 9.2 and recalling once again that $a_0 = 0$, the *global stiffness* and *mass matrices* can be displayed in the

schematic form

$$K = \begin{bmatrix} K_1 & & & & & \\ & K_2 & & & & \\ & & K_3 & & & \\ & & & \dots & & \\ & & & & K_{n-1} & \\ & & & & & K_n \end{bmatrix}, \quad M = \begin{bmatrix} M_1 & & & & & \\ & M_2 & & & & \\ & & M_3 & & & \\ & & & \dots & & \\ & & & & M_{n-1} & \\ & & & & & M_n \end{bmatrix} \quad (9.30)$$

The shaded matrix elements correspond to the external nodes and the elements in-between correspond to internal nodes. Both matrices are banded, with a half-bandwidth equal to two.

For uniform tension, the element stiffness matrices reduce to

$$K_j = \frac{T}{3h} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} + \delta_{jnk} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad j = 1, 2, \dots, n \quad (9.31a)$$

which are all the same except K_n , and for uniform mass density the element mass matrices become

$$M_j = \frac{h\rho}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}, \quad j = 1, 2, \dots, n \quad (9.31b)$$

and these are all the same. Using the scheme given by Eqs. (9.30), the global stiffness matrix and global mass matrix can be shown to have the form

$$K = \frac{T}{3h} \begin{bmatrix} 16 & -8 & 0 & \dots & 0 & 0 & 0 \\ & 14 & -8 & \dots & 0 & 0 & 0 \\ & & 16 & \dots & 0 & 0 & 0 \\ & & & \dots & & & \\ & & & \text{symm} & & & \\ & & & & 14 & -8 & 1 \\ & & & & & 16 & -8 \\ & & & & & & 7 + 3kh/T \end{bmatrix} \quad (9.32a)$$

and

$$M = \frac{h\rho}{30} \begin{bmatrix} 16 & 2 & 0 & \dots & 0 & 0 & 0 \\ & 8 & 2 & \dots & 0 & 0 & 0 \\ & & 16 & \dots & 0 & 0 & 0 \\ & & & \dots & & & \\ & & & \text{symm} & & & \\ & & & & 8 & 2 & -1 \\ & & & & & 16 & 2 \\ & & & & & & 4 \end{bmatrix} \quad (9.32b)$$

respectively.

The same procedure can be used to derive the *cubic interpolation functions*. To this end, we begin with the polynomials

$$\phi_i = c_{i1} + c_{i2}\xi + c_{i3}\xi^2 + c_{i4}\xi^3, \quad i = 1, 2, 3, 4 \quad (9.33)$$

and observe that now we must have four nodes, *two external and two internal nodes*. Then, assuming that the approximate displacement has the expression

$$w(\xi) = \phi_1(\xi)a_{j-1} + \phi_2(\xi)a_{j-2/3} + \phi_3(\xi)a_{j-1/3} + \phi_4(\xi)a_j = \boldsymbol{\Phi}^T \mathbf{a}_j \quad (9.34)$$

where the four-dimensional vectors $\boldsymbol{\Phi}$ and \mathbf{a}_j are obvious, and following the established pattern, we obtain the *cubic elements* (Problem 9.7)

$$\begin{aligned} \phi_1 &= \frac{1}{2}\xi(2 - 9 + 9\xi^2), & \phi_2 &= -\frac{9}{2}\xi(1 - 4\xi + 3\xi^2) \\ \phi_3 &= \frac{9}{2}\xi(2 - 5\xi + 3\xi^2), & \phi_4 &= 1 - \frac{11}{2}\xi + 9\xi^2 - \frac{9}{2}\xi^3 \end{aligned} \quad (9.35)$$

They are shown in Fig. 9.5. The calculation of the 4×4 element stiffness and mass matrices for cubic elements and given system parameters can be carried out by inserting Eqs. (9.35) into Eqs. (9.17) and performing the indicated integrations (Problem 9.8).

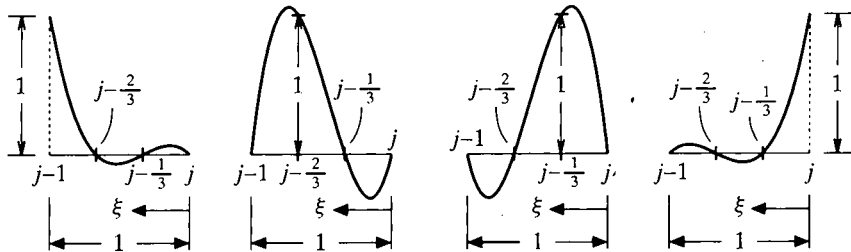


Figure 9.5 Cubic interpolation functions

In general, approximate eigenvalues computed on the basis of quadratic interpolation functions tend to be more accurate than those computed using linear interpolation functions, and the same can be said for approximate eigenvalues based on cubic elements compared to those based on quadratic elements.

Example 9.2

Solve the eigenvalue problem for the rod in axial vibration of Example 9.1 by means of the finite element method in two ways, first using quadratic interpolation functions and then using cubic interpolation functions. Compare the results with those obtained in Sec. 8.6 by the classical Rayleigh-Ritz method in conjunction with quasi-comparison functions and in Example 9.1 by the finite element method using linear interpolation functions and draw conclusions.

The 3×3 element stiffness and mass matrices for quadratic elements are obtained by inserting the system parameters given by Eqs. (a) of Example 9.1 into Eqs. (9.29) with $T_j(\xi)$ and $\rho_j(\xi)$ replaced by $EA_j(\xi)$ and $m_j(\xi)$, respectively, and carrying out the indicated integrations. Then, the global stiffness and mass matrices are assembled by

TABLE 9.2 First Three Natural Frequencies Computed by the Finite Element Method Using Quadratic (FEMQ) and Cubic (FEMC) Interpolation Functions

n	$\omega_1^{(n)} \sqrt{mL^2/E A}$		$\omega_2^{(n)} \sqrt{mL^2/E A}$		$\omega_3^{(n)} \sqrt{mL^2/E A}$	
	FEMQ	FEMC	FEMQ	FEMC	FEMQ	FEMC
1	—	—	—	—	—	—
2	2.23433	—	6.27984	—	—	—
3	—	2.1558	—	5.25278	—	10.76486
4	2.21705	—	5.18817	—	9.13233	—
5	—	—	—	—	—	—
6	2.21584	2.21553	5.12170	5.10352	8.30344	8.21161
7	—	—	—	—	—	—
8	2.21563	—	5.10695	—	8.18985	—
9	—	2.21553	—	5.09988	—	8.12705
10	2.21557	—	5.10265	—	8.14851	—
11	—	—	—	—	—	—
12	2.21555	2.21552	5.10105	5.09959	8.13242	8.11835
13	—	—	—	—	—	—
14	2.21554	—	5.10036	—	8.12520	—
15	—	—	—	5.09954	—	8.11688
16	2.21553	—	5.10002	—	8.12160	—
17	—	—	—	—	—	—
18	2.21553	—	5.09983	5.09953	8.11965	8.11651
19	—	—	—	—	—	—
20	2.21553	—	5.09973	—	8.11852	—
21	—	—	—	5.09953	—	8.11640
22	2.21553	—	5.09966	—	8.11783	—
23	—	—	—	—	—	—
24	2.21553	—	5.09962	5.09953	8.11739	8.11635
25	—	—	—	—	—	—
26	2.21553	—	5.09960	—	8.11710	—
27	—	—	—	5.09953	—	8.11634
28	2.21553	—	5.09958	—	8.11690	—
29	—	—	—	—	—	—
30	2.21552	—	5.09956	5.09953	8.11676	8.11633

using the pattern exhibited in Eqs. (9.30). We note that, as the number of finite elements increases by one the dimension of the global matrices increases by two.

In a similar fashion, the 4×4 element stiffness and mass matrices for cubic elements are computed by introducing the same system parameters and Eq. (9.35) into

Eqs. (9.17) and performing the prescribed integrations. Then, the global stiffness and mass matrices are assembled in the customary way. But, because in the case of cubic elements there are two internal nodes for every external node, the global matrices are characterized by two nonshaded entries on the main diagonal separating any pair of shaded entries. Consistent with this, as the number of finite elements increases by one, the dimension of the global matrices increases by three. The derivation of the global matrices follows the established pattern and is the subject of Problem 9.8.

The eigenvalue problems corresponding to quadratic and cubic elements have been solved in Ref. 15, and the results are displayed in Table 9.2. In the first place, we note that, in using quadratic elements, the number of degrees of freedom increases by two at a time, and so does the number of computed natural frequencies. In the case of cubic elements, the number of computed natural frequencies increases by three at a time. This explains the empty spaces in Table 9.2. A comparison of Tables 9.1 and 9.2 confirms the expectation that the finite element model based on quadratic elements has better convergence characteristics than the model based on linear elements, and the model using cubic elements converges faster than the model using quadratic elements. This statement must be tempered by the realization that the relatively good results obtained by the finite element method in conjunction with cubic interpolation functions fall far short of the results obtained by the enhanced Rayleigh-Ritz method using quasi-comparison functions. Indeed, a comparison of Tables 8.3 and 9.2 reveals convergence of the first three natural frequencies computed by the enhanced Rayleigh-Ritz method with six-, six- and thirteen-degree-of-freedom models. By contrast, the first natural frequency computed by the finite element method using cubic elements converges with a twelve-degree-of-freedom model, whereas the second and third natural frequencies have not achieved convergence with a thirty-degree-of-freedom model.

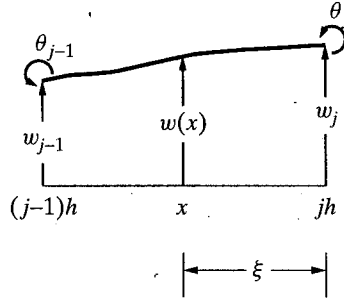
9.4 BEAMS IN BENDING VIBRATION

According to the Rayleigh-Ritz theory, the lowest degree polynomials admissible for beams in bending are quadratic, so that we consider them as possible candidates for interpolation functions. To this end, we recognize that in bending both the displacement and the slope must be continuous at nodal points. Because there are two nodes, it follows that every interpolation function must satisfy four end conditions. But, as shown in Sec. 9.3, quadratic interpolation functions are defined by only three constants, from which it follows that second-degree polynomials cannot be used as interpolation functions. Hence, the lowest-degree polynomials admissible are cubic, which are defined by four constants.

The derivation of the interpolation functions for beams in bending can be carried out by considering the typical finite element shown in Fig. 9.6, in which w_{j-1} , θ_{j-1} and w_j , θ_j denote the translation and rotation of the beam at the nodal points $j-1$ and j , respectively. Using the same approach as in Secs. 9.2 and 9.3, we express the displacement at point ξ in the form

$$w(\xi) = \phi_1(\xi)w_{j-1} + \phi_2(\xi)h\theta_{j-1} + \phi_3(\xi)w_j + \phi_4(\xi)h\theta_j = \Phi^T \mathbf{a}_j \quad (9.36)$$

where $\Phi = [\phi_1 \ \phi_2 \ \phi_3 \ \phi_4]^T$ and $\mathbf{a}_j = [w_{j-1} \ h\theta_{j-1} \ w_j \ h\theta_j]^T$. Note that we multiplied the rotations θ_{j-1} and θ_j by h so as to ensure that the interpolation functions ϕ_1 , ϕ_2 , ϕ_3 and ϕ_4 are all dimensionless.

Figure 9.6 Beam displacement over element j

As in Sec. 9.3, cubic elements have the form

$$\phi_i(\xi) = c_{i1} + c_{i2}\xi + c_{i3}\xi^2 + c_{i4}\xi^3, \quad i = 1, 2, 3, 4 \quad (9.37)$$

in which c_{i1} , c_{i2} , c_{i3} and c_{i4} ($i = 1, 2, 3, 4$) are coefficients to be determined. To this end, we replace the rotations θ_{j-1} and θ_j by the slopes of the displacement curve at the nodes $j-1$ and j , respectively, recall Eq. (9.15b) and write

$$\theta = \frac{dw}{dx} = \frac{dw}{d\xi} \frac{d\xi}{dx} = -\frac{1}{h} \frac{dw}{d\xi} = -\frac{1}{h} w' \quad (9.38)$$

Then, following the pattern of Secs. 9.2 and 9.3 and considering Eq. (9.38), we can write

$$A = \begin{bmatrix} 1 & \xi_1 & \xi_1^2 & \xi_1^3 \\ 0 & -1 & -2\xi_1 & -3\xi_1^2 \\ 1 & \xi_2 & \xi_2^2 & \xi_2^3 \\ 0 & -1 & -2\xi_2 & -3\xi_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \quad (9.39)$$

in which we substituted $\xi_1 = 1$ and $\xi_2 = 0$. Moreover, we observe that the second row of A is the negative of the derivative of the first row with respect to ξ_1 and an analogous statement can be made about the fourth and third rows. Hence, the desired coefficients are obtained as the elements of the columns of A^{-1} , or

$$[c_1 \ c_2 \ c_3 \ c_4] = A^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 3 & 1 & -3 & 2 \\ -2 & -1 & 2 & -1 \end{bmatrix} \quad (9.40)$$

Inserting the elements of the columns of A^{-1} into Eqs. (9.37), we obtain the interpolation functions

$$\phi_1 = 3\xi^2 - 2\xi^3, \quad \phi_2 = \xi^2 - \xi^3, \quad \phi_3 = 1 - 3\xi^2 + 2\xi^3, \quad \phi_4 = -\xi + 2\xi^2 - \xi^3 \quad (9.41)$$

The interpolation functions given by Eqs. (9.41) are known as *Hermite cubics*. They are displayed in Fig. 9.7.

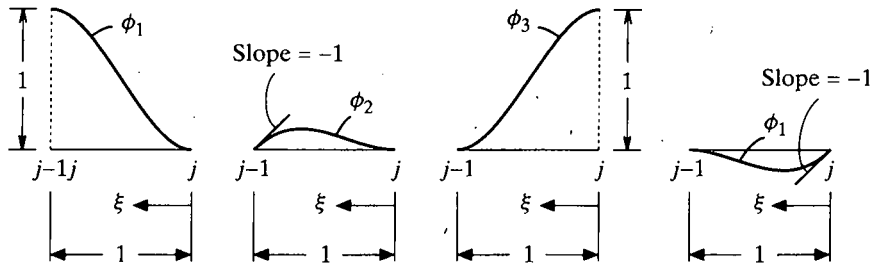


Figure 9.7 Hermite cubics

The derivation of the global stiffness and mass matrices follows a pattern similar to that established in Sec. 9.2. To demonstrate the process, we consider a beam in bending subjected to an axial force and recall the definition of Rayleigh's quotient

$$R = \frac{[w, w]}{(\sqrt{m}w, \sqrt{m}w)} = \frac{\sum_{j=1}^n N_j}{\sum_{j=1}^n D_j} \quad (9.42)$$

first encountered in Sec. 9.2, where, from Secs. 7.2 and 7.5,

$$N_j = [w, w]_j = \int_{(j-1)h}^{jh} \left\{ EI(x) \left[\frac{d^2 w(x)}{dx^2} \right]^2 + P(x) \left[\frac{dw(x)}{dx} \right]^2 \right\} dx, \quad j = 1, 2, \dots, n \quad (9.43a)$$

$$D_j = (\sqrt{m}w, \sqrt{m}w)_j = \int_{(j-1)h}^{jh} m(x) w^2(x) dx, \quad j = 1, 2, \dots, n \quad (9.43b)$$

Then, changing variables from x to the local coordinate ξ according to Eqs. (9.15) and using Eq. (9.36), we have

$$N_j = \mathbf{a}_j^T K_j \mathbf{a}_j, \quad D_j = \mathbf{a}_j^T M_j \mathbf{a}_j, \quad j = 1, 2, \dots, n \quad (9.44a, b)$$

in which

$$K_j = \frac{1}{h^3} \int_0^1 [EI_j(\xi) \Phi'' \Phi'^T + h^2 P_j(\xi) \Phi' \Phi'^T] d\xi, \quad j = 1, 2, \dots, n \quad (9.45a)$$

$$M_j = h \int_0^1 m_j(\xi) \Phi \Phi^T d\xi, \quad j = 1, 2, \dots, n \quad (9.45b)$$

are the element stiffness matrix and element mass matrix, respectively.

Finally, there is the assembly process, which requires the system boundary conditions. As an illustration, we consider a beam clamped at $x = 0$ and free at $x = L$. In this case, the assembly process yields global stiffness and mass matrices of the form depicted in Eqs. (9.20), except that now matrices K_1 and M_1 are 2×2 and matrices K_j and M_j are 4×4 ($j = 2, 3, \dots, n$).

Example 9.3

Use the finite element method to derive the global stiffness and mass matrices for a uniform helicopter blade rotating with the constant angular velocity Ω .

From Example 7.4, the energy inner product has the expression

$$[w, w] = \int_0^L \left\{ EI \left[\frac{d^2 w(x)}{dx^2} \right]^2 + P(x) \left[\frac{dw(x)}{dx} \right]^2 \right\} dx \quad (a)$$

where, from Example 7.2, the axial force is given by

$$P(x) = \int_x^L m \Omega^2 \xi d\xi = \frac{1}{2} m \Omega^2 L^2 \left[1 - \left(\frac{x}{L} \right)^2 \right] \quad (b)$$

The weighted inner product is simply

$$(\sqrt{m}w, \sqrt{m}w) = \int_0^L m w^2(x) dx \quad (c)$$

Hence, using Eqs. (9.45), the element stiffness and mass matrices have the form

$$K_j = \frac{1}{h^3} \left\{ EI \int_0^1 \Phi'' \Phi''^T d\xi + \frac{1}{2} m \Omega^2 L^2 h^2 \int_0^1 \left[1 - \left(\frac{h}{L} \right)^2 (j - \xi)^2 \right] \Phi' \Phi'^T d\xi \right\}, \quad j = 1, 2, \dots, n \quad (d)$$

and

$$M_j = hm \int_0^L \Phi \Phi^T d\xi, \quad j = 1, 2, \dots, n \quad (e)$$

respectively, where, from Eqs. (9.41), the vector of interpolation functions is

$$\Phi = [3\xi^2 - 2\xi^3 \quad \xi^2 - \xi^3 \quad 1 - 3\xi^2 + 2\xi^3 \quad -\xi + 2\xi^2 - \xi^3]^T \quad (f)$$

Inserting Eqs. (f) into Eqs. (d) and (e) and carrying out the appropriate integrations, we obtain the explicit element stiffness matrices

$$K_j = \frac{EI}{h^3} \begin{bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ & \text{symm} & 12 & -6 \\ & & & 4 \end{bmatrix} + \frac{1}{2} m \Omega^2 L n \left(\frac{1}{30} \left[1 - \left(\frac{j}{n} \right)^2 \right] \begin{bmatrix} 36 & -3 & -36 & 3 \\ & 4 & -3 & 1 \\ & \text{symm} & 36 & -3 \\ & & & 4 \end{bmatrix} - \frac{j}{30n} \begin{bmatrix} 36 & 0 & -36 & 6 \\ & 6 & 0 & -1 \\ & \text{symm} & 36 & -6 \\ & & & 2 \end{bmatrix} + \frac{1}{210n^2} \begin{bmatrix} 72 & -6 & -72 & -15 \\ & 18 & 6 & -3 \\ & \text{symm} & 72 & 15 \\ & & & 4 \end{bmatrix} \right), \quad j = 1, 2, \dots, n \quad (g)$$

and element mass matrices

$$M_j = \frac{mL}{420n} \begin{bmatrix} 156 & 22 & 54 & -13 \\ & 4 & 13 & -3 \\ & \text{symm} & 156 & -22 \\ & & & 4 \end{bmatrix}, \quad j = 1, 2, \dots, n \quad (h)$$

$$+ \frac{1}{210n^2} \begin{bmatrix} 144 & 9 & -72 & -15 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ & 22 & 6 & -3 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ & & 144 & 9 & -72 & -15 & \dots & 0 & 0 & 0 & 0 \\ & & & 22 & 6 & -3 & \dots & 0 & 0 & 0 & 0 \\ & & & & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & & & & 144 & 9 & -72 & -15 \\ & & & & & & & \text{symm} & & 22 & 6 & -3 \\ & & & & & & & & & & 72 & 15 \\ & & & & & & & & & & & 4 \end{bmatrix} \quad (i)$$

and the global mass matrix is

$$M = \frac{mL}{420n} \begin{bmatrix} 312 & 0 & 54 & -13 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ & 8 & 13 & -3 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ & & 312 & 0 & 54 & -13 & \dots & 0 & 0 & 0 & 0 \\ & & & 8 & 13 & -3 & \dots & 0 & 0 & 0 & 0 \\ & & & & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & & & & 312 & 0 & 54 & -13 \\ & & & & & & & \text{symm} & & 8 & 13 & -3 \\ & & & & & & & & & & 156 & -22 \\ & & & & & & & & & & & 4 \end{bmatrix} \quad (j)$$

9.5 VIBRATION OF MEMBRANES. TRIANGULAR ELEMENTS

The transverse vibration of membranes is described by two-dimensional boundary-value problems, which are materially more complex than one-dimensional ones. Indeed, as explained in Sec. 7.12, an important consideration in two-dimensional problems is the shape of the boundary. In the relatively few cases in which closed-form solutions are possible, the boundary shape dictates the choice of coordinates. More often than not, however, the shape of the boundary places closed-form solutions beyond reach, making approximate solutions a virtual necessity. It is here that the versatility of the finite element method becomes evident, as it permits solutions where other methods fail. In using the finite element method, the choice of coordinates tends to fade as an issue, at least in the classical sense.

In seeking a closed-form solution, a smooth boundary can be taken as an indication that the solution may be smooth. In contrast, in approximate solutions by the finite element method, a smooth boundary creates a new problem in that the choice of the finite element mesh must not only permit an accurate approximate solution but also the perimeter of the bounding polygon must approximate well the boundary itself. Indeed, when the boundary is smooth it is generally not feasible to devise a finite element mesh covering the domain D exactly, so that there is a difference between D and the domain $D^{(n)}$ covered by the finite element mesh, where the latter is referred to as the *finite element domain*. An important question relates to the shape of the elements minimizing the difference $D - D^{(n)}$ between the actual domain bounded by a smooth curve and the finite element domain bounded by a polygon. Experience shows that triangular elements are particularly suited to the task of filling tightly domains with smooth boundaries, thus minimizing $D - D^{(n)}$. Of course, this implies the use of increasingly smaller elements as we approach the boundary. This is demonstrated in Fig. 9.8, in which $D - D^{(n)}$ represents the union

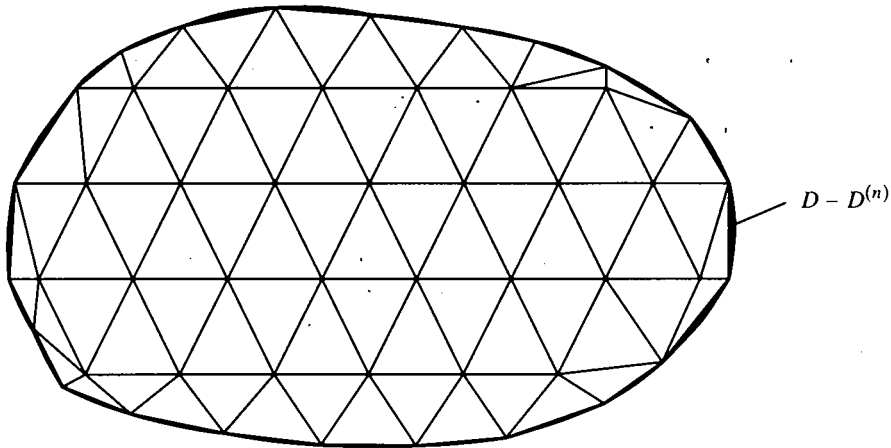


Figure 9.8 Finite element mesh for a two-dimensional domain

of the small shaded areas. It is clear that, as the number n of elements increases, the difference $D - D^{(n)}$ tends to disappear.

The problem of approximating a domain D with a smooth boundary S by a domain $D^{(n)}$ with a polygonal boundary $S^{(n)}$ can be regarded as the problem of approximating S by $S^{(n)}$. The latter problem is very old indeed, as mathematicians in ancient times were able to calculate π with a high degree of accuracy by calculating the perimeter of a polygon with equal sides inscribed in a circle and progressively increasing the number of sides. This purely geometric idea was given physical content by Courant (Ref. 8), who used triangular elements to produce an approximate solution to the plane torsion problem for multiconnected domains, thus anticipating the finite element method by over a decade.

Triangular elements are equally useful in cases in which the boundaries possess corners. There are cases, however, in which quadrilateral elements may prove superior, as they are able to produce a finite element mesh with fewer elements.

Another question arising in two-dimensional problems not encountered in one-dimensional ones relates to the choice of a numbering scheme for the elements and the nodal points. In the one-dimensional case, the nodes are along a straight line and the numbering of the elements and nodes progresses uninterruptedly from one boundary point to the other. The resulting mass and stiffness matrices are banded, which is due to the fact that the interpolation functions are nearly orthogonal. Whereas a similar situation exists in the case of two-dimensional problems, the bandedness is not guaranteed. Indeed, in two-dimensional problems there is a large variety of choices in the numbering of the elements and nodes, and the bandwidth tends to differ from choice to choice. Of course, the most desirable numbering scheme is the one minimizing the bandwidth of the mass and stiffness matrices.

The procedure for obtaining the stiffness and mass matrices for systems defined over two-dimensional domains parallels the procedure for one-dimensional domains, except that certain details are different. In particular, we write once again Rayleigh's quotient in the form given by Eq. (9.2), where $N_j = [w, w]_j$ is the element energy

inner product and $D_j = (\sqrt{m}w, \sqrt{m}w)_j$ is the element weighted inner product. Then, we use $[w, w]_j$ and $(\sqrt{m}w, \sqrt{m}w)_j$ to derive element stiffness and mass matrices K_j and M_j , respectively. Finally, we assemble the element stiffness and mass matrices to obtain global stiffness and mass matrices. Differences arise in the determination of local, natural coordinates (in the finite element sense) and in the assembly process, both being more involved for two-dimensional problems than for one-dimensional ones. Perhaps this is the time to explain a statement made earlier in this section that the choice of coordinates is not an issue when the finite element method is applied to two-dimensional problems. Indeed, the shape of the boundary plays no particular role in choosing coordinates and, almost as a rule, $[w, w]_j$ and $(\sqrt{m}w, \sqrt{m}w)_j$ are expressed in terms of rectangular coordinates prior to the transformation to natural coordinates. This is the case even for boundaries with nice analytical form, such as circular and elliptic.

Next, we consider the problem of deriving stiffness and mass matrices for a membrane in transverse vibration by means of the finite element method. From Eq. (7.252), we conclude that the elements must be from \mathcal{K}_G^1 , so that linear elements are admissible. In fact, they are the simplest elements admissible. *Linear triangular elements* have some very desirable features. One of them is that the plane

$$w(x, y) = a_1 + a_2x + a_3y \quad (9.46)$$

defining the displacement at any point x, y of the element, is determined uniquely by the values w_1, w_2 and w_3 of the nodal displacements, namely, the values of w at the three vertices of the triangle (Fig. 9.9). Moreover, we will show that the value of w along an element edge reduces to a linear function of a single variable, a local coordinate defined uniquely by the nodal displacements of the two end points of the edge, which implies that the nodal displacement at the third point does not affect the value of w along the edge in question. It follows that *the continuity of w across the edge is guaranteed by continuity at the nodal points*.

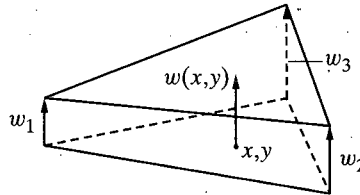


Figure 9.9 Planar displacement over triangular element

Using the analogy with Eq. (9.1), and dropping the superscript (n) for simplicity, we write the finite element solution in the general form

$$w(x, y) = \sum_{j=1}^n a_j \phi_j(x, y) \quad (9.47)$$

where $\phi_j(x, y)$ are trial functions. For linear elements, the trial functions have the form of *pyramid functions*, such as that depicted in Fig. 9.10, and they represent the

two-dimensional counterpart of the roof functions of Fig. 9.2. We take the height of the pyramid equal to unity, so that the coefficients a_j can be identified as the displacement of the membrane at the nodal points.

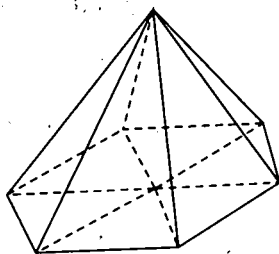


Figure 9.10 Pyramid function

At this point, we turn our attention to the derivation of local coordinates. To this end, we consider the triangular element shown in Fig. 9.11 and denote the vectors from the origin O of the rectangular system x, y to the vertices 1, 2 and 3 by \mathbf{r}_1 , \mathbf{r}_2 and \mathbf{r}_3 , respectively. Then, letting $P(x, y)$ be a typical point inside the triangle and denoting the radius vector from O to P by \mathbf{r} , the vectors from 1, 2 and 3 to P are simply $\mathbf{r} - \mathbf{r}_1$, $\mathbf{r} - \mathbf{r}_2$ and $\mathbf{r} - \mathbf{r}_3$, respectively. Because the three vectors lie in the same plane, they must satisfy the relation

$$c_1(\mathbf{r} - \mathbf{r}_1) + c_2(\mathbf{r} - \mathbf{r}_2) + c_3(\mathbf{r} - \mathbf{r}_3) = \mathbf{0} \quad (9.48)$$

which can be solved for \mathbf{r} with the result

$$\mathbf{r} = \frac{c_1\mathbf{r}_1 + c_2\mathbf{r}_2 + c_3\mathbf{r}_3}{c_1 + c_2 + c_3} = \sum_{i=1}^3 \xi_i \mathbf{r}_i \quad (9.49)$$

where

$$\xi_i = c_i / \sum_{j=1}^3 c_j, \quad i = 1, 2, 3 \quad (9.50)$$

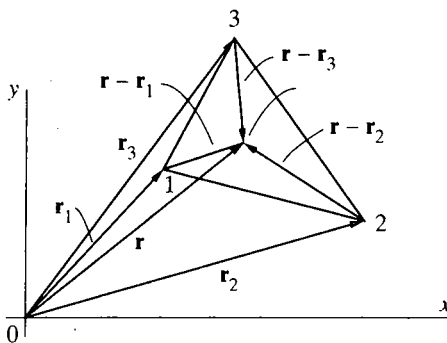


Figure 9.11 Triangular element with vectors defining the vertices and point $P(x, y)$

Equations (9.49) and (9.50) can be written in the more explicit form

$$\begin{aligned}x_1\xi_1 + x_2\xi_2 + x_3\xi_3 &= x \\y_1\xi_1 + y_2\xi_2 + y_3\xi_3 &= y \\ \xi_1 + \xi_2 + \xi_3 &= 1\end{aligned}\tag{9.51}$$

Equations (9.51) represent three equations in the unknowns ξ_1 , ξ_2 and ξ_3 and have the solution

$$\begin{aligned}\xi_1(x, y) &= \frac{1}{\Delta} [x_2y_3 - x_3y_2 + (y_2 - y_3)x + (x_3 - x_2)y] \\ \xi_2(x, y) &= \frac{1}{\Delta} [x_3y_1 - x_1y_3 + (y_3 - y_1)x + (x_1 - x_3)y] \\ \xi_3(x, y) &= \frac{1}{\Delta} [x_1y_2 - x_2y_1 + (y_1 - y_2)x + (x_2 - x_1)y]\end{aligned}\tag{9.52}$$

where

$$\Delta = (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1)\tag{9.53}$$

The functions ξ_1 , ξ_2 and ξ_3 possess some very interesting and useful properties. We observe that for $x = x_1$, $y = y_1$ Eqs. (9.52) yield $\xi_1 = 1$, $\xi_2 = \xi_3 = 0$, for $x = x_2$, $y = y_2$ they yield $\xi_1 = 0$, $\xi_2 = 1$, $\xi_3 = 0$, and for $x = x_3$, $y = y_3$ they give $\xi_1 = \xi_2 = 0$, $\xi_3 = 1$. It follows that the vertices (x_1, y_1) , (x_2, y_2) and (x_3, y_3) of the triangle are defined by the triplets $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, respectively. This suggests that the functions ξ_i ($i = 1, 2, 3$) can be used as coordinates, as shown in Fig. 9.12. Indeed, they represent the local coordinates mentioned earlier in this section. For example, the edge 2-3 is defined as $\xi_1 = 0$ and the vertex 1 as $\xi_1 = 1$. A line parallel to the edge 2-3 is described by $\xi_1 = c = \text{constant}$, where the constant c is proportional to the distance between 2-3 and the line in question.

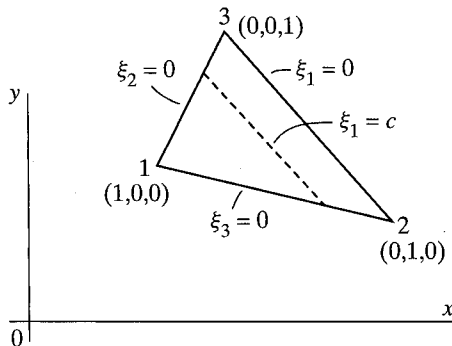


Figure 9.12 Local coordinates for triangular element

Similar geometric interpretations can be given to ξ_2 and ξ_3 . The coordinates ξ_1 , ξ_2 and ξ_3 can be identified as the *natural coordinates* for the triangular element. The natural coordinates ξ_1 , ξ_2 , ξ_3 can be given a different geometric interpretation by observing that $\Delta = 2A$, where A is the area of the triangular element. Letting $x_i = x$, $y_i = y$ ($i = 1, 2, 3$) in Eq. (9.53), in sequence, we obtain

$$\begin{aligned}(x_2y_3 - x_3y_2) + (x_3y - x_1y_3) + (x_1y_2 - x_2y_1) &= 2A_1 \\(x_1y_3 - x_3y_1) + (x_3y_2 - x_2y_3) + (x_2y - x_1y_2) &= 2A_2 \\(x_2y - x_1y_2) + (x_1y_1 - x_1y_2) + (x_1y_2 - x_2y_1) &= 2A_3\end{aligned}\quad (9.54)$$

where A_i is the area of the triangle formed by point $P(x, y)$ with the side opposite to vertex i ($i = 1, 2, 3$), as shown in Fig. 9.13. Then, comparing Eqs. (9.52) and (9.54), we conclude that

$$\begin{aligned}\xi_1(x, y) &= \frac{1}{2A} [(y_2 - y_3)x + (x_3 - x_2)y + x_2y_3 - x_3y_2] = \frac{A_1}{A} \\ \xi_2(x, y) &= \frac{1}{2A} [(y_3 - y_1)x + (x_1 - x_3)y + x_3y_1 - x_1y_3] = \frac{A_2}{A} \\ \xi_3(x, y) &= \frac{1}{2A} [(y_1 - y_2)x + (x_2 - x_1)y + x_1y_2 - x_2y_1] = \frac{A_3}{A}\end{aligned}\quad (9.55)$$

In view of Eqs. (9.55), ξ_1 , ξ_2 and ξ_3 are also called *area coordinates*.

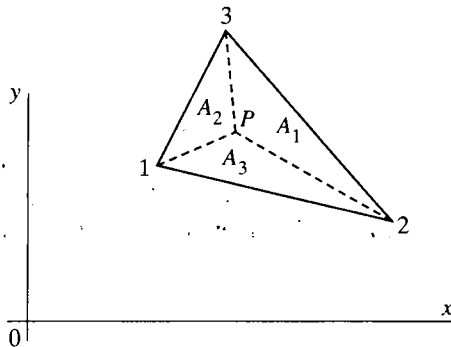


Figure 9.13 Area coordinates for triangular element

By analogy with the linear interpolation functions for one-dimensional domains, the natural coordinates ξ_1 , ξ_2 , ξ_3 can be used as linear elements for our membrane, the lowest-degree polynomials admissible. Letting

$$\phi_i = \xi_i, \quad i = 1, 2, 3 \quad (9.56)$$

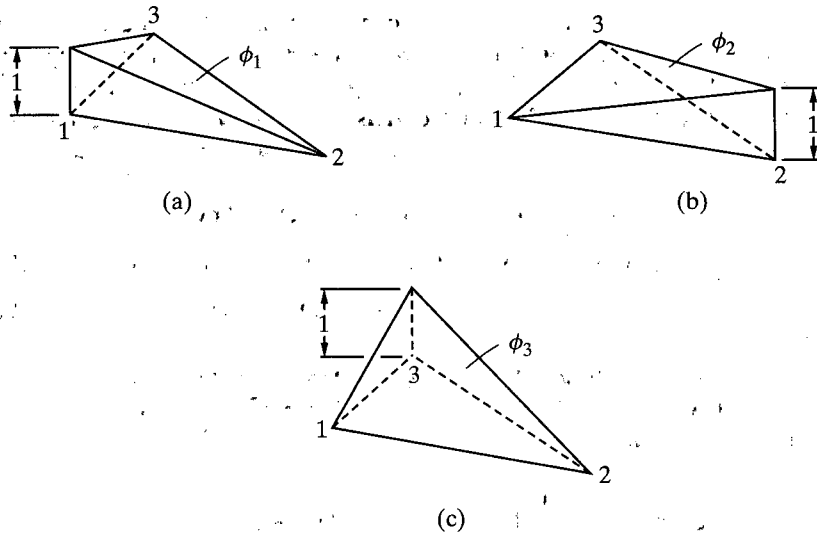


Figure 9.14 Linear interpolation functions for triangular element

and referring to Fig. 9.12, we can depict ϕ_1 , ϕ_2 and ϕ_3 as the pyramid sections shown in Figs. 9.14a, b and c, respectively, so that they really represent linear interpolation functions for the triangular element. The analogy can be extended by representing the displacement at any point ξ_1, ξ_2, ξ_3 inside the triangle as a linear combination of the interpolation functions ϕ_1, ϕ_2 and ϕ_3 multiplied by the nodal displacements w_1, w_2 and w_3 , respectively, or

$$w(\xi_1, \xi_2, \xi_3) = \sum_{i=1}^3 \phi_i w_i = \boldsymbol{\phi}^T \mathbf{w} \quad (9.57)$$

where $\boldsymbol{\phi} = [\xi_1 \ \xi_2 \ \xi_3]^T$ and $\mathbf{w} = [w_1 \ w_2 \ w_3]^T$. The function $w(\xi_1, \xi_2, \xi_3)$ is displayed in Fig. 9.15. Regarding Fig. 9.15 as representing the displacement over a given element D_j , the displacement of the whole membrane can be expressed in the form

$$w(\xi_1, \xi_2, \xi_3) = \boldsymbol{\phi}^T \mathbf{w}_j \quad x, y \text{ in } D_j \quad (9.58)$$

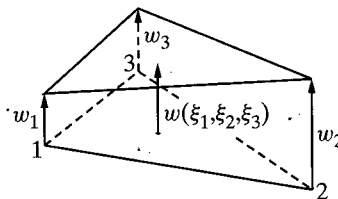


Figure 9.15 Membrane displacement in terms of linear interpolation functions

where \mathbf{w}_j is the nodal vector corresponding to the j th finite element. Hence, by analogy with the piecewise linear string displacement profile of Fig. 9.1, the displacement of the membrane consists of a surface made up of flat triangular surfaces joined along the edges. Moreover, the components of the nodal vectors \mathbf{w}_j represent the actual displacements of the membrane at the nodal points in question. It should be remarked here that this solution of the membrane vibration problem is similar to Courant's solution of the plane torsion problem (Ref. 8).

As suggested earlier in this section, we can write the energy inner product in the form

$$[w, w] = \sum_{j=1}^n [w, w]_j \quad (9.59)$$

where, using results from Sec. 7.12 and assuming that the tension T is constant,

$$[w, w]_j = T \int_{A_j} \left[\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right] dA_j \quad (9.60)$$

in which A_j is the area of the j th element. Moreover, for constant mass density ρ , the weighted inner product can be written as

$$(\sqrt{m}w, \sqrt{m}w) = \sum_{j=1}^n (\sqrt{m}w, \sqrt{m}w)_j \quad (9.61)$$

where

$$(\sqrt{m}w, \sqrt{m}w)_j = \rho \int_{A_j} w^2 dA_j \quad (9.62)$$

To evaluate the integrals in Eqs. (9.60) and (9.62), it is necessary to carry out a transformation from rectangular coordinates to natural coordinates. Hence, using Eqs. (9.55) and (9.58), we can write

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{\partial w}{\partial \xi_1} \frac{d\xi_1}{dx} + \frac{\partial w}{\partial \xi_2} \frac{d\xi_2}{dx} + \frac{\partial w}{\partial \xi_3} \frac{d\xi_3}{dx} \\ &= \frac{1}{2A_j} \left[\frac{\partial \Phi^T}{\partial \xi_1} (y_2 - y_3) + \frac{\partial \Phi^T}{\partial \xi_2} (y_3 - y_1) + \frac{\partial \Phi^T}{\partial \xi_3} (y_1 - y_2) \right] \mathbf{w}_j \\ &= \frac{1}{2A_j} [y_2 - y_3 \quad y_3 - y_1 \quad y_1 - y_2] \mathbf{w}_j \end{aligned} \quad (9.63a)$$

$$\begin{aligned} \frac{\partial w}{\partial y} &= \frac{\partial w}{\partial \xi_1} \frac{\partial \xi_1}{\partial y} + \frac{\partial w}{\partial \xi_2} \frac{\partial \xi_2}{\partial y} + \frac{\partial w}{\partial \xi_3} \frac{\partial \xi_3}{\partial y} \\ &= \frac{1}{2A_j} \left[\frac{\partial \Phi^T}{\partial \xi_1} (x_3 - x_2) + \frac{\partial \Phi^T}{\partial \xi_2} (x_1 - x_3) + \frac{\partial \Phi^T}{\partial \xi_3} (x_2 - x_1) \right] \mathbf{w}_j \\ &= \frac{1}{2A_j} [x_3 - x_2 \quad x_1 - x_3 \quad x_2 - x_1] \mathbf{w}_j \end{aligned} \quad (9.63b)$$

so that, introducing Eqs. (9.63) into Eq. (9.60), we obtain

$$[w, w]_j = \mathbf{w}_j^T K^{(j)} \mathbf{w}_j \quad (9.64)$$

where

$$\begin{aligned} K^{(j)} &= \frac{T}{4A_j^2} \int_{A_j} \left\{ \begin{bmatrix} y_2 - y_3 \\ y_3 - y_1 \\ y_1 - y_2 \end{bmatrix} \begin{bmatrix} y_2 - y_3 \\ y_3 - y_1 \\ y_1 - y_2 \end{bmatrix}^T + \begin{bmatrix} x_3 - x_2 \\ x_1 - x_3 \\ x_2 - x_1 \end{bmatrix} \begin{bmatrix} x_3 - x_2 \\ x_1 - x_3 \\ x_2 - x_1 \end{bmatrix}^T \right\} dA_j \\ &= \frac{T}{4A_j} \begin{bmatrix} (y_2 - y_3)^2 + (x_3 - x_2)^2 & (y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3) \\ \text{symm} & (y_3 - y_1)^2 + (x_1 - x_3)^2 \\ & (y_2 - y_3)(y_1 - y_2) + (x_3 - x_2)(x_2 - x_1) \\ & (y_3 - y_1)(y_1 - y_2) + (x_1 - x_3)(x_2 - x_1) \\ & (y_1 - y_2)^2 + (x_2 - x_1)^2 \end{bmatrix} \quad (9.65) \end{aligned}$$

is the *element stiffness matrix*, and we note that the integral was trivially evaluated because the integrand did not depend on the natural coordinates. On the other hand, the integral in the element mass matrix does involve the natural coordinates, and its evaluation can be rendered routine by means of the formula (Ref. 11)

$$\int_{A_j} \xi_1^m \xi_2^n \xi_3^p dA_j = \frac{m!n!p!}{(m+n+p+2)!} 2A_j \quad (9.66)$$

Inserting Eq. (9.58) into Eq. (9.62), we have

$$(\sqrt{m}w, \sqrt{m}w)_j = \mathbf{w}_j^T M^{(j)} \mathbf{w}_j \quad (9.67)$$

where $M^{(j)}$ is the *element mass matrix*, which, upon using Eq. (9.66), can be written in the explicit form

$$\begin{aligned} M^{(j)} &= \rho \int_{A_j} \Phi \Phi^T dA_j = \rho \int_{A_j} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}^T dA_j \\ &= \rho \int_{A_j} \begin{bmatrix} \xi_1^2 & \xi_1 \xi_2 & \xi_1 \xi_3 \\ \xi_1 \xi_2 & \xi_2^2 & \xi_2 \xi_3 \\ \xi_1 \xi_3 & \xi_2 \xi_3 & \xi_3^2 \end{bmatrix} dA_j \\ &= \frac{\rho A_j}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \quad (9.68) \end{aligned}$$

The general ideas behind the assembly process for two-dimensional domains are essentially the same as for one-dimensional domains, but the details are more involved, which can be attributed to the fact that there is no longer a simple correspondence between the node number and element number. To introduce the ideas, we consider part of a uniform membrane consisting of four triangular elements, as shown in Fig. 9.16, in which the encircled numbers represent the element number, the outside numbers the global node number and the smaller size inside numbers the

local node numbers for each element. The membrane is free on all sides. Inserting the coordinates corresponding to the local node numbers into Eq. (9.65), we obtain two types of element stiffness matrices, as follows:

$$\begin{aligned}
 K^{(j)} &= \frac{T}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, & j = 1, 3 \\
 K^{(j)} &= \frac{T}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}, & j = 2, 4
 \end{aligned}
 \tag{9.69}$$

where we note that, to make the assembly process easier to implement, we identify the element number by a superscript. On the other hand, letting $A_j = h^2/2$ in Eq. (9.68), we obtain the element mass matrices

$$M^{(j)} = \frac{mh^2}{24} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad j = 1, 2, 3, 4
 \tag{9.70}$$

Because the motion of the membrane is defined by the global nodal displacements and the entries of the element stiffness and mass matrices correspond to local nodal displacements, it is necessary to develop a scheme for placing the element entries in the proper position in the global matrices. To this end, we define the *connectivity array* $C = [c_{jk}]$, where the subscript j identifies the element number and the subscript k the global node numbers listed in the order specified by the local nodes. The connectivity array for the system of Fig. 9.16 is simply

$$C = \begin{bmatrix} 1 & 4 & 5 \\ 1 & 5 & 2 \\ 2 & 5 & 6 \\ 2 & 6 & 3 \end{bmatrix}
 \tag{9.71}$$

As an illustration of the use of the connectivity array, we construct the 6×6 global stiffness matrix for the system of Fig. 9.16. The first row in C represents the first element and it instructs us to place the entries $k_{11}^{(1)}$, $k_{12}^{(1)}$, $k_{13}^{(1)}$, $k_{22}^{(1)}$, $k_{23}^{(1)}$ and $k_{33}^{(1)}$ of the element stiffness matrix $K^{(1)}$ in the positions (1, 1), (1, 4), (1, 5), (4, 4), (4, 5) and (5, 5) of the global stiffness matrix K . Of course, placement of the symmetric entries is implied. Repeating the process for the remaining three rows of C , we obtain the global stiffness matrix

$$K = \begin{bmatrix} k_{11}^{(1)} + k_{11}^{(2)} & k_{13}^{(2)} & 0 & k_{12}^{(1)} & k_{13}^{(1)} + k_{12}^{(2)} & 0 \\ & k_{33}^{(2)} + k_{11}^{(3)} + k_{11}^{(4)} & k_{13}^{(4)} & 0 & k_{23}^{(2)} + k_{12}^{(3)} & k_{13}^{(3)} + k_{12}^{(4)} \\ \text{symm} & & k_{33}^{(4)} & 0 & 0 & k_{23}^{(4)} \\ & & & k_{22}^{(1)} & k_{23}^{(1)} & 0 \\ & & & & k_{33}^{(1)} + k_{22}^{(2)} + k_{22}^{(3)} & k_{23}^{(3)} \\ & & & & & k_{33}^{(3)} + k_{22}^{(4)} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 \\ & 4 & -1 & 0 & -2 & 0 \\ & & 2 & 0 & 0 & -1 \\ \text{symm} & & & 2 & -1 & 0 \\ & & & & 4 & -1 \\ & & & & & 2 \end{bmatrix} \quad (9.72)$$

The same process applies to the construction of the global mass matrix.

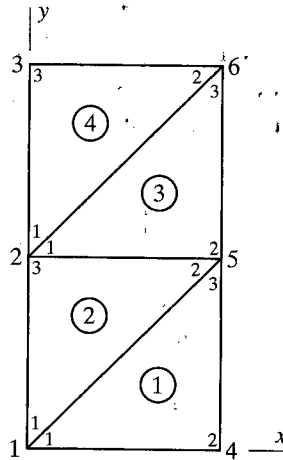


Figure 9.16 Four triangular elements with numbering scheme

The generation of the finite element mesh amounts to dividing the domain D into triangular domains D_j in such a way that no vertex of one triangle lies on the edge of another triangle. It is common practice to begin with a coarse mesh and refine the mesh so as to improve the accuracy of the eigensolutions. A mesh refinement guaranteeing that no vertex of one triangle lies on the edge of another consists of dividing each triangle into four similar triangles obtained by joining the midpoints of the edges. With each refinement of the mesh, a check verifying that a reduction in h leads indeed to a suitable reduction in the approximate eigenvalues is highly desirable.

As an alternative to refining the finite element mesh, the accuracy can be improved by refining the elements. This implies higher-degree polynomials, such as quadratic, cubic, etc. The various polynomials can be arranged in the so-called *Pascal's triangle* displayed in Fig. 9.17. Of course, the top two rows represent the linear polynomial, the top three the quadratic, etc. We consider first the *quadratic elements*

$$w(x, y) = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 \quad (9.73)$$

Because there are now six coefficients, we need six nodes. We choose three nodes at the vertices of the triangle and three at the midpoints of the edges, as shown in Fig. 9.18.

1	constant	3 terms
x y	linear	6 terms
x ² xy y ²	quadratic	10 terms
x ³ x ² y xy ² y ³	cubic	15 terms
x ⁴ x ³ y x ² y ² xy ³ y ⁴	quartic	

Figure 9.17 Pascal's triangle

As in the case of linear elements, it is convenient to use local, natural coordinates in the form of the area coordinates ξ_1, ξ_2, ξ_3 , instead of global rectangular coordinates. Recalling that the natural coordinates satisfy the relation $\xi_1 + \xi_2 + \xi_3 = 1$, only two of the coordinates should be regarded as independent. It does not matter which of the three coordinates are chosen as the independent ones, as the final result is the same. Hence, we choose ξ_1 and ξ_2 as the independent coordinates and express the interpolation functions in the form

$$\phi_i(\xi_1, \xi_2, \xi_3) = c_{i1} + c_{i2}\xi_1 + c_{i3}\xi_2 + c_{i4}\xi_1^2 + c_{i5}\xi_1\xi_2 + c_{i6}\xi_2^2, \quad i = 1, 2, \dots, 6 \tag{9.74}$$

where the dependence on ξ_3 is only implicit. Equations (9.74) require six conditions on each interpolation function, which can be generated by considering the six nodes shown in Fig. 9.18 and insisting that each ϕ_i be equal to 1 at node i and equal to zero at the remaining five nodes. Then, if we denote the values of ξ_1 and ξ_2 at the node k by $\xi_{1,k}$ and $\xi_{2,k}$, respectively, we can extend the procedure given by Eqs. (9.11)–(9.14) to the case of two variables and write

$$A = \begin{bmatrix} 1 & \xi_{1,1} & \xi_{2,1} & \xi_{1,1}^2 & \xi_{1,1}\xi_{2,1} & \xi_{2,1}^2 \\ 1 & \xi_{1,2} & \xi_{2,2} & \xi_{1,2}^2 & \xi_{1,2}\xi_{2,2} & \xi_{2,2}^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \xi_{1,6} & \xi_{2,6} & \xi_{1,6}^2 & \xi_{1,6}\xi_{2,6} & \xi_{2,6}^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1/2 & 1/2 & 1/4 & 1/4 & 1/4 \\ 1 & 0 & 1/2 & 0 & 0 & 1/4 \\ 1 & 1/2 & 0 & 1/4 & 0 & 0 \end{bmatrix} \tag{9.75}$$

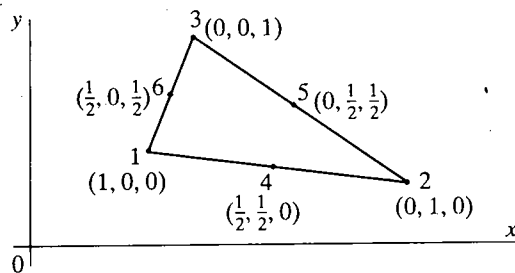


Figure 9.18 Triangular element with six nodes

so that

$$[c_1 \ c_2 \ \dots \ c_6] = A^{-1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -3 & 0 & 0 & 4 \\ 0 & -1 & -3 & 0 & 4 & 0 \\ 2 & 0 & 2 & 0 & 0 & -4 \\ 0 & 0 & 4 & 4 & -4 & -4 \\ 0 & 2 & 2 & 0 & -4 & 0 \end{bmatrix} \quad (9.76)$$

Inserting the elements in the columns of Eq. (9.76) into Eqs. (9.74) and recalling that $\xi_1 + \xi_2 + \xi_3 = 1$, we obtain the interpolation functions

$$\begin{aligned} \phi_1 &= -\xi_1 + 2\xi_1^2 = \xi_1(2\xi_1 - 1) \\ \phi_2 &= -\xi_2 + 2\xi_2^2 = \xi_2(2\xi_2 - 1) \\ \phi_3 &= 1 - 3\xi_1 - 3\xi_2 + 2\xi_1^2 + 4\xi_1\xi_2 + 2\xi_2^2 = \xi_3(2\xi_3 - 1) \\ \phi_4 &= 4\xi_1\xi_2 \\ \phi_5 &= 4\xi_2 - 4\xi_1\xi_2 - 4\xi_2^2 = 4\xi_2\xi_3 \\ \phi_6 &= 4\xi_1 - 4\xi_1^2 - 4\xi_1\xi_2 = 4\xi_1\xi_3 \end{aligned} \quad (9.77)$$

The functions ϕ_1 and ϕ_4 are displayed in Figs. 9.19a and 9.19b, respectively. The functions ϕ_2 and ϕ_3 are similar to ϕ_1 , except that the unit displacement is at the nodes 2 and 3, respectively, and the same statement can be made concerning the similarity between the functions ϕ_5 and ϕ_6 and the function ϕ_4 .

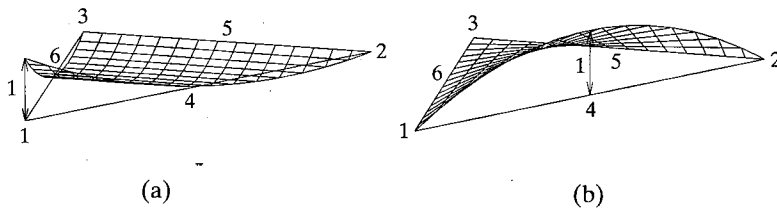


Figure 9.19 Quadratic interpolation functions

Cubic elements have the form

$$\begin{aligned} \phi_i(\xi_1, \xi_2, \xi_3) &= c_{i1} + c_{i2}\xi_1 + c_{i3}\xi_2 + c_{i4}\xi_1^2 + c_{i5}\xi_1\xi_2 + c_{i6}\xi_2^2 + c_{i7}\xi_1^3 \\ &\quad + c_{i8}\xi_1^2\xi_2 + c_{i9}\xi_1\xi_2^2 + c_{i10}\xi_2^3, \quad i = 1, 2, \dots, 10 \end{aligned} \quad (9.78)$$

so that the triangular element must have ten nodes. An element satisfying this requirement is depicted in Fig. 9.20. Using the same process as for quadratic elements,

the interpolation functions can be shown to be (Prob. 9.17)

$$\begin{aligned}
 \phi_i &= \frac{1}{2}\xi_i(3\xi_i - 1)(3\xi_i - 2), \quad i = 1, 2, 3 \\
 \phi_4 &= \frac{9}{2}\xi_2\xi_1(3\xi_1 - 1), \quad \phi_5 = \frac{9}{2}\xi_1\xi_2(3\xi_2 - 1), \quad \phi_6 = \frac{9}{2}\xi_3\xi_2(3\xi_2 - 1) \\
 \phi_7 &= \frac{9}{2}\xi_2\xi_3(3\xi_3 - 1), \quad \phi_8 = \frac{9}{2}\xi_1\xi_3(3\xi_3 - 1), \quad \phi_9 = \frac{9}{2}\xi_3\xi_1(3\xi_1 - 1) \\
 \phi_{10} &= 27\xi_1\xi_2\xi_3
 \end{aligned} \tag{9.79}$$

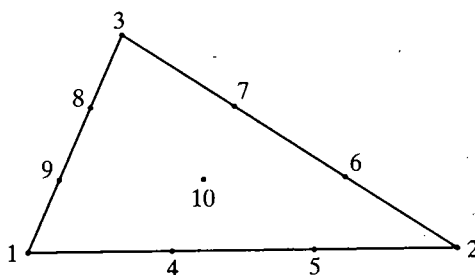


Figure 9.20 Triangular element with ten nodes

Example 9.4

Derive the global stiffness and mass matrices for a $2h \times 3h$ rectangular membrane. The membrane is free on all sides.

We divide the membrane into 12 triangular elements, as shown in Fig. 9.21, and note that the membrane of Fig. 9.16 represents a mere one third of that of Fig. 9.21. In fact, the numbering of the elements, global and local nodes in Fig. 9.21 is entirely consistent with that of Fig. 9.16. It can be verified that the element stiffness matrices are still of two types, as in Eqs. (9.69), or

$$K^{(j)} = \frac{T}{2} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \quad j \text{ odd}, \quad K^{(j)} = \frac{T}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \quad j \text{ even} \tag{a}$$

Moreover, from Eqs. (9.70), the element mass matrices are

$$M^{(j)} = \frac{mh^2}{24} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad j = 1, 2, \dots, 12 \tag{b}$$

From Fig. 9.21, the connectivity array is

$$C = \begin{bmatrix} 1 & 1 & 2 & 2 & 4 & 4 & 5 & 5 & 7 & 7 & 8 & 8 \\ 4 & 5 & 5 & 6 & 7 & 8 & 8 & 9 & 10 & 11 & 11 & 12 \\ 5 & 2 & 6 & 3 & 8 & 5 & 9 & 6 & 11 & 8 & 12 & 9 \end{bmatrix}^T \tag{c}$$

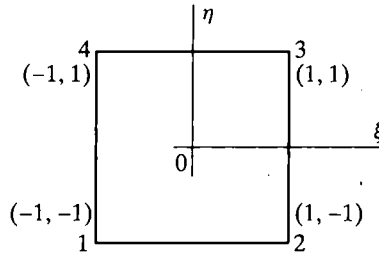


Figure 9.22 Rectangular element with natural coordinates

Both K and M are banded with half-bandwidth equal to three and four, respectively. For banded matrices, it is more efficient to list the entries by the nonzero diagonals.

9.6 RECTANGULAR ELEMENTS

When the membrane boundary possesses corners, *quadrilateral elements* may be able to yield a more economical finite element mesh. In this section, we consider the simplest quadrilateral elements, namely, *rectangular elements*. Figure 9.22 shows a typical rectangular element together with the corresponding natural coordinates ξ, η . Regarding the four corners as nodal points, the lowest-degree polynomials admissible are the *bilinear*, given by

$$\phi_i(\xi, \eta) = c_{i1} + c_{i2}\xi + c_{i3}\eta + c_{i4}\xi\eta, \quad i = 1, 2, 3, 4 \quad (9.80)$$

Hence, following the standard procedure, we obtain the bilinear interpolation functions (Problem 9.19).

$$\begin{aligned} \phi_1 &= \frac{1}{4}(1 - \xi)(1 - \eta), & \phi_2 &= \frac{1}{4}(1 + \xi)(1 - \eta) \\ \phi_3 &= \frac{1}{4}(1 + \xi)(1 + \eta), & \phi_4 &= \frac{1}{4}(1 - \xi)(1 + \eta) \end{aligned} \quad (9.81)$$

The function ϕ_1 is shown in Fig. 9.23. The other three functions are relatively easy to visualize.

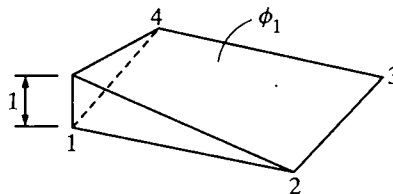


Figure 9.23 Bilinear interpolation function for rectangular element

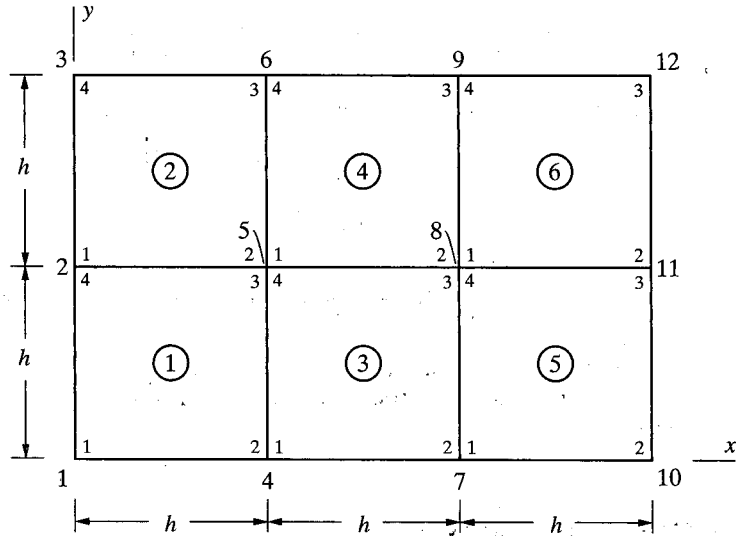


Figure 9.24 Membrane with six rectangular elements

Next, we derive the stiffness and mass matrices for the membrane of Example 9.4 by means of the bilinear elements. We begin with the derivation of the element stiffness and mass matrices. To this end, we express the displacement of the membrane at any point of the rectangle in the form

$$w(\xi, \eta) = \Phi^T \mathbf{w}_j \quad (9.82)$$

where $\Phi = [\phi_1 \phi_2 \phi_3 \phi_4]^T$ is a four-dimensional vector of interpolation functions and $\mathbf{w}_j = [w_1 w_2 w_3 w_4]^T$ is the corresponding nodal vector for element j . We recall from Sec. 9.5 that the stiffness matrix involves the terms $\partial w / \partial x$ and $\partial w / \partial y$. Because w is in terms of the natural coordinates ξ, η , we must first write the relations between the rectangular coordinates x, y and the local, natural coordinates ξ, η for each element. From Fig. 9.24, we have simply

$$\begin{aligned} x &= \frac{h}{2}(j + \xi), & y &= \frac{h}{2}(1 + \eta), & j &= 1, 3, 5 \\ x &= \frac{h}{2}(j - 1 + \xi), & y &= \frac{h}{2}(3 + \eta), & j &= 2, 4, 6 \end{aligned} \quad (9.83)$$

Hence, using Eqs. (9.81)–(9.83), we can write

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{\partial \xi}{\partial x} \frac{\partial w}{\partial \xi} = \frac{2}{h} \frac{\partial \Phi^T}{\partial \xi} \mathbf{w}_j \\ &= \frac{1}{2h} [-(1 - \eta) \quad 1 - \eta \quad 1 + \eta \quad -(1 + \eta)]^T \mathbf{w}_j \end{aligned} \quad (9.84a)$$

$$\frac{\partial w}{\partial y} = \frac{\partial \eta}{\partial y} \frac{\partial w}{\partial \eta} = \frac{2}{h} \frac{\partial \Phi^T}{\partial \eta} \mathbf{w}_j$$

$$= \frac{1}{2h} [-(1-\xi) \quad -(1+\xi) \quad 1+\xi \quad 1-\xi]^T \mathbf{w}_j \quad (9.84b)$$

Introducing Eqs. (9.84) into Eq. (9.60) and recalling Eq. (9.64), we obtain the element stiffness matrices

$$\begin{aligned} K^{(j)} &= \frac{T}{4h^2} \int_{A_j} \left(\frac{\partial \Phi}{\partial \xi} \frac{\partial \Phi^T}{\partial \xi} + \frac{\partial \Phi}{\partial \eta} \frac{\partial \Phi^T}{\partial \eta} \right) dA_j \\ &= \frac{T}{16} \int_{-1}^1 \int_{-1}^1 \left\{ \begin{bmatrix} -(1-\eta) \\ 1-\eta \\ 1+\eta \\ -(1+\eta) \end{bmatrix} \begin{bmatrix} -(1-\eta) \\ 1-\eta \\ 1+\eta \\ -(1+\eta) \end{bmatrix}^T \right. \\ &\quad \left. + \begin{bmatrix} -(1-\xi) \\ -(1+\xi) \\ 1+\xi \\ 1-\xi \end{bmatrix} \begin{bmatrix} -(1-\xi) \\ -(1+\xi) \\ 1+\xi \\ 1-\xi \end{bmatrix}^T \right\} d\xi d\eta \\ &= \frac{T}{6} \begin{bmatrix} 4 & -1 & -2 & -1 \\ & 4 & -1 & -2 \\ \text{symm} & & 4 & -1 \\ & & & 4 \end{bmatrix} \quad (9.85) \end{aligned}$$

Moreover, inserting Eqs. (9.81)–(9.83) into Eq. (9.62) and recalling Eq. (9.67), we obtain the element mass matrix

$$\begin{aligned} M^{(j)} &= \rho \int_{A_j} \Phi \Phi^T dA_j \\ &= \frac{\rho h^2}{64} \int_{-1}^1 \int_{-1}^1 \begin{bmatrix} (1-\xi)(1-\eta) \\ (1+\xi)(1-\eta) \\ (1+\xi)(1+\eta) \\ (1-\xi)(1+\eta) \end{bmatrix} \begin{bmatrix} (1-\xi)(1-\eta) \\ (1+\xi)(1-\eta) \\ (1+\xi)(1+\eta) \\ (1-\xi)(1+\eta) \end{bmatrix}^T d\xi d\eta \\ &= \frac{\rho h^2}{36} \begin{bmatrix} 4 & 2 & 1 & 2 \\ & 4 & 2 & 1 \\ \text{symm} & & 4 & 2 \\ & & & 4 \end{bmatrix} \quad (9.86) \end{aligned}$$

The assembly process is the same as that for triangular elements described in Sec. 9.5. Hence, using the numbering scheme of Fig. 9.24, we obtain the connectivity array

$$C = \begin{bmatrix} 1 & 2 & 4 & 5 & 7 & 8 \\ 4 & 5 & 7 & 8 & 10 & 11 \\ 5 & 6 & 8 & 9 & 11 & 12 \\ 2 & 3 & 5 & 6 & 8 & 9 \end{bmatrix} \quad (9.87)$$

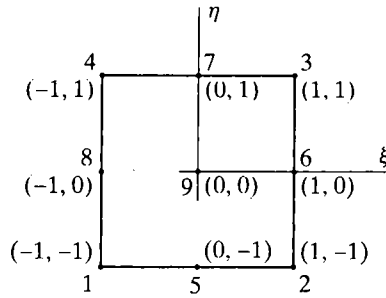


Figure 9.25 Rectangular element with nine nodes

functions can be shown to have the form

$$\begin{aligned}
 \phi_1 &= \frac{1}{4}\xi\eta(1-\xi)(1-\eta), & \phi_2 &= -\frac{1}{4}\xi\eta(1+\xi)(1-\eta) \\
 \phi_3 &= \frac{1}{4}\xi\eta(1+\xi)(1+\eta), & \phi_4 &= -\frac{1}{4}\xi\eta(1-\xi)(1+\eta) \\
 \phi_5 &= -\frac{1}{2}\eta(1-\xi^2)(1-\eta), & \phi_6 &= \frac{1}{2}\xi(1+\xi)(1-\eta^2) \\
 \phi_7 &= \frac{1}{2}\eta(1-\xi^2)(1+\eta), & \phi_8 &= -\frac{1}{2}\xi(1-\xi)(1-\eta^2) \\
 \phi_9 &= (1-\xi^2)(1-\eta^2)
 \end{aligned} \tag{9.91}$$

The quadratic interpolation functions ϕ_1 , ϕ_5 and ϕ_9 are shown in Figs. 9.26a,b and c, respectively.

The quadratic interpolation functions given by Eqs. (9.91) are characterized by eight external nodes and one internal node. Because internal nodes do not contribute to the interelement connectivity, their usefulness has come into question. A family of rectangular elements known as *serendipity elements* contains only external nodes (Ref. 9). Of course, the four-node element was already examined earlier in this section. The eight-node element is obtained by simply omitting the ninth node from Fig. 9.25 and the last term in Eq. (9.90). Following the same procedure as above, we obtain the interpolation functions

$$\begin{aligned}
 \phi_1 &= -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta), & \phi_2 &= -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta) \\
 \phi_3 &= -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta), & \phi_4 &= -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta) \\
 \phi_5 &= \frac{1}{2}(1-\xi^2)(1-\eta), & \phi_6 &= \frac{1}{2}(1+\xi)(1-\eta^2) \\
 \phi_7 &= \frac{1}{2}(1-\xi^2)(1+\eta), & \phi_8 &= \frac{1}{2}(1-\xi)(1-\eta^2)
 \end{aligned} \tag{9.92}$$

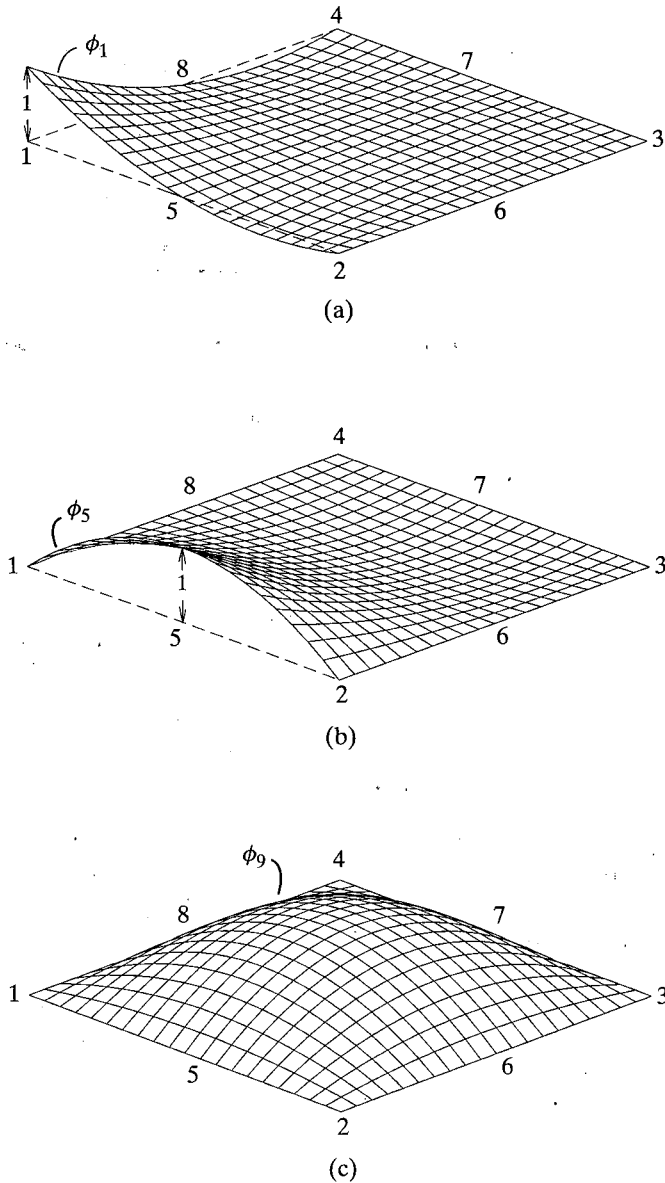


Figure 9.26 Quadratic interpolation functions for rectangular elements

Figures 9.27a and 9.27b show the eight-node interpolation functions ϕ_1 and ϕ_5 , respectively.

Comparing Figs. 9.26 a and 9.27a, we conclude that there is not much difference between the functions ϕ_1 with eight nodes and nine nodes. On the other hand, from Figs. 9.26b and 9.27b, we see that there is significant difference between the functions ϕ_5 with eight nodes and nine nodes.

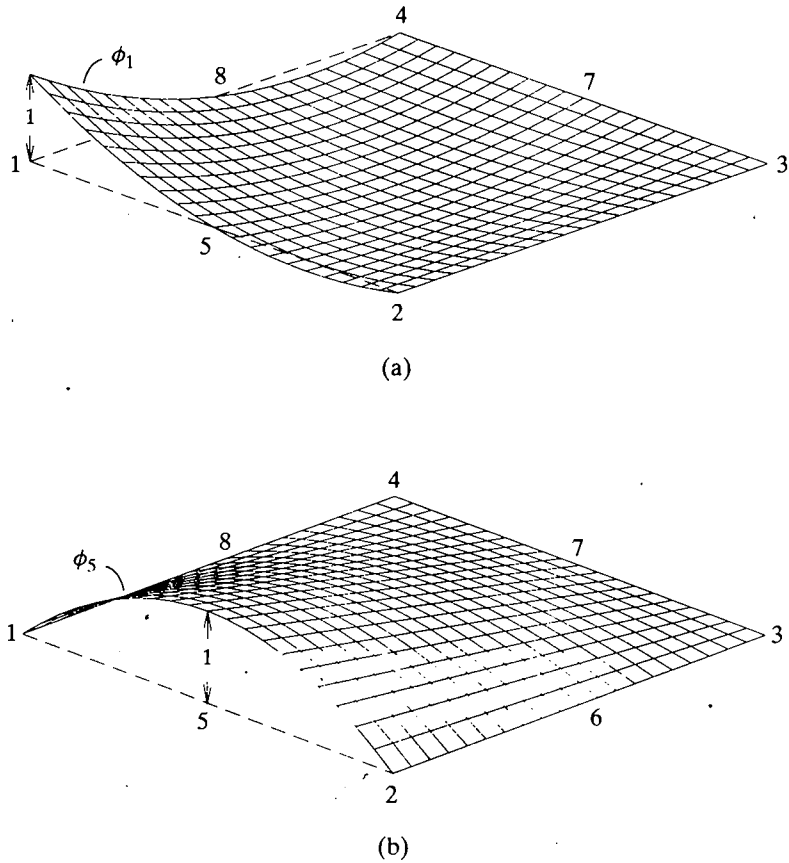


Figure 9.27 Eight-node interpolation functions for rectangular elements

9.7 ISOPARAMETRIC ELEMENTS

At times, the shape of the membrane boundary requires more versatile elements, such as general quadrilateral elements and elements with curved boundaries. Yet, it is difficult to forsake triangular and rectangular elements with their many advantages. Fortunately, it is not necessary to abandon them due to some ingenious coordinate transformations introduced by Taig (Ref. 24) and generalized by Irons and Ergatoudis et al. (Refs. 12 and 9, respectively).

The coordinate transformation

$$x = x(\xi, \eta), \quad y = y(\xi, \eta) \tag{9.93}$$

represents a mapping of the points (ξ, η) in the ξ, η -plane onto points (x, y) in the x, y -plane. In particular, the objective is to use a coordinate transformation capable of straightening out curved elements. Figure 9.28a shows a rectangular element in the ξ, η -plane and Fig. 9.28b shows a curved element in the x, y -plane. We refer to the element in the ξ, η -plane as the *master element* and to the element in the x, y -plane as an *isoparametric element*. Because in the finite element method displacements are

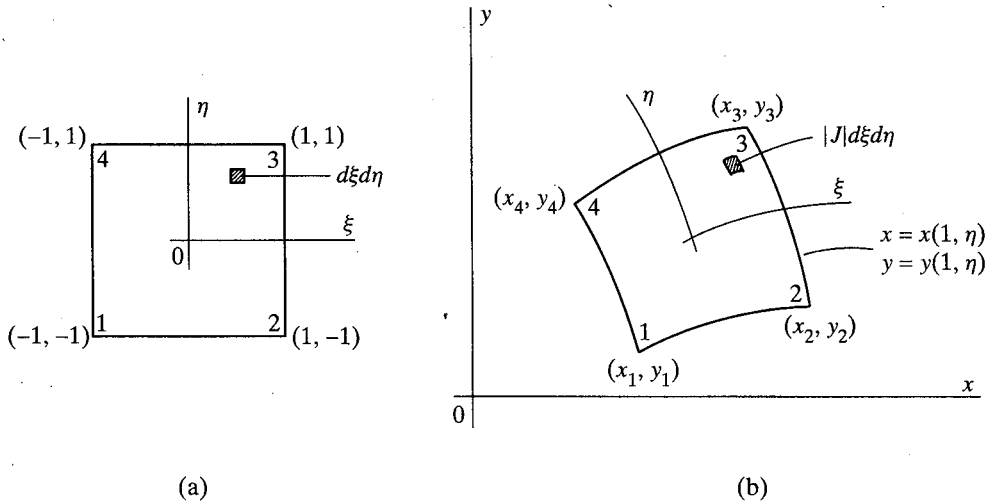


Figure 9.28 (a) Master element (b) Isoparametric element

approximated by piecewise polynomials, there are reasons to believe that piecewise polynomials can also be used for mapping one element onto another. We assume that the finite element approximation of the displacement w over a given element with n nodes has the form

$$w(\xi, \eta) = \sum_{i=1}^n \phi_i(\xi, \eta) w_i = \Phi^T(\xi, \eta) \mathbf{w} \quad (9.94)$$

where Φ is an n -vector of interpolation functions and \mathbf{w} an n -vector of nodal displacements. Moreover, we assume that the mapping of (ξ, η) onto (x, y) is given by

$$x = \sum_{i=1}^n x_i \phi_i(x, \eta) = \Phi^T(\xi, \eta) \mathbf{x}, \quad y = \sum_{i=1}^n y_i \phi_i(\xi, \eta) = \Phi^T(\xi, \eta) \mathbf{y} \quad (9.95)$$

in which $\mathbf{x} = [x_1 \ x_2 \ \dots \ x_n]^T$ and $\mathbf{y} = [y_1 \ y_2 \ \dots \ y_n]^T$ are n -vectors with entries equal to the x - and y -components, respectively, of the nodal points (x_i, y_i) . Equations (9.95) are said to represent an *isoparametric* mapping. If the dimension of Φ in Eq. (9.95) is lower, or higher, than the dimension of Φ in Eq. (9.94), the transformation is said to be *subparametric*, or *superparametric*, respectively. We are concerned only with isoparametric transformations.

Next, we consider the problem of deriving the stiffness and mass matrices for isoparametric elements. As in the case of rectangular elements, the element stiffness matrix requires the transformation from rectangular coordinates to natural coordinates. In particular, we recall from Sec. 9.6 that the stiffness matrix involves the partial derivatives $\partial w / \partial x$ and $\partial w / \partial y$, as well as the differential element of area

$dx dy$. Hence, for the general transformation described by Eqs. (9.93), we can write

$$dx = \frac{\partial x}{\partial \xi} d\xi + \frac{\partial x}{\partial \eta} d\eta, \quad dy = \frac{\partial y}{\partial \xi} d\xi + \frac{\partial y}{\partial \eta} d\eta \quad (9.96)$$

Equations (9.96) can be rewritten in the matrix form

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = J^T \begin{bmatrix} d\xi \\ d\eta \end{bmatrix} \quad (9.97)$$

where

$$J = \begin{bmatrix} \partial x/\partial \xi & \partial y/\partial \xi \\ \partial x/\partial \eta & \partial y/\partial \eta \end{bmatrix} \quad (9.98)$$

represents the *Jacobian matrix*. Then, the differential element of area can be shown to transform according to (Ref. 22)

$$dx dy = |J| d\xi d\eta \quad (9.99)$$

in which

$$|J| = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} \quad (9.100)$$

is the *Jacobian determinant*, or simply the Jacobian of the transformation. The inverse of the transformation (9.93) has the form

$$\xi = \xi(x, y), \quad \eta = \eta(x, y) \quad (9.101)$$

so that we can write

$$\begin{bmatrix} d\xi \\ d\eta \end{bmatrix} = \bar{J}^T \begin{bmatrix} dx \\ dy \end{bmatrix} \quad (9.102)$$

where

$$\bar{J} = \begin{bmatrix} \partial \xi/\partial x & \partial \eta/\partial x \\ \partial \xi/\partial y & \partial \eta/\partial y \end{bmatrix} \quad (9.103)$$

Comparing Eqs. (9.97) and (9.102), we conclude that

$$\begin{bmatrix} \partial \xi/\partial x & \partial \eta/\partial x \\ \partial \xi/\partial y & \partial \eta/\partial y \end{bmatrix} = J^{-1} = \frac{1}{|J|} \begin{bmatrix} \partial y/\partial \eta & -\partial y/\partial \xi \\ -\partial x/\partial \eta & \partial x/\partial \xi \end{bmatrix} \quad (9.104)$$

Next we consider

$$\frac{\partial w}{\partial x} = \frac{\partial \xi}{\partial x} \frac{\partial w}{\partial \xi} + \frac{\partial \eta}{\partial x} \frac{\partial w}{\partial \eta}, \quad \frac{\partial w}{\partial y} = \frac{\partial \xi}{\partial y} \frac{\partial w}{\partial \xi} + \frac{\partial \eta}{\partial y} \frac{\partial w}{\partial \eta} \quad (9.105)$$

in which $\partial \xi/\partial x$, $\partial \xi/\partial y$, $\partial \eta/\partial x$ and $\partial \eta/\partial y$ can be obtained from Eq. (9.104). Equations (9.105) in conjunction with Eq. (9.104) can be arranged in the matrix form

$$\begin{aligned} \begin{bmatrix} \partial w/\partial x \\ \partial w/\partial y \end{bmatrix} &= \begin{bmatrix} \partial \xi/\partial x & \partial \eta/\partial x \\ \partial \xi/\partial y & \partial \eta/\partial y \end{bmatrix} \begin{bmatrix} \partial w/\partial \xi \\ \partial w/\partial \eta \end{bmatrix} \\ &= \frac{1}{|J|} \begin{bmatrix} \partial y/\partial \eta & -\partial y/\partial \xi \\ -\partial x/\partial \eta & \partial x/\partial \xi \end{bmatrix} \begin{bmatrix} \partial w/\partial \xi \\ \partial w/\partial \eta \end{bmatrix} \end{aligned} \quad (9.106)$$

At this point, we are ready to write expressions for the isoparametric element stiffness and mass matrices. To this end, we assume that Fig. 9.28 represents the transformation between the master element and a typical isoparametric element j , insert Eqs. (9.99) and (9.106) into Eq. (9.60), consider Eq. (9.94) with w replaced by w_j , as well as Eqs. (9.95) with x and y replaced by x_j and y_j , respectively, and obtain

$$\begin{aligned}
 [w, w]_j &= T \int_{A_j} \begin{bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{bmatrix}^T \begin{bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{bmatrix} dx dy \\
 &= T \int_{-1}^1 \int_{-1}^1 \frac{1}{|J|_j} \begin{bmatrix} \frac{\partial w}{\partial \xi} \\ \frac{\partial w}{\partial \eta} \end{bmatrix}^T \begin{bmatrix} \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 \\ -\left(\frac{\partial x}{\partial \xi} \frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi} \frac{\partial y}{\partial \eta}\right) \\ -\left(\frac{\partial x}{\partial \xi} \frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi} \frac{\partial y}{\partial \eta}\right) \\ \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial \xi} \\ \frac{\partial w}{\partial \eta} \end{bmatrix} d\xi d\eta \\
 &= \mathbf{w}_j^T K^{(j)} \mathbf{w}_j
 \end{aligned} \tag{9.107}$$

where

$$K^{(j)} = T \int_{-1}^1 \int_{-1}^1 \frac{1}{|J|_j} \begin{bmatrix} \frac{\partial \Phi}{\partial \xi} & \frac{\partial \Phi}{\partial \eta} \end{bmatrix} B \begin{bmatrix} \frac{\partial \Phi^T}{\partial \xi} \\ \frac{\partial \Phi^T}{\partial \eta} \end{bmatrix} d\xi d\eta \tag{9.108}$$

is the isoparametric element stiffness matrix, in which, from Eq. (9.100),

$$|J|_j = \mathbf{x}_j^T \begin{bmatrix} \frac{\partial \Phi}{\partial \xi} & \frac{\partial \Phi^T}{\partial \eta} \\ \frac{\partial \Phi}{\partial \eta} & \frac{\partial \Phi^T}{\partial \xi} \end{bmatrix} \mathbf{y}_j \tag{9.109}$$

and

$$B = \begin{bmatrix} \mathbf{x}_j^T \frac{\partial \Phi}{\partial \eta} \frac{\partial \Phi^T}{\partial \eta} \mathbf{x}_j + \mathbf{y}_j^T \frac{\partial \Phi}{\partial \eta} \frac{\partial \Phi^T}{\partial \eta} \mathbf{y}_j & -\left(\mathbf{x}_j^T \frac{\partial \Phi}{\partial \xi} \frac{\partial \Phi^T}{\partial \eta} \mathbf{x}_j + \mathbf{y}_j^T \frac{\partial \Phi}{\partial \xi} \frac{\partial \Phi^T}{\partial \eta} \mathbf{y}_j \right) \\ \text{symm} & \mathbf{x}_j^T \frac{\partial \Phi}{\partial \xi} \frac{\partial \Phi^T}{\partial \xi} \mathbf{x}_j + \mathbf{y}_j^T \frac{\partial \Phi}{\partial \xi} \frac{\partial \Phi^T}{\partial \xi} \mathbf{y}_j \end{bmatrix} \tag{9.110}$$

Moreover, introducing Eqs. (9.94) and (9.99) into Eq. (9.62), we can write

$$(\sqrt{m}w, \sqrt{m}w)_j = \rho \int_{A_j} w^2 dx dy = \rho \int_{-1}^1 \int_{-1}^1 |J|_j w^2 d\xi d\eta = \mathbf{w}_j^T M^{(j)} \mathbf{w}_j \tag{9.111}$$

where

$$M^{(j)} = \rho \int_{-1}^1 \int_{-1}^1 |J|_j \phi \phi^T d\xi d\eta \quad (9.112)$$

is the isoparametric element mass matrix.

The simplest and most common isoparametric transformation is that in which the interpolation functions are *bilinear* and the rectangular element in the ξ, η -plane has four nodes. In this case, from Eqs. (9.81), the vector of interpolation functions is simply

$$\phi = \frac{1}{4} [(1 - \xi)(1 - \eta) \quad (1 + \xi)(1 - \eta) \quad (1 + \xi)(1 + \eta) \quad (1 - \xi)(1 + \eta)]^T \quad (9.113)$$

Of course, the vectors x_j and y_j in Eqs. (9.109) and (9.110) are four-dimensional. From Eq. (9.113), we write

$$\frac{\partial \phi}{\partial \xi} = \frac{1}{4} \begin{bmatrix} -(1 - \eta) \\ 1 - \eta \\ 1 + \eta \\ -(1 + \eta) \end{bmatrix}, \quad \frac{\partial \phi}{\partial \eta} = \frac{1}{4} \begin{bmatrix} -(1 - \xi) \\ -(1 + \xi) \\ 1 + \xi \\ 1 - \xi \end{bmatrix} \quad (9.114)$$

so that the Jacobian determinant, Eq. (9.109), can be written in the explicit form

$$|J|_j = \frac{1}{8} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_j^T \begin{bmatrix} 0 & \xi(1 - \eta) & -(\xi - \eta) & -(1 - \xi) \\ & 0 & 1 + \xi & -(\xi + \eta) \\ & \text{symm} & 0 & 1 + \eta \\ & & & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}_j \quad (9.115)$$

Similarly, we can write

$$\frac{\partial \phi}{\partial \xi} \frac{\partial \phi^T}{\partial \xi} = \frac{1}{16} \begin{bmatrix} (1 - \eta)^2 & -(1 - \eta)^2 & -(1 - \eta^2) & 1 - \eta^2 \\ & (1 - \eta)^2 & 1 - \eta^2 & -(1 - \eta^2) \\ & \text{symm} & (1 + \eta)^2 & -(1 + \eta)^2 \\ & & & (1 + \eta)^2 \end{bmatrix} \quad (9.116a)$$

$$\frac{\partial \phi}{\partial \xi} \frac{\partial \phi^T}{\partial \eta} =$$

$$\frac{1}{16} \begin{bmatrix} (1 - \xi)(1 - \eta) & -(1 - \xi)(1 - \eta) & -(1 - \xi)(1 + \eta) & (1 - \xi)(1 + \eta) \\ (1 + \xi)(1 - \eta) & -(1 + \xi)(1 - \eta) & -(1 + \xi)(1 + \eta) & (1 + \xi)(1 + \eta) \\ -(1 + \xi)(1 - \eta) & (1 + \xi)(1 - \eta) & (1 + \xi)(1 + \eta) & -(1 + \xi)(1 + \eta) \\ -(1 - \xi)(1 - \eta) & (1 - \xi)(1 - \eta) & (1 - \xi)(1 + \eta) & -(1 - \xi)(1 + \eta) \end{bmatrix} \quad (9.116b)$$

$$\frac{\partial \Phi}{\partial \eta} \frac{\partial \Phi^T}{\partial \eta} = \frac{1}{16} \begin{bmatrix} (1 - \xi)^2 & 1 - \xi^2 & -(1 - \xi^2) & -(1 - \xi)^2 \\ & (1 + \xi)^2 & -(1 + \xi)^2 & -(1 - \xi^2) \\ \text{symm} & & (1 + \xi)^2 & 1 - \xi^2 \\ & & & (1 - \xi)^2 \end{bmatrix} \quad (9.116c)$$

The element stiffness matrix is obtained by inserting Eq. (9.110) in conjunction with Eqs. (9.116), as well as Eqs. (9.114) and (9.115), into Eq. (9.108) and carrying out the necessary integrations for given vectors x_j and y_j of global nodal positions. Similarly, the element mass matrix is obtained by inserting Eqs. (9.113) and (9.115) into Eq. (9.112) and performing the indicated integrations. The nature of the integrals in Eqs. (9.108) and (9.112) dictates that the integrations be carried out numerically, which is consistent with the idea of assigning as much of the routine work as possible to the computer.

The assembly process is similar to the one described in Sec. 9.5. The element stiffness and mass matrices are 4×4 and require the global nodal positions. Then, the global stiffness and mass matrices are assembled through the use of a connectivity array, as in Sec. 9.5. As an illustration, we consider the quadrilateral membrane shown in Fig. 9.29. The numbering scheme is as for the rectangular membrane of Fig. 9.24, so that the connectivity array remains in the form of Eq. (9.87). On the other hand, the nodal positions are different and are given in Table 9.3

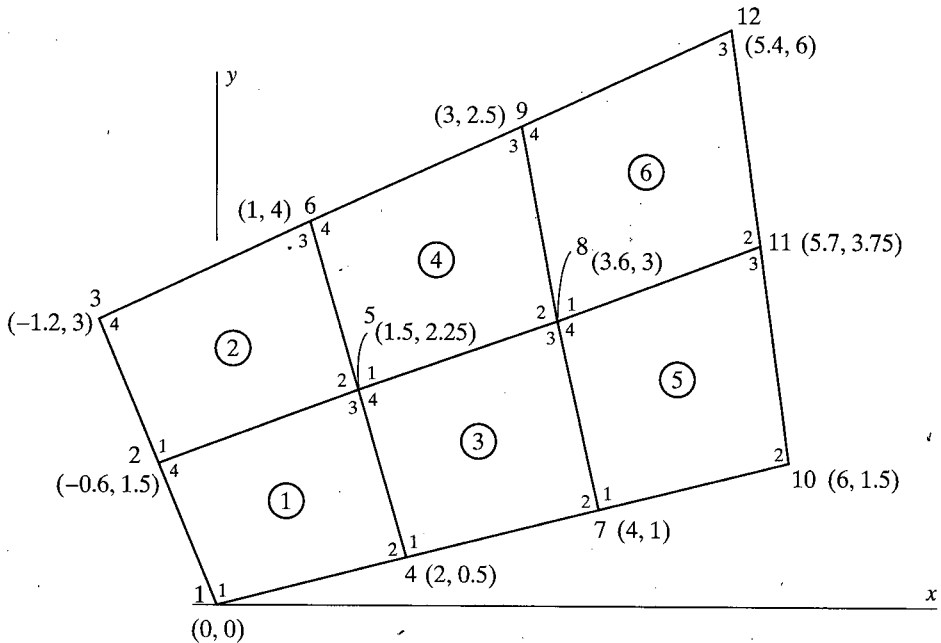


Figure 9.29 Quadrilateral membrane

If a given element has a curved edge, then the isoparametric transformation must involve higher-degree interpolation functions, which complicates matters appreciably. For a discussion of this subject, see Ref. 22 (p. 158).

TABLE 9.3 Nodal Position Vectors

		j					
		1	2	3	4	5	6
i							
x_j	1	0	-0.6	2	1.5	4	3.6
	2	2	1.5	4	3.6	6	5.7
	3	1.5	1	3.6	3.2	5.7	5.4
	4	-0.6	-1.2	1.5	1	3.6	3.2
y_j	1	0	1.5	0.5	2.25	1	3
	2	0.5	2.25	1	3	1.5	3.75
	3	2.25	4	3	5	3.75	6
	4	1.5	3	2.25	4	3	5

9.8 VIBRATION OF PLATES

As can be concluded from Sec. 7.13, plates are significantly more complex than membranes, and the same can be said about finite element solutions to plate problems versus membrane problems. The difficulties can be traced to the fact that the stiffness operator L is the fourth-order biharmonic operator ∇^4 for plates and only the second-order Laplacian operator ∇^2 for membranes. Hence, whereas for membranes the continuity conditions between elements are imposed on the displacement w alone, for plates the continuity conditions are imposed on the displacement and its derivatives. Usually three continuity conditions, one concerning the displacement and two concerning rotations about two orthogonal axes, must be satisfied at each node. If complete interelement slope continuity is imposed, the derivation of interpolation functions becomes an intimidating task. The task is significantly simpler if one insists only on interelement displacement continuity and slope continuity at nodes alone, which gives rise to so-called *nonconforming* elements. Continuity at interface points can be restored by introducing new interpolation functions, thus obtaining *conforming* elements, but accuracy tends to suffer (Ref. 28, p. 227).

The process of deriving element stiffness and mass matrices involves the choice of finite elements. We begin with rectangular elements, which are the simplest for plate vibration. To this end, we consider the master element of Fig. 9.30, characterized by three degrees of freedom at each node, one translation and two rotations. Hence, the element nodal displacement vector is twelve-dimensional, having the form

$$\mathbf{w} = [w_1 \ \theta_{\xi 1} \ \theta_{\eta 1} \ w_2 \ \theta_{\xi 2} \ \dots \ \theta_{\eta 4}]^T \tag{9.117}$$

The relation between the translation and rotations in terms of natural coordinates is simply

$$\theta_{\xi} = \partial w / \partial \eta, \quad \theta_{\eta} = -\partial w / \partial \xi \tag{9.118}$$

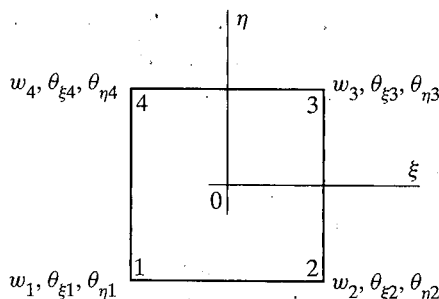


Figure 9.30 Rectangular element with three degrees of freedom at each node

The derivation of the interpolation functions follows the usual procedure, but the details are different. Because there are twelve degrees of freedom, the polynomials must contain twelve constants. A complete fourth-order polynomial has fourteen terms that together with a constant term makes fifteen terms. Hence, certain terms must be omitted. A good choice for the interpolation functions is (Ref. 28)

$$\begin{aligned} \phi_i(\xi, \eta) = & c_{i1} + c_{i2}\xi + c_{i3}\eta + c_{i4}\xi^2 + c_{i5}\xi\eta + c_{i6}\eta^2 + c_{i7}\xi^3 + c_{i8}\xi^2\eta \\ & + c_{i9}\xi\eta^2 + c_{i10}\eta^3 + c_{i11}\xi^3\eta + c_{i12}\xi\eta^3, \quad i = 1, 2, \dots, 12 \end{aligned} \quad (9.119)$$

This polynomial has the advantage that along any line $\xi = \text{constant}$ or $\eta = \text{constant}$, such as the element boundaries, it reduces to a cubic containing four constants. But, a cubic is defined uniquely by four constants, so that the displacements and slopes at the two ends of an element boundary define the displacement along this boundary uniquely. Because the nodal displacements and slopes are shared by adjacent elements, displacement continuity is ensured at all interface points. This does not guarantee continuity of the slope in the direction normal to the boundary, so that this is a nonconforming element (Ref. 28). As usual, we assume that the displacement at any point of the master element is given by

$$w(\xi, \eta) = \boldsymbol{\phi}^T \mathbf{w} \quad (9.120)$$

where $\boldsymbol{\phi} = [\phi_1 \ \phi_2 \ \dots \ \phi_{12}]^T$ is a vector of interpolation functions, whose components are given by Eqs. (9.119). The coefficients $c_{i1}, c_{i2}, \dots, c_{i12}$ in Eqs. (9.119) can be obtained by modifying the approach used in Sec. 9.5 for membranes so as to take into account Eqs. (9.118). Consistent with this, we write

$$A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix} \quad (9.121)$$

in which

$$A_k = \begin{bmatrix} 1 & \xi_k & \eta_k & \xi_k^2 & \xi_k \eta_k & \eta_k^2 & \xi_k^3 & \xi_k^2 \eta_k & \xi_k \eta_k^2 & \eta_k^3 & \xi_k^3 \eta_k & \xi_k \eta_k^3 \\ 0 & 0 & 1 & 0 & \xi_k & 2\eta_k & 0 & \xi_k^2 & 2\xi_k \eta_k & 3\eta_k^2 & \xi_k^3 & 3\xi_k \eta_k^2 \\ 0 & -1 & 0 & -2\xi_k & -\eta_k & 0 & -3\xi_k^2 & -2\xi_k \eta_k & -\eta_k^2 & 0 & -3\xi_k^2 \eta_k & -\eta_k^3 \end{bmatrix}, \quad k = 1, 2, 3, 4 \quad (9.122)$$

The terms ξ_k and η_k ($k = 1, 2, 3, 4$) in Eqs. (9.122) represent the natural coordinates of the nodal points, or

$$\xi_1 = \eta_1 = -1, \quad \xi_2 = 1, \quad \eta_2 = -1, \quad \xi_3 = \eta_3 = 1, \quad \xi_4 = -1, \quad \eta_4 = 1 \quad (9.123)$$

Hence, inserting Eqs. (9.123) into Eqs. (9.122) and the result into Eq. (9.121), we have

$$A_{..} = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 & -2 & 0 & 1 & 2 & 3 & -1 & -3 \\ 0 & -1 & 0 & 2 & 1 & 0 & -3 & -2 & -1 & 0 & 3 & 1 \\ 1 & 1 & -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 1 & -2 & 0 & 1 & -2 & 3 & 1 & 3 \\ 0 & -1 & 0 & -2 & 1 & 0 & -3 & 2 & -1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 0 & 1 & 2 & 3 & 1 & 3 \\ 0 & -1 & 0 & -2 & -1 & 0 & -3 & -2 & -1 & 0 & -3 & -1 \\ 1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 2 & 0 & 1 & -2 & 3 & -1 & -3 \\ 0 & -1 & 0 & 2 & -1 & 0 & -3 & 2 & -1 & 0 & -3 & -1 \end{bmatrix} \quad (9.124)$$

which has the inverse

$$A^{-1} = \frac{1}{8} \begin{bmatrix} 2 & 1 & -1 & 2 & 1 & 1 & 2 & -1 & 1 & 2 & -1 & -1 \\ -3 & -1 & 1 & 3 & 1 & 1 & 3 & -1 & 1 & -3 & 1 & 1 \\ -3 & -1 & 1 & -3 & -1 & -1 & 3 & -1 & 1 & 3 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 4 & 1 & -1 & -4 & -1 & -1 & 4 & -1 & 1 & -4 & 1 & 1 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 1 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & -1 \\ -1 & -1 & 0 & 1 & 1 & 0 & -1 & 1 & 0 & 1 & -1 & 0 \end{bmatrix} \quad (9.125)$$

Inserting the columns of A^{-1} into Eqs. (9.119), we obtain the interpolation functions

$$\begin{aligned}\phi_1 &= \frac{1}{8} (2 - 3\xi - 3\eta + 4\xi\eta + \xi^3 + \eta^3 - \xi^3\eta - \xi\eta^3) \\ \phi_2 &= \frac{1}{8} (1 - \xi - \eta + \xi\eta - \eta^2 + \xi\eta^2 + \eta^3 - \xi\eta^3) \\ \phi_3 &= -\frac{1}{8} (1 - \xi - \eta - \xi^2 + \xi\eta + \xi^3 + \xi^2\eta - \xi^3\eta) \\ &\dots\dots\dots \\ \phi_{12} &= -\frac{1}{8} (1 - \xi + \eta - \xi^2 - \xi\eta + \xi^3 - \xi^2\eta + \xi^3\eta)\end{aligned}\tag{9.126}$$

Next we address the task of deriving the element stiffness and mass matrices. To this end, we consider Eqs. (7.316) and (7.318) and write the element energy inner product

$$[w, w]_j = \int_{A_j} D_E \left\{ (\nabla^2 w)^2 + (1 - \nu) \left[2 \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 - \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w}{\partial x^2} \right] \right\} dA_j, \quad x, y \text{ in } D_j \tag{9.127}$$

where D_E is the plate flexural rigidity, Eq. (7.317), and the weighted inner product

$$(\sqrt{m}w, \sqrt{m}w)_j = \int_{A_j} m w^2 dx dy, \quad x, y \text{ in } D_j \tag{9.128}$$

To transform the integrals to ones in terms of natural coordinates, we refer to Fig. 9.31 and write the relations

$$x = x_j + a\xi, \quad y = y_j + b\eta \tag{9.129}$$

Then, following the usual procedure, we obtain the element stiffness matrix

$$\begin{aligned}K^{(j)} &= ab \int_{-1}^1 \int_{-1}^1 D_{Ej} \left\{ \left(\frac{1}{a^2} \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{1}{b^2} \frac{\partial^2 \Phi}{\partial \eta^2} \right) \left(\frac{1}{a^2} \frac{\partial^2 \Phi^T}{\partial \xi^2} + \frac{1}{b^2} \frac{\partial^2 \Phi^T}{\partial \eta^2} \right) \right. \\ &\quad \left. + \frac{1 - \nu}{a^2 b^2} \left[2 \frac{\partial^2 \Phi}{\partial \xi \partial \eta} \frac{\partial^2 \Phi^T}{\partial \xi \partial \eta} - \left(\frac{\partial^2 \Phi}{\partial \xi^2} \frac{\partial^2 \Phi^T}{\partial \eta^2} + \frac{\partial^2 \Phi}{\partial \eta^2} \frac{\partial^2 \Phi^T}{\partial \xi^2} \right) \right] \right\} d\xi d\eta\end{aligned}\tag{9.130}$$

in which D_{Ej} is the flexural rigidity for element j , and the element mass matrix

$$M^{(j)} = ab \int_{-1}^1 \int_{-1}^1 m_j \Phi \Phi^T d\xi d\eta \tag{9.131}$$

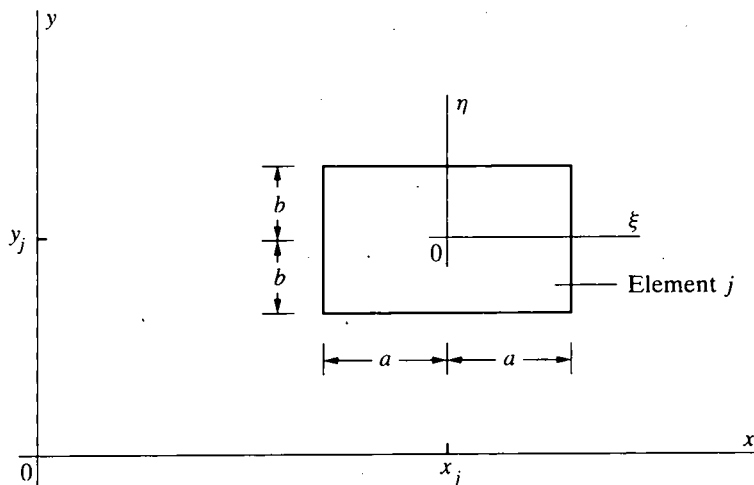


Figure 9.31 Global and local coordinates for a rectangular element

where m_j is the mass density for element j . The element stiffness and mass matrices are obtained by introducing Eqs. (9.126) into Eqs. (9.130) and (9.131), respectively, and performing the indicated integrations.

The assembly process for rectangular plate elements is similar to that for rectangular membrane elements, the main difference being that for plates there are three degrees of freedom per node. Keeping this in mind, the connectivity array for rectangular plate elements remains the same as for rectangular membrane elements.

The generalization of rectangular elements to quadrilateral elements for plate vibration is not trivial, and serious difficulties can be anticipated (Ref. 28, p. 240).

At this point, we turn our attention to triangular plate elements. In view of our discussion earlier in this section, we consider a triangular element with three nodes and three degrees of freedom per node, as shown in Fig. 9.32, for a total of nine degrees of freedom per element. Here we encounter immediately a problem, as the lowest-degree polynomial admissible is the cubic, which has nine terms. Adding the constant term, this would yield interpolation functions in terms of area coordinates

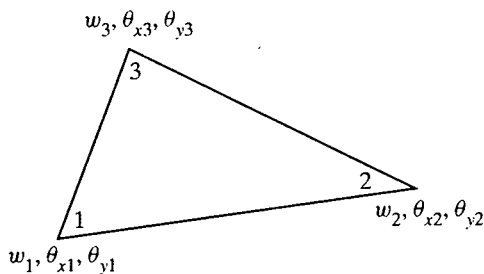


Figure 9.32 Triangular element with three degrees of freedom at each node

of the form

$$\phi_i = c_{i1} + c_{i2}\xi_1 + c_{i3}\xi_2 + c_{i4}\xi_1^2 + c_{i5}\xi_1\xi_2 + c_{i6}\xi_2^2 + c_{i7}\xi_1^3 + c_{i8}\xi_1^2\xi_2 + c_{i9}\xi_1\xi_2^2 + c_{i10}\xi_2^3 \tag{9.132}$$

where the dependence on the third area coordinate ξ_3 is implicit. Because there are ten constants and only nine degrees of freedom, there are several alternatives. One alternative is to add an internal node with only the transverse displacement as a degree of freedom (Ref. 22). Another alternative is to take arbitrarily $c_{i8} = c_{i9}$. An alternative with superior convergence characteristics consists of the interpolation functions (Ref. 28, p. 244)

$$\begin{aligned} \phi_1 &= \xi_1 + \xi_1^2\xi_2 + \xi_1^2\xi_3 - \xi_1\xi_2^2 - \xi_1\xi_2^2 \\ \phi_2 &= (y_3 - y_1)(\xi_3\xi_1^2 + \frac{1}{2}\xi_1\xi_2\xi_3) - (y_1 - y_2)(\xi_1^2\xi_2 + \frac{1}{2}\xi_1\xi_2\xi_3) \\ \phi_3 &= (x_1 - x_3)(\xi_3\xi_1^2 + \frac{1}{2}\xi_1\xi_2\xi_3) - (x_2 - x_1)(\xi_1^2\xi_2 + \frac{1}{2}\xi_1\xi_2\xi_3) \\ &\dots\dots\dots \\ \phi_9 &= (x_3 - x_2)(\xi_2\xi_3^2 + \frac{1}{2}\xi_1\xi_2\xi_3) - (x_1 - x_3)(\xi_3^2\xi_1 + \frac{1}{2}\xi_1\xi_2\xi_3) \end{aligned} \tag{9.133}$$

and note that a difference in the sign of $\phi_2, \phi_3, \phi_5, \dots, \phi_9$ is due to the fact that the rotations defined in Ref. 28 are the negative of those defined by Eqs. (9.118).

The element stiffness and mass matrices are as given by Eqs. (9.127) and (9.128), respectively, where

$$w = \Phi^T \mathbf{w} \tag{9.134}$$

in which Φ is a nine-dimensional vector with components given by Eqs. (9.133) and \mathbf{w} is the nine-dimensional nodal vector

$$\mathbf{w} = [w_1 \theta_{x1} \theta_{y1} \ w_2 \theta_{x2} \theta_{y2} \ w_3 \theta_{x3} \theta_{y3}]^T \tag{9.135}$$

and note that the rotations are defined in terms of rectangular coordinates as

$$\theta_{xi} = (\partial w / \partial y)_i, \quad \theta_{yi} = -(\partial w / \partial x)_i, \quad i = 1, 2, 3 \tag{9.136}$$

The relation between the rectangular coordinates and area coordinates are given by Eqs. (9.55). Hence, using Eqs. (9.55) with A replaced by A_j and Eq. (9.134) with \mathbf{w}

replaced by w_j , we can write

$$\begin{aligned} \frac{\partial w}{\partial x} &= \frac{\partial w}{\partial \xi_1} \frac{\partial \xi_1}{\partial x} + \frac{\partial w}{\partial \xi_2} \frac{\partial \xi_2}{\partial x} + \frac{\partial w}{\partial \xi_3} \frac{\partial \xi_3}{\partial x} \\ &= \frac{1}{2A_j} \left[\frac{\partial \Phi^T}{\partial \xi_1} (y_2 - y_3) + \frac{\partial \Phi^T}{\partial \xi_2} (y_3 - y_1) + \frac{\partial \Phi^T}{\partial \xi_3} (y_1 - y_2) \right] w_j \end{aligned} \quad (9.137)$$

$$\begin{aligned} \frac{\partial w}{\partial y} &= \frac{\partial w}{\partial \xi_1} \frac{\partial \xi_1}{\partial y} + \frac{\partial w}{\partial \xi_2} \frac{\partial \xi_2}{\partial y} + \frac{\partial w}{\partial \xi_3} \frac{\partial \xi_3}{\partial y} \\ &= \frac{1}{2A_j} \left[\frac{\partial \Phi^T}{\partial \xi_1} (x_3 - x_2) + \frac{\partial \Phi^T}{\partial \xi_2} (x_1 - x_3) + \frac{\partial \Phi^T}{\partial \xi_3} (x_2 - x_1) \right] w_j \end{aligned}$$

Then, using the chain rule again, we obtain

$$\begin{aligned} \frac{\partial^2 w}{\partial x^2} &= \frac{1}{4A_j^2} \left[\frac{\partial^2 \Phi^T}{\partial \xi_1^2} (y_2 - y_3)^2 + \frac{\partial^2 \Phi^T}{\partial \xi_2^2} (y_3 - y_1)^2 + \frac{\partial^2 \Phi^T}{\partial \xi_3^2} (y_1 - y_2)^2 \right. \\ &\quad + 2 \frac{\partial^2 \Phi^T}{\partial \xi_1 \partial \xi_2} (y_2 - y_3)(y_3 - y_1) + 2 \frac{\partial^2 \Phi^T}{\partial \xi_1 \partial \xi_3} (y_2 - y_3)(y_1 - y_2) \\ &\quad \left. + 2 \frac{\partial^2 \Phi^T}{\partial \xi_2 \partial \xi_3} (y_3 - y_1)(y_1 - y_2) \right] w_j \\ \frac{\partial^2 w}{\partial y^2} &= \frac{1}{4A_j^2} \left[\frac{\partial^2 \Phi^T}{\partial \xi_1^2} (x_3 - x_2)^2 + \frac{\partial^2 \Phi^T}{\partial \xi_2^2} (x_1 - x_3)^2 + \frac{\partial^2 \Phi^T}{\partial \xi_3^2} (x_2 - x_1)^2 \right. \\ &\quad + 2 \frac{\partial^2 \Phi^T}{\partial \xi_1 \partial \xi_2} (x_3 - x_2)(x_1 - x_3) + 2 \frac{\partial^2 \Phi^T}{\partial \xi_1 \partial \xi_3} (x_3 - x_2)(x_2 - x_1) \\ &\quad \left. + 2 \frac{\partial^2 \Phi^T}{\partial \xi_2 \partial \xi_3} (x_1 - x_3)(x_2 - x_1) \right] w_j \end{aligned} \quad (9.138)$$

$$\begin{aligned} \frac{\partial^2 w}{\partial x \partial y} &= \frac{1}{4A_j^2} \left\{ \frac{\partial^2 \Phi^T}{\partial \xi_1^2} (x_3 - x_2)(y_2 - y_3) + \frac{\partial^2 \Phi^T}{\partial \xi_2^2} (x_1 - x_3)(y_3 - y_1) \right. \\ &\quad + \frac{\partial^2 \Phi^T}{\partial \xi_3^2} (x_2 - x_1)(y_1 - y_2) \\ &\quad + \frac{\partial^2 \Phi^T}{\partial \xi_1 \partial \xi_2} [(x_3 - x_2)(y_3 - y_1) + (x_1 - x_3)(y_2 - y_3)] \\ &\quad + \frac{\partial^2 \Phi^T}{\partial \xi_1 \partial \xi_3} [(x_3 - x_2)(y_1 - y_2) + (x_2 - x_1)(y_2 - y_3)] \\ &\quad \left. + \frac{\partial^2 \Phi^T}{\partial \xi_2 \partial \xi_3} [(x_1 - x_3)(y_1 - y_2) + (x_2 - x_1)(y_3 - y_1)] \right\} w_j \end{aligned}$$

The element stiffness matrix can be obtained by inserting Eqs. (9.138) in conjunction with Eqs. (9.133) into Eq. (9.127) and performing the indicated integrations. Moreover, the element mass matrix can be produced by introducing Eqs. (9.133) into Eq. (9.131) and carrying out the required integrations. In both regards, Eq. (9.66) should prove useful.

The assembly process for triangular plate elements, including the connectivity array, is similar to that for triangular membrane elements, with the exception of the fact that for plates there are three degrees of freedom at each node instead of one.

The generation of the element stiffness and mass matrices and the assembly process are routine and can be carried out conveniently on a computer.

9.9 ERRORS IN THE APPROXIMATE EIGENVALUES AND EIGENFUNCTIONS

In view of the fact that for self-adjoint systems the finite element method represents a Rayleigh-Ritz method, if the elements are conformable, then its convergence is assured, although convergence is possible even with nonconformable elements (Ref. 28, p. 244). Clearly, the problem of nonconformable elements arises in plates in bending. It appears that convergence can be assumed if the elements pass the *patch test* (see Ref. 28, Secs. 2.7 and 11.2). It has been shown by Irons (Ref. 4) in conjunction with the interpolation functions given by Eqs. (9.133) that a mesh generated by three sets of equally spaced parallel lines, such as that of Fig. 9.33a, passes the patch test and a mesh of the type shown in Fig. 9.33b does not. The issue of convergence is to a large extent only of academic interest, as for most systems the finite element method is known to converge. Indeed, more important is the rate of convergence. It is here that the finite element method pays the price for the simplicity arising from the use of low-degree polynomials as admissible functions.

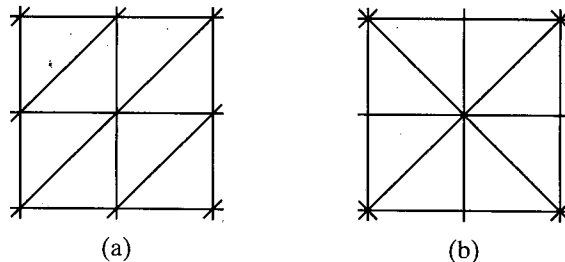


Figure 9.33 (a) Mesh passing the patch test (b) Mesh failing the patch test

Convergence is a qualitative concept. A more quantitative measure of the accuracy of the approximate solution can be obtained from error estimates. In the case of approximate solutions derived by the finite element method, the error estimates can be quantified to some extent by tying them not only to the dimension of the Ritz space but also to the degree of the elements. In this section, we give a summary of results presented in Ref. 22 (Sec. 6.3).

We are concerned with eigenvalue problems described by the differential equation

$$Lw = \lambda mw \text{ over } D \quad (9.139)$$

where L is a linear homogeneous self-adjoint differential operator of order $2p$, and the boundary conditions

$$B_i w = 0 \text{ on } S, \quad i = 1, 2, \dots, p \quad (9.140)$$

in which B_i are boundary operators of maximum order $2p - 1$. Multiplying Eq. (9.139) through by w and integrating by parts with due consideration to boundary conditions (9.140), we obtain the weak form of the eigenvalue problem

$$[w, w] = \lambda(\sqrt{m}w, \sqrt{m}w) \quad (9.141)$$

where $[w, w]$ is an energy inner product, a measure of the potential energy, and $(\sqrt{m}w, \sqrt{m}w)$ is a weighted inner product, a measure of the kinetic energy. Then, dividing Eq. (9.141) by $(\sqrt{m}w, \sqrt{m}w)$, we obtain the Rayleigh's quotient

$$\lambda = R(w) = \frac{[w, w]}{(\sqrt{m}w, \sqrt{m}w)} \quad (9.142)$$

As shown in Sec. 7.14, rendering Rayleigh's quotient stationary is equivalent to solving the weak form of the eigenvalue problem.

Throughout this chapter, we have been concerned with finite element approximate solutions to the eigenvalue problem obtained by rendering Rayleigh's quotient stationary. Such solutions, denoted by $w^{(n)}$, have been assumed to have the form of linear combinations of admissible functions ϕ_i ($i = 1, 2, \dots, n$) from the subspace \mathcal{R}_n of the energy space \mathcal{K}_G^p , or

$$w^{(n)} = \boldsymbol{\phi}^T \mathbf{a} \quad (9.143)$$

where $\boldsymbol{\phi}$ is an n -vector of admissible functions and \mathbf{a} an n -vector of undetermined coefficients. Inserting Eq. (9.143) into Eq. (9.142), we obtain the discretized version of Rayleigh's quotient

$$R(w^{(n)}) = \frac{[w^{(n)}, w^{(n)}]}{(\sqrt{m}w^{(n)}, \sqrt{m}w^{(n)})} = \frac{\mathbf{a}^T \mathbf{K} \mathbf{a}}{\mathbf{a}^T \mathbf{M} \mathbf{a}} \quad (9.144)$$

in which \mathbf{K} and \mathbf{M} are stiffness and mass matrices. Rayleigh's quotient has stationary values in the form of approximate eigenvalues $\lambda_r^{(n)}$ at the eigenvectors \mathbf{a}_r . The eigenvectors are orthogonal with respect to \mathbf{M} and \mathbf{K} and can be normalized so that

$$\mathbf{a}_s^T \mathbf{M} \mathbf{a}_r = \delta_{rs}, \quad \mathbf{a}_s^T \mathbf{K} \mathbf{a}_r = \lambda_r^{(n)} \delta_{rs}, \quad r, s = 1, 2, \dots, n \quad (9.145)$$

Introducing the eigenvectors \mathbf{a}_r into Eq. (9.143), we obtain the approximate eigenfunctions

$$w_r^{(n)} = \boldsymbol{\phi}^T \mathbf{a}_r, \quad r = 1, 2, \dots, n \quad (9.146)$$

which satisfy the orthonormality relations

$$(\sqrt{m}w_s^{(n)}, \sqrt{m}w_r^{(n)}) = \delta_{rs}, \quad [w_s^{(n)}, w_r^{(n)}] = \lambda_r^{(n)} \delta_{rs}, \quad r, s = 1, 2, \dots, n \quad (9.147)$$

Then, using the analogy with Eq. (7.404), we can write the maximin theorem for the approximate eigensolutions in the form

$$\lambda_{r+1}^{(n)} = \max_{v_i} \min_{w^{(n)}} R(w^{(n)}), \quad (w^{(n)}, v_i) = 0, \quad i = 1, 2, \dots, r \quad (9.148)$$

where v_i are r independent but otherwise arbitrary functions.

Using the maximin theorem, it is shown in Ref. 22 (Sec. 6.3) that the approximate eigenvalues are bounded for small h by

$$\lambda_r \leq \lambda_r^{(n)} \leq \lambda_r + 2\delta h^{2(k-p)} (\lambda_r^{(n)})^{k/p}, \quad r = 1, 2, \dots, n \quad (9.149)$$

in which h is the longest side of the finite elements, $k-1$ is the degree of the elements and δ is some constant. The factor $h^{2(k-p)}$ in inequalities (9.149) implies that, for a given small h , the errors decrease as the degree of the elements increases. On the other hand, the factor $(\lambda_r^{(n)})^{k/p}$ indicates that the errors increase with the mode number, so that higher approximate eigenvalues tend to be inaccurate. As a rule of thumb, more than one half of the approximate eigenvalues must be regarded as unreliable (Ref. 22, p. 227). Hence, the dimension of the Ritz space \mathcal{R}_n must be at least twice as large as the number of accurate eigenvalues desired.

As shown in Ref. 22 (Sec. 6.3) error estimates for approximate eigenfunctions are also possible. These are not pointwise error estimates, but estimates in an average sense. Indeed, it is demonstrated in Ref. 22 that for small h

$$\|\sqrt{m}(w_r - w_r^{(n)})\| \leq c[h^k + h^{2(k-p)}] (\lambda_r^{(n)})^{k/2p}, \quad r = 1, 2, \dots, n \quad (9.150a)$$

and

$$(w_r - w_r^{(n)}, w_r - w_r^{(n)}) \leq c'h^{2(k-p)} (\lambda_r^{(n)})^{k/p}, \quad r = 1, 2, \dots, n \quad (9.150b)$$

Errors of a different type arise when the mass matrix is generated by a different process than the stiffness matrix, giving rise to so-called *inconsistent mass matrices*. The most common example of an inconsistent mass matrix is the *lumped mass matrix*. The use of a lumped matrix tends to raise the value of the denominator in the Rayleigh's quotient, thus lowering the approximate eigenvalues compared to the eigenvalues computed using a consistent mass matrix. Because the eigenvalues computed by means of a Rayleigh-Ritz process tend to be higher than the actual ones, this may appear as a good way of improving the estimates. This is not the case, however, as the use of inconsistent mass matrices violates the Rayleigh-Ritz code, so that the Rayleigh-Ritz theory can no longer be counted on to argue that the actual eigenvalues represent lower bounds for the approximate eigenvalues. As a result, the use of lumped masses can cause significant errors (Ref. 22, p. 228, and Ref. 25).

Before abandoning this section, it should be noted that, whereas the maximin theorem does apply to the finite element method, in general the separation theorem (Sec. 8.5) cannot be demonstrated to hold true. However, the fact that no proof exists should not be construed to imply that the separation theorem does not hold for the finite element method.

9.10 THE HIERARCHICAL FINITE ELEMENT METHOD

As demonstrated in Sec. 8.5, the accuracy of the approximate eigenvalues obtained by means of the Rayleigh-Ritz method can be improved by simply increasing the number of admissible functions in the linear combination representing the approximate solution. On the other hand, accuracy can be improved in the finite element method by refining the mesh, which implies a reduction in the width h of the element, or equivalently an increase in the number of elements. The procedure is characterized by the fact that the degree p of the elements is a fixed, generally low number. No confusion should arise from the fact that earlier in this text we used the symbol p in connection with the order of the stiffness operator L . Another way of improving the accuracy of the finite element approximation is to keep the width h constant and to increase the degree p of the polynomials. To distinguish between the two, the approach whereby the accuracy is improved by refining the finite element mesh is referred to as the h -version of the finite element method, and the approach whereby the degree of the polynomials is increased is known as the p -version (Refs. 2, 3 and 23).

Because in the p -version of the finite element method accuracy is improved through an increase in the degree of the polynomials, which implies an increase in the number of admissible functions in the approximation, the p -version has something in common with the classical Rayleigh-Ritz method. Of course, one of the main differences remains, as the admissible functions used in the p -version of the finite element method are local functions and in the classical Rayleigh-Ritz method they are global functions. This gives the p -version greater versatility than the h -version. As a result, the rate of convergence of the p -version can be higher than that of the h -version.

In the p -version of the finite element method it is possible to choose from a variety of different sets of polynomials, provided the sets are complete. Particularly desirable are the so-called *hierarchical* polynomials, which have the property that the set of polynomials in the approximation of degree p represents a subset of the polynomials in the approximation of degree $p + 1$. This version is referred to as the *hierarchical finite element method* (Refs. 18, 27 and 29) and is characterized by the fact that the mass and stiffness matrices possess the embedding property indicated by Eqs. (8.100). As a result, the separation theorem holds true for the hierarchical finite element method (Ref. 14).

Next, we demonstrate the hierarchical finite element method by applying it to the eigenvalue problem of a beam in bending. In Sec. 9.4, we have shown that the most common polynomials for beams in bending are the Hermite cubics

$$\phi_1 = 3\xi^2 - \xi^3, \quad \phi_2 = \xi^2 - \xi^3, \quad \phi_3 = 1 - 3\xi^2 + 2\xi^3, \quad \phi_4 = -\xi + 2\xi^2 - \xi^3$$

(9.151)

A suitable set of hierarchical functions consists of the polynomials

$$\begin{aligned}
 \phi_5 &= \xi^2(1 - \xi)^2 \\
 \phi_6 &= \xi^2(1 - \xi)^2(1 - 2\xi) \\
 \phi_7 &= \xi^2(1 - \xi)^2(1 - 3\xi)(2 - 3\xi) \\
 &\dots\dots\dots \\
 \phi_{4+i} &= \xi^2(1 - \xi)^2 \prod_{j=2}^i (j - 1 - i\xi)
 \end{aligned} \tag{9.152}$$

and we note that all hierarchical functions and their slope have zero values at the nodes $\xi = 0$ and $\xi = 1$. As a result, when one hierarchical function is added to the approximating series for a given element, one row and one column are added to the element stiffness and mass matrices without disturbing the previously computed entries, so that they possess the *embedding property*. Hence, the separation theorem applies.

To develop an appreciation for the hierarchical finite element method, we consider the numerical example of Ref. 14, in which the five lowest eigenvalues of a uniform cantilever beam were computed by means of the h -version and p -version of the finite element method. Results are displayed in Tables 9.4 and 9.5, respectively. To place matters in proper perspective, we note that in the h -version 4, 6, 8 and 10 elements imply 8, 12, 16 and 20 degrees of freedom, respectively. On the other hand, the number of degrees of freedom for the p -version ranges from 9 to 12 in the upper columns in Table 9.5, from 10 to 16 in the middle columns and from 11 to 20 in the bottom columns. Hence, a direct comparison can be made between columns 2,3 and 4 in Table 9.4 and the extreme right upper, middle and bottom columns in Table 9.5, respectively. Clearly, the hierarchical finite element method gives significantly more accurate results than the ordinary finite element method, particularly for the higher eigenvalues. Examining results in the upper columns in Table 9.5, it is clear that the separation theorem holds true for the hierarchical finite element method. It turns out that the same is true for the eigenvalues in Table 9.4, the middle columns in Table 9.5 and the lower columns in Table 9.5. Not too much should be read into this, however. Indeed, the explanation is that for one-dimensional domains the eigenvalues tend to be well spaced, so it would be quite difficult to violate the separation theorem.

TABLE 9.4 The Five Lowest Eigenvalues Computed by the h -Version

Four Elements	Six Elements	Eight Elements	Ten Elements
0.14065	0.14064	0.14064	0.14064
0.88241	0.88160	0.88145	0.88141
2.48700	2.47240	2.46991	2.46852
4.90631	4.86724	4.84691	4.84068
9.12550	8.11453	8.04064	8.01453

TABLE 9.5 The Five Lowest Eigenvalues Computed by the p -Version Using Four Finite Elements

$\phi_1, \phi_2, \dots, \phi_4$ on 1, 2, 3	$\phi_1, \phi_2, \dots, \phi_4$ on 1, 2	$\phi_1, \phi_2, \dots, \phi_4$ on 1	$\phi_1, \phi_2, \dots, \phi_5$ on all
$\phi_1, \phi_2, \dots, \phi_5$ on 4	$\phi_1, \phi_2, \dots, \phi_5$ on 3, 4	$\phi_1, \phi_2, \dots, \phi_5$ on 2, 3, 4	
0.14065	0.14064	0.14064	0.14064
0.88221	0.88190	0.88142	0.88140
2.48687	2.48008	2.47145	2.46875
4.89846	4.89464	4.89295	4.85038
8.92140	8.53673	8.26909	8.01927
$\phi_1, \phi_2, \dots, \phi_4$ on 1, 2, 3	$\phi_1, \phi_2, \dots, \phi_4$ on 1, 2	$\phi_1, \phi_2, \dots, \phi_4$ on 1	$\phi_1, \phi_2, \dots, \phi_6$ on all
$\phi_1, \phi_2, \dots, \phi_6$ on 4	$\phi_1, \phi_2, \dots, \phi_6$ on 3, 4	$\phi_1, \phi_2, \dots, \phi_6$ on 2, 3, 4	
0.14064	0.14064	0.14064	0.14064
0.88220	0.88188	0.88141	0.88138
2.48633	2.47948	2.47074	2.46790
4.89359	4.88554	4.87950	4.83619
8.90123	8.51833	8.24920	7.99920
$\phi_1, \phi_2, \dots, \phi_4$ on 1, 2, 3	$\phi_1, \phi_2, \dots, \phi_4$ on 1, 2	$\phi_1, \phi_2, \dots, \phi_4$ on 1	$\phi_1, \phi_2, \dots, \phi_7$ on all
$\phi_1, \phi_2, \dots, \phi_7$ on 4	$\phi_1, \phi_2, \dots, \phi_7$ on 3, 4	$\phi_1, \phi_2, \dots, \phi_7$ on 2, 3, 4	
0.14064	0.14064	0.14064	0.14064
0.88220	0.88188	0.88141	0.88138
2.48633	2.47947	2.47073	2.46789
4.89351	4.88545	4.87941	4.83609
8.89793	8.51363	8.24405	7.99442

9.11 SYSTEM RESPONSE

The response of systems discretized by the finite element method can be obtained by an approach similar to that used in Sec. 8.10 in conjunction with the classical Rayleigh-Ritz method. However, in view of the fact that our treatment of the finite element method is based on the variational approach, we will modify the derivation accordingly. As a result, the approach adopted here resembles that used in Sec. 8.7 in conjunction with the assumed-modes method, which is really a variant of the Rayleigh-Ritz method.

The kinetic energy of a distributed-parameter system can be written as a weighted inner product in the form

$$T(t) = \frac{1}{2} \left(\sqrt{m(P)} \dot{w}(P, t), \sqrt{m(P)} \dot{w}(P, t) \right) \quad (9.153)$$

Similarly, the potential energy can be expressed as the energy inner product

$$V(t) = \frac{1}{2} [w(P, t), w(P, t)] \quad (9.154)$$

Moreover, the virtual work of the nonconservative forces has the form

$$\overline{\delta W_{nc}}(t) = \int_D f(P, t) \delta w(P, t) dD(P) \quad (9.155)$$

where $f(P, t)$ is the distributed force.

The response of a system discretized by the finite element method can be obtained by simply letting the nodal coordinates be functions of time. Hence, by analogy with Eq. (9.5), we write the response in the form

$$w(P, t) \cong w^{(n)}(P, t) = \Phi^T(P) \mathbf{q}_j(t) \text{ over } D_j, \quad j = 1, 2, \dots, n \quad (9.156)$$

where Φ is a vector of interpolation functions and $\mathbf{q}_j(t)$ a vector of time-dependent nodal displacements for element j . Introducing Eq. (9.156) into Eq. (9.153), we obtain the discretized kinetic energy

$$T(t) \cong \frac{1}{2} \sum_{j=1}^n \dot{\mathbf{q}}_j^T (\sqrt{m} \Phi, \sqrt{m} \Phi^T)_j \dot{\mathbf{q}}_j = \frac{1}{2} \sum_{j=1}^n \dot{\mathbf{q}}_j^T M_j \dot{\mathbf{q}}_j \quad (9.157)$$

in which

$$M_j = (\sqrt{m} \Phi, \sqrt{m} \Phi^T)_j, \quad j = 1, 2, \dots, n \quad (9.158)$$

are recognized as the element mass matrices. In the same manner, the discretized potential energy can be written as

$$V(t) \cong \frac{1}{2} \sum_{j=1}^n \mathbf{q}_j^T [\Phi, \Phi^T]_j \mathbf{q}_j = \frac{1}{2} \sum_{j=1}^n \mathbf{q}_j^T K_j \mathbf{q}_j \quad (9.159)$$

where

$$K_j = [\Phi, \Phi^T]_j, \quad j = 1, 2, \dots, n \quad (9.160)$$

are the element stiffness matrices. Similarly, the discretized virtual work has the expression

$$\overline{\delta W_{nc}}(t) \cong \sum_{j=1}^n \int_{D_j} f \Phi^T \delta \mathbf{q}_j dD = \sum_{j=1}^n \mathbf{Q}_j^T \delta \mathbf{q}_j \quad (9.161)$$

in which

$$\mathbf{Q}_j(t) = \int_{D_j} f(P, t) \Phi(P) dD(P), \quad j = 1, 2, \dots, n \quad (9.162)$$

are the element nodal force vectors.

The assembly process amounts to using the connectivity array (Sec. 9.5) to eliminate redundant coordinates. It is not difficult to show that, following the assembly, the kinetic energy, potential energy and virtual work assume the usual form

$$T = \frac{1}{2} \mathbf{q}^T M \dot{\mathbf{q}}, \quad V = \frac{1}{2} \mathbf{q}^T K \mathbf{q} \quad (9.163a, b)$$

and

$$\overline{\delta W_{nc}} = \mathbf{Q}^T \delta \mathbf{q} \quad (9.163c)$$

where \mathbf{q} is the global nodal displacement vector, M and K are the global mass and stiffness matrices and \mathbf{Q} is the global nodal force vector.

Using Lagrange's equations, the nodal equations of motion for the system are simply

$$M \ddot{\mathbf{q}} + K \mathbf{q} = \mathbf{Q} \quad (9.164)$$

The solution of Eq. (9.164) was discussed in Sec. 4.10.

Example 9.5

Determine the global mass matrix, stiffness matrix and nodal force vector defining the response of a uniform simply supported beam to the distributed force $f(x, t) = (1 - x/2L)f(t)$ derived by the finite element method using Hermite cubics as interpolation functions.

We begin by determining the element mass matrix, stiffness matrix and nodal force vector. To this end, we recall from Eqs. (9.41) that the Hermite cubics have the form

$$\phi_1 = 3\xi^2 - 2\xi^3, \quad \phi_2 = \xi^2 - \xi^3, \quad \phi_3 = 1 - 3\xi^2 + 2\xi^3, \quad \phi_4 = -\xi + 2\xi^2 - \xi^3 \quad (a)$$

so that, using Eqs. (9.158) and recalling that $x = (j - \xi)h$, the element mass matrices are

$$\begin{aligned} M_j &= (\sqrt{m}\phi, \sqrt{m}\phi^T)_j = \int_{(j-1)h}^{jh} m\phi\phi^T dx = mh \int_0^1 \phi\phi^T d\xi \\ &= mh \int_0^1 \begin{bmatrix} 3\xi^2 - 2\xi^3 \\ \xi^2 - \xi^3 \\ 1 - 3\xi^2 + 2\xi^3 \\ -\xi + 2\xi^2 - \xi^3 \end{bmatrix} \begin{bmatrix} 3\xi^2 - 2\xi^3 \\ \xi^2 - \xi^3 \\ 1 - 3\xi^2 + 2\xi^3 \\ -\xi + 2\xi^2 - \xi^3 \end{bmatrix}^T d\xi \\ &= \frac{mL}{420n} \begin{bmatrix} 156 & 22 & 54 & -13 \\ & 4 & 13 & -3 \\ & & \text{symm} & \\ & & & 156 & -22 \\ & & & & 4 \end{bmatrix}, \quad j = 1, 2, \dots, n \end{aligned} \quad (b)$$

and, using Eqs. (9.160), the element stiffness matrices are

$$\begin{aligned} K_j &= [\phi, \phi^T]_j = \int_{(j-1)h}^{jh} EI \frac{d^2\phi}{dx^2} \frac{d^2\phi^T}{dx^2} dx = \frac{EI}{h^3} \int_0^1 \phi''\phi''^T d\xi \\ &= \frac{EI}{h^3} \int_0^1 \begin{bmatrix} 6 - 12\xi \\ 2 - 6\xi \\ -6 + 12\xi \\ 4 - 6\xi \end{bmatrix} \begin{bmatrix} 6 - 12\xi \\ 2 - 6\xi \\ -6 + 12\xi \\ 4 - 6\xi \end{bmatrix}^T d\xi \end{aligned}$$

$$= \frac{EIn^3}{L^3} \begin{bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ \text{symm} & & 12 & -6 \\ & & & 4 \end{bmatrix}, \quad j = 1, 2, \dots, n \quad (c)$$

and we observe that M_j and K_j are consistent with results obtained in Example 9.3. Then, from Eqs. (9.162), the element nodal force vectors are

$$\begin{aligned} \mathbf{Q}_j(t) &= \int_{(j-1)h}^{jh} f(x, t) \boldsymbol{\Phi} dx \\ &= hf(t) \int_0^1 \left[1 - \frac{(j-\xi)h}{2L} \right] [3\xi^2 - 2\xi^3 \quad \xi^2 - \xi^3 \quad 1 - 3\xi^2 + 2\xi^3 \quad -\xi + 2\xi^2 - \xi^3]^T d\xi \\ &= hf(t) \left\{ \left(1 - \frac{jh}{2L} \right) \int_0^1 \begin{bmatrix} 3\xi^2 - 2\xi^3 \\ \xi^2 - \xi^3 \\ 1 - 3\xi^2 + 2\xi^3 \\ -\xi + 2\xi^2 - \xi^3 \end{bmatrix} d\xi + \frac{h}{2L} \int_0^1 \begin{bmatrix} 3\xi^3 - 2\xi^4 \\ \xi^3 - \xi^4 \\ \xi - 3\xi^3 + 2\xi^4 \\ -\xi^2 + 2\xi^3 - \xi^4 \end{bmatrix} d\xi \right\} \\ &= \frac{Lf(t)}{120n^2} \left\{ (2n - j) \begin{bmatrix} 30 \\ 5 \\ 30 \\ -5 \end{bmatrix} + \begin{bmatrix} 21 \\ 3 \\ 9 \\ -2 \end{bmatrix} \right\}, \quad j = 1, 2, \dots, n \quad (d) \end{aligned}$$

At this point, we are ready for the assembly process. In view of the fact that for a simply supported beam the displacement is zero at both ends, $w_0 = w_n = 0$, the structure of the global mass and stiffness matrices is somewhat different from that indicated in Sec. 9.4. Indeed, in the case at hand, we must strike out the first and $(2n - 1)$ st rows and columns from the global mass and stiffness matrices. Hence, using Eqs. (b), we obtain the global mass matrix

$$M = \frac{mL}{420n} \begin{bmatrix} 4 & 13 & -3 & 0 & 0 & \dots & 0 & 0 & 0 \\ & 312 & 0 & 54 & -13 & \dots & 0 & 0 & 0 \\ & & 8 & 13 & -3 & \dots & 0 & 0 & 0 \\ & & & 312 & 0 & \dots & 0 & 0 & 0 \\ \text{symm} & & & & 8 & \dots & 0 & 0 & 0 \\ & & & & & \dots & 312 & 0 & -13 \\ & & & & & & & 8 & -3 \\ & & & & & & & & 4 \end{bmatrix} \quad (e)$$

Similarly, using Eqs. (c), we can write the global stiffness matrix

$$K = \frac{EIn^3}{L^3} \begin{bmatrix} 4 & -6 & 2 & 0 & 0 & \dots & 0 & 0 & 0 \\ & 24 & 0 & -12 & 6 & \dots & 0 & 0 & 0 \\ & & 8 & -6 & 2 & \dots & 0 & 0 & 0 \\ & & & 24 & 0 & \dots & 0 & 0 & 0 \\ \text{symm} & & & & 8 & \dots & 0 & 0 & 0 \\ & & & & & \dots & 24 & 0 & 6 \\ & & & & & & & 8 & 2 \\ & & & & & & & & 4 \end{bmatrix} \quad (f)$$

Finally, from Eqs. (d), striking out the first and $(2n - 1)$ st components and adding the contributions from \mathbf{Q}_{j-1} and \mathbf{Q}_j corresponding to shared nodal forces, we obtain the global nodal force vector

$$\mathbf{Q}(t) = \frac{Lf(t)}{120n^2} [10n - 2 \quad 120n - 60 \quad -4 \quad 120n - 120 \quad -4 \quad \dots \quad 60n + 60 \quad 5n + 1 \quad -5n - 2]^T \quad (g)$$

and we recognize that the global nodal displacement vector has the form

$$\mathbf{q}(t) = [h\theta_0(t) \quad u_1(t) \quad h\theta_1(t) \quad w_2(t) \quad h\theta_2(t) \quad \dots \quad w_{n-1}(t) \quad h\theta_{n-1}(t) \quad h\theta_n(t)]^T \quad (h)$$

9.12 SYNOPSIS

As far as vibration theory is concerned, the finite element method represents a Rayleigh-Ritz method. The main difference between the original Rayleigh-Ritz method, referred to as the classical Rayleigh-Ritz method, and the finite element method lies in the nature of the admissible functions. Indeed, the classical Rayleigh-Ritz method uses global admissible functions and the finite element method uses local admissible functions, generally called interpolation functions. This difference has profound implications and is responsible for the success of the finite element method. In the first place, because the method works with small subdomains, i.e., the finite elements over which the local functions are defined, systems with very complex parameter distributions and irregular geometries can be accommodated. We recall that one of the weaknesses of the classical Rayleigh-Ritz method and the weighted residuals methods, such as Galerkin's method and the collocation method, is the difficulty in handling systems with irregular boundaries. Moreover, because the finite elements are generally very small, good representation of the motion can be achieved with interpolation functions in the form of low-degree polynomials. An important aspect of this is that the interpolation functions can be prescribed for given classes of systems, a clear advantage over the classical Rayleigh-Ritz method and Galerkin's method, for which the generation of suitable admissible and comparison functions, respectively, requires experience and ingenuity. Another important aspect is that the whole finite element spatial discretization process lends itself to easy computer coding. This includes the generation of a finite element mesh, the computation of the element mass matrix, stiffness matrix and nodal force vector and the assembly of these element quantities into global quantities. With the power of computers to carry out the various steps increasing at a dizzying rate, finite element models involving thousands of degrees of freedom are not uncommon.

The enthusiasm for the finite element method should be tempered somewhat by other considerations. Some of the advantages cited above are more important to static stress and structural analyses than to vibrations. In particular, stress analysis problems are more typical of three-dimensional systems, which tend to be bulkier and hence less prone to vibrate than one-dimensional and two-dimensional systems. Hence complex geometries are more frequently encountered in stress analysis than in vibrations. Then, for problems for which the classical Rayleigh-Ritz method is able to produce a solution, it is reasonable to expect that, for a given accuracy, such

a solution requires appreciably fewer degrees of freedom than a solution produced by the finite element method.

On balance, however, the advantages of the finite element method far outweigh any disadvantages, as the method is capable of producing solutions where other methods fail. Already very popular, the use of the finite element method can only increase as vibration problems become progressively more complex.

PROBLEMS

(to be solved by the finite element method)

- 9.1 Solve Problem 8.1 using linear interpolation functions and determine the order of the problem required to match the accuracy of the lowest natural frequency computed in Problem 8.1.
- 9.2 Solve Problem 8.16 using linear interpolation functions and determine the order of the problem required to match the accuracy of the lowest natural frequency computed in Problem 8.16 with $n = 6$.
- 9.3 Solve Problem 9.2 with Problem 8.17 replacing Problem 8.16.
- 9.4 Solve Problem 9.1 using quadratic interpolation functions. Discuss convergence characteristics compared with the solution obtained in Problem 9.1.
- 9.5 Solve Problem 9.4 with Problem 9.2 replacing Problem 9.1.
- 9.6 Solve Problem 9.4 with Problem 9.3 replacing Problem 9.1.
- 9.7 Use the approach of Sec. 9.3 to derive the cubic elements given by Eqs. (9.35).
- 9.8 Use Eqs. (9.17) to derive the element stiffness and mass matrices for a nonuniform string fixed at $x = 0$ and supported by a spring at $x = L$ using cubic interpolation functions. Then, indicate the structure of the global stiffness and mass matrices and give the matrices in explicit form.
- 9.9 Solve Problem 9.4 using cubic interpolation functions.
- 9.10 Solve Problem 9.5 using cubic interpolation functions.
- 9.11 Solve Problem 9.6 using cubic interpolation functions.
- 9.12 Solve Problem 8.4 using Hermite cubics and determine the order of the problem required to match the accuracy of the lowest natural frequency computed in Problem 8.4.
- 9.13 Solve Problem 9.12 with Problem 8.18 with $n = 6$ replacing Problem 8.4.
- 9.14 Solve Problem 9.12 with Problem 8.19 with $n = 6$ replacing Problem 8.4.
- 9.15 Solve Problem 7.39 with $a = 2b = 4h$ using sixteen triangular elements in conjunction with linear interpolation functions. Compare the three lowest natural frequencies with those obtained in Problem 7.39 and draw conclusions.
- 9.16 Solve Problem 9.15 with quadratic interpolation functions instead of linear.
- 9.17 Use the approach of Sec. 9.5 to derive the cubic interpolation functions given by Eqs. (9.79).
- 9.18 Determine the element stiffness and mass matrices for the cubic interpolation functions given by Eqs. (9.79).
- 9.19 Use the approach of Sec. 9.5 to derive the bilinear interpolation functions given by Eqs. (9.81).
- 9.20 Solve Problem 9.15 using eight rectangular elements in conjunction with bilinear interpolation functions. Compare results with those obtained in Problem 9.15 and draw conclusions.
- 9.21 Determine the element stiffness and mass matrices for the serendipity elements given by Eqs. (9.92).

- 9.22 Use Eqs. (9.108) and (9.112) to determine the element stiffness matrix and mass matrix, respectively, for the quadrilateral membrane of Fig. 9.29 using the bilinear interpolation functions given by Eq. (9.113).
- 9.23 Determine the global stiffness and mass matrices for the membrane of Problem 9.22.
- 9.24 Determine the element stiffness and mass matrices for triangular plate elements using the interpolation functions given by Eqs. (9.133).
- 9.25 Determine the global stiffness and mass matrices for the plate of Problem 8.23 using results from Problem 9.24.
- 9.26 Derive the response of the shaft of Problem 9.3 to the distributed torque $m(x, t) = m_0x(L - x)u(t)$, where $u(t)$ is the unit step function.
- 9.27 Solve Problem 9.26 with Problem 9.6 replacing Problem 9.3.
- 9.28 Derive the response of the cantilever beam of Problem 9.12 to the concentrated force $F(t) = F_0[u(t) - u(t - T)]$ applied at $x = L$, where $u(t)$ is the unit step function.
- 9.29 Derive the response of the beam of Problem 9.14 to the distributed force $f(x, t) = f_0(1 - x/2L)\delta(t)$, where $\delta(t)$ is the unit impulse.
- 9.30 Derive the response of the membrane of Problem 9.15 to the force per unit area $f(x, y, t) = f_0y(b - y)u(t)$, where $u(t)$ is the unit step function.
- 9.31 Solve Problem 9.30 with Problem 9.16 replacing Problem 9.15.
- 9.32 Derive the response of the plate of Problem 9.25 to the force $f(x, y, t) = f_0[r(t) - r(t - T)]$ distributed uniformly over the rectangular area defined by $a/2 < x < 3a/4$, $b/4 < y < 3b/4$, where $r(t)$ is the unit ramp function (Sec. 1.7).

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A

ELEMENTS OF LAPLACE TRANSFORMATION

The solution of difficult mathematical problems can often be simplified greatly by means of a suitable transformation. The process consists of transforming a difficult problem into a simpler problem, solving the simpler problem and inverse transforming the solution to the simpler problem to obtain the solution to the original problem. Transformations are widely used in vibrations. In particular, the Laplace transformation can be used to obtain efficient solutions to ordinary differential equations of motion for linear time-invariant systems, also known as systems with constant coefficients.

A.1 INTEGRAL TRANSFORMATIONS

We consider a function $f(t)$ defined by an ordinary differential equation and certain initial conditions and propose to obtain a solution by transforming the problem for $f(t)$ into a problem for $F(s)$ given by the *integral transformation*

$$F(s) = \int_a^b f(t)K(s, t) dt \quad (\text{A.1})$$

where $K(s, t)$ is a given function of s and t called the *kernel* of the transformation. When the limits a and b are finite, $F(s)$ is a finite transformation. Such an integral transformation converts the differential equation into an algebraic equation in terms of the transformed function $F(s)$, where s is a parameter. In the process, initial conditions are accounted for automatically. The algebraic equation for $F(s)$ can in general be solved without much difficulty, and the function $f(t)$ is obtained by inverse transforming $F(s)$.

Integral transform methods can also be used to solve boundary-value problems defined by partial differential equations. In such a case one transformation reduces the number of independent variables by one. Hence, if two independent variables are involved, instead of solving a partial differential equation one must solve an ordinary differential equation, which is in general a considerably easier task.

The main difficulty in using integral transform methods lies in the inverse transformation, which for the most part involves evaluation of an integral. The transform and its inverse constitute a *transform pair*, and for many transformations in use there are tables of transform pairs available. More often than not, it is possible to find the inverse transformation in transform tables, thus eliminating the need to evaluate an inversion integral. One of the most widely used integral transformations is the Laplace transformation.

A.2 THE LAPLACE TRANSFORMATION

The Laplace transformation method provides a very convenient means of solving linear ordinary differential equations for linear time-invariant systems, or systems with constant coefficients. Such equations arise frequently in the study of vibrations of discrete linear systems. This type of problem was treated by Heaviside by means of his operational calculus. Much of Heaviside's work was based on intuition and the mathematical treatment was often obscure. The Laplace transformation method, although similar to Heaviside's operational calculus, is mathematically rigorous. The method accounts automatically for initial conditions and provides a great deal of insight into the physics of the system.

We consider a function $f(t)$ given for all values of time larger than zero, $t \geq 0$, and define the *unilateral Laplace transformation* of $f(t)$ as

$$\mathcal{L}f(t) = F(s) = \int_0^{\infty} e^{-st} f(t) dt \quad (\text{A.2})$$

We note that the kernel of the transformation is $K(s, t) = e^{-st}$, where s is a *subsidiary variable*, which is in general a complex quantity. The complex plane defined by s is referred to as the *s-plane*, or the *Laplace plane*.

The function $f(t)$ is subject to certain restrictions, because the integral, Eq. (A.2), must converge. If $f(t)$ is such that

$$|e^{-st} f(t)| < Ce^{-(s-a)t}, \quad \text{Re } s > a \quad (\text{A.3})$$

where C is a constant and $\text{Re } s$ denotes the real part of s , then the Laplace transformation of $f(t)$ exists. Condition (A.3) implies that $f(t)$ does not increase more rapidly than Ce^{at} with increasing t . Such a function $f(t)$ is said to be of *exponential order* and is denoted $f(t) = \mathcal{O}(e^{at})$. Another condition for the existence of the Laplace transformation is that $f(t)$ be piecewise continuous, which means that in a given interval it has a finite number of finite discontinuities and no infinite discontinuity. Most functions describing physical phenomena satisfy these conditions.

A.3 TRANSFORMATION OF DERIVATIVES

Our interest lies in solving ordinary differential equations by means of the Laplace transformation method, which requires the transformation of derivatives of $f(t)$. To this end, we assume that the Laplace transformation of $f(t)$ exists and consider first the transform of $df(t)/dt$. Integrating by parts, we can write

$$\begin{aligned} \mathcal{L} \frac{df(t)}{dt} &= \int_0^\infty e^{-st} \frac{df(t)}{dt} dt \\ &= e^{-st} f(t) \Big|_0^\infty - \int_0^\infty (-se^{-st}) f(t) dt = -f(0) + sF(s) \end{aligned} \quad (\text{A.4})$$

In general, for the n th derivative of $f(t)$, we obtain

$$\mathcal{L} \frac{d^n f(t)}{dt^n} = \mathcal{L} f^{(n)}(t) = -f^{(n-1)}(0) - sf^{(n-2)}(0) - \dots - s^{n-1} f(0) + s^n F(s) \quad (\text{A.5})$$

where we adopted the notation

$$\left. \frac{d^{(n-j)} f(t)}{dt^{n-j}} \right|_{t=0} = f^{(n-j)}(0) \quad (\text{A.6})$$

Equation (A.5) is valid only if $f(t)$ satisfies the conditions prescribed in Sec. A.3 and all its derivatives through the $(n - 1)$ st are continuous.

A.4 TRANSFORMATION OF ORDINARY DIFFERENTIAL EQUATIONS

The differential equation of motion of a mass-damper-spring system is (Sec. 2.3)

$$m \frac{d^2 x(t)}{dt^2} + c \frac{dx(t)}{dt} + kx(t) = f(t) \quad (\text{A.7})$$

where m , c and k are the mass, coefficient of viscous damping and spring constant, respectively, $x(t)$ is the displacement of m and $F(t)$ the force acting on m . Transforming both sides of Eq. (A.7) and using Eq. (A.5), we can write

$$m[s^2 X(s) - sx(0) - \dot{x}(0)] + c[sX(s) - x(0)] + kX(s) = F(s) \quad (\text{A.8})$$

where $x(0)$ and $\dot{x}(0)$ are the initial displacement and velocity, respectively. Solving Eq. (A.8) for $X(s)$, we obtain

$$X(s) = \frac{1}{ms^2 + cs + k} F(s) + \frac{ms + c}{ms^2 + cs + k} x(0) + \frac{m}{ms^2 + cs + k} \dot{x}(0) \quad (\text{A.9})$$

Equation (A.9) is called the *subsidiary equation* of the differential equation, Eq. (A.7). To obtain the response $x(t)$, we must evaluate the inverse transformation of $X(s)$. Note that the Laplace transformation method provides automatically for initial conditions.

A.5 THE INVERSE TRANSFORMATION

Equation (A.9) gives the transform $X(s)$ of $x(t)$, which is a function of s . To obtain the actual solution $x(t)$ of Eq. (A.7), we must carry out an inverse transformation, which is denoted symbolically as

$$\mathcal{L}^{-1}X(s) = x(t) \quad (\text{A.10})$$

meaning that the inverse transform of $X(s)$ is $x(t)$.

In general, the operation $\mathcal{L}^{-1}F(s)$ involves the evaluation of the integral

$$f(t) = \mathcal{L}^{-1}F(s) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} F(s) ds \quad (\text{A.11})$$

where the path of integration is a line parallel to the imaginary axis crossing the real axis at $\text{Re } s = \gamma$ and extending from $-\infty$ to $+\infty$. In many cases, however, we can carry out the inverse Laplace transformation without having to recourse to line integrals. This is the case when Jordan's lemma can be used to replace the line integral by a closed contour integral, which in turn can be evaluated by means of the residue theorem (Ref. 1, Sec. 9-15). In most cases, however, it is possible to obtain the inverse transformation by means of tables of Laplace transform pairs (see Sec. A.9). Quite often, to increase the usefulness of the tables, we can use the method of partial fractions to reduce seemingly complicated functions to a form listed in tables.

A.6 SHIFTING THEOREMS

We consider the function

$$f_1(t) = f(t)e^{at} \quad (\text{A.12})$$

where a is a real or complex number, and evaluate its Laplace transformation as follows:

$$\begin{aligned} F_1(s) &= \int_0^{\infty} [f(t)e^{at}]e^{-st} dt \\ &= \int_0^{\infty} f(t)e^{-(s-a)t} dt = F(s-a) \end{aligned} \quad (\text{A.13})$$

It follows that

$$\mathcal{L}f(t)e^{at} = F(s-a) \quad (\text{A.14})$$

Hence, the effect of multiplying $f(t)$ by e^{at} in the real domain is to shift the transform $F(s)$ of $f(t)$ by an amount a in the s -domain. Because the s -domain is a complex plane, this theorem is also called the *complex shifting theorem*.

Next, we consider the Laplace transformation

$$F(s) = \int_0^{\infty} e^{-s\lambda} f(\lambda) d\lambda \quad (\text{A.15})$$

and let $\lambda = t - a$, so that

$$F(s) = \int_0^\infty e^{-s(t-a)} f(t-a) dt = e^{as} \int_0^\infty e^{-st} [f(t-a)\mu(t-a)] dt \quad (\text{A.16})$$

where $\mu(t-a)$ is the unit step function initiated at $t = a$. Multiplying both sides of Eq. (A.16) by e^{-as} , we obtain

$$e^{-as} F(s) = \int_0^\infty e^{-st} [f(t-a)\mu(t-a)] dt \quad (\text{A.17})$$

Hence,

$$\mathcal{L}^{-1} e^{-as} F(s) = f(t-a)\mu(t-a) \quad (\text{A.18})$$

Equation (A.18) represents the *shifting theorem in the real domain*.

A.7 METHOD OF PARTIAL FRACTIONS

Under consideration is the inverse transformation of the function

$$F(s) = \frac{A(s)}{B(s)} \quad (\text{A.19})$$

where both $A(s)$ and $B(s)$ are polynomials in s , and we recall that Eq. (A.9) is of this type. In general, $B(s)$ is a polynomial of higher degree than $A(s)$. We assume that $B(s)$ is a polynomial of n th degree and write it in the factored form

$$B(s) = (s - a_1)(s - a_2) \dots (s - a_n) \quad (\text{A.20})$$

in which a_1, a_2, \dots, a_n are the roots. We consider first the case in which all n roots are distinct, in which case Eq. (A.19) can be expressed as

$$F(s) = \frac{A(s)}{B(s)} = \frac{C_1}{s - a_1} + \frac{C_2}{s - a_2} + \dots + \frac{C_n}{s - a_n} \quad (\text{A.21})$$

where the coefficients C_k are given by

$$C_k = \lim_{s \rightarrow a_k} [(s - a_k)F(s)] = \left. \frac{A(s)}{B'(s)} \right|_{s=a_k} \quad (\text{A.22})$$

in which B' is the derivative of B with respect to s . But, using the table of Laplace transforms in Sec. A.9 and considering the complex shifting theorem, we have

$$\mathcal{L}^{-1} \frac{1}{s - a_k} = e^{a_k t} \quad (\text{A.23})$$

so that the inverse transform of Eq. (A.21) is

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}F(s) = C_1e^{a_1t} + C_2e^{a_2t} + \dots + C_n e^{a_nt} \\ &= \sum_{k=1}^n C_k e^{a_k t} \\ &= \sum_{k=1}^n \lim_{s \rightarrow a_k} [(s - a_k)F(s)e^{st}] = \sum_{k=1}^n \frac{A(s)}{B'(s)} e^{st} \Big|_{s=a_k} \end{aligned} \quad (\text{A.24})$$

The roots a_1, a_2, \dots, a_n are called *simple poles* of $F(s)$. It should be noted that Eq. (A.24) can be used for functions other than ratios of two polynomials, provided the function has simple poles. Poles are points at which the function $F(s)$ becomes infinite.

Next, we consider the case in which $B(s)$ has a multiple root of order k , i.e., $F(s)$ has a pole of order k as opposed to poles of first order, or simple poles, examined previously. Hence, we consider

$$B(s) = (s - a_1)^k (s - a_2)(s - a_3) \dots (s - a_n) \quad (\text{A.25})$$

The partial-fractions expansion in this case is of the form

$$\begin{aligned} F(s) = \frac{A(s)}{B(s)} &= \frac{C_{11}}{(s - a_1)^k} + \frac{C_{12}}{(s - a_1)^{k-1}} + \dots + \frac{C_{1k}}{s - a_1} \\ &+ \frac{C_2}{s - a_2} + \frac{C_3}{s - a_3} + \dots + \frac{C_n}{s - a_n} \end{aligned} \quad (\text{A.26})$$

It is not difficult to show that the coefficients corresponding to the repeated root a_1 are

$$C_{1r} = \frac{1}{(r - 1)!} \frac{d^{r-1}}{ds^{r-1}} [(s - a_1)^k F(s)]_{s=a_1}, \quad r = 1, 2, \dots, k \quad (\text{A.27})$$

The simple poles of $F(s)$ are treated by means of (A.22), as previously. For the higher-order pole, we use the table of Laplace transforms in conjunction with the complex shifting theorem and write

$$\mathcal{L}^{-1} \frac{1}{(s - a_1)^r} = \frac{t^{r-1}}{(r - 1)!} e^{a_1 t} \quad (\text{A.28})$$

so that the inverse transform of Eq. (A.26) becomes

$$\begin{aligned} f(t) &= \left[C_{11} \frac{t^{k-1}}{(k - 1)!} + C_{12} \frac{t^{k-2}}{(k - 2)!} + \dots + C_{1k} \right] e^{a_1 t} \\ &+ C_2 e^{a_2 t} + C_3 e^{a_3 t} + \dots + C_n e^{a_n t} \end{aligned} \quad (\text{A.29})$$

Equation (A.29) can be shown to be equal to

$$f(t) = \frac{1}{(k - 1)!} \frac{d^{k-1}}{ds^{k-1}} [(s - a_1)^k F(s)e^{st}]_{s=a_1} + \sum_{i=2}^n [(s - a_i)F(s)e^{st}]_{s=a_i} \quad (\text{A.30})$$

A.8 THE CONVOLUTION INTEGRAL

We consider two functions $f_1(t)$ and $f_2(t)$, both defined for $t \geq 0$, and assume that $f_1(t)$ and $f_2(t)$ possess the Laplace transforms $F_1(s)$ and $F_2(s)$, respectively. Then, we consider the integral

$$x(t) = \int_0^t f_1(\tau) f_2(t - \tau) d\tau = \int_0^\infty f_1(\tau) f_2(t - \tau) d\tau \quad (\text{A.31})$$

The function $x(t)$ given by these integrals, sometimes denoted by $x(t) = f_1(t) * f_2(t)$, is called the *convolution* of the functions f_1 and f_2 over the interval $(0, \infty)$. The validity of the change in the upper limit of the first integral can be explained by the fact $f_2(t - \tau) = 0$ for $\tau > t$, or $t - \tau < 0$. Transforming both sides of Eq. (A.31), we obtain

$$\begin{aligned} X(s) &= \int_0^\infty e^{-st} \left[\int_0^\infty f_1(\tau) f_2(t - \tau) d\tau \right] dt \\ &= \int_0^\infty f_1(\tau) d\tau \int_0^\infty e^{-st} f_2(t - \tau) dt \\ &= \int_0^\infty f_1(\tau) d\tau \int_\tau^\infty e^{-st} f_2(t - \tau) dt \end{aligned} \quad (\text{A.32})$$

where the limit in the second integral was changed because $f_2(t - \tau) = 0$ for $t < \tau$.

Next, we let $t - \tau = \lambda$ in the second integral and note that $\lambda = 0$ for $t = \tau$, so that

$$\begin{aligned} X(s) &= \int_0^\infty f_1(\tau) d\tau \int_\tau^\infty e^{-st} f_2(t - \tau) dt \\ &= \int_0^\infty f_1(\tau) d\tau \int_0^\infty e^{-s(\tau+\lambda)} f_2(\lambda) d\lambda \\ &= \int_0^\infty e^{-s\tau} f_1(\tau) d\tau \int_0^\infty e^{-s\lambda} f_2(\lambda) d\lambda = F_1(s) F_2(s) \end{aligned} \quad (\text{A.33})$$

From Eqs. (A.31) and (A.33), it follows that

$$\begin{aligned} x(t) &= \mathcal{L}^{-1} X(s) \\ &= \mathcal{L}^{-1} F_1(s) F_2(s) \\ &= \int_0^t f_1(\tau) f_2(t - \tau) d\tau = \int_0^t f_1(t - \tau) f_2(\tau) d\tau. \end{aligned} \quad (\text{A.34})$$

This is true because it does not matter which of the functions $f_1(t)$ and $f_2(t)$ is shifted. Equation (A.34) can be stated in words as the following theorem: *The inverse transformation of the product of two transforms is equal to the convolution of their inverse transforms.* The integrals in (A.34) are called *convolution integrals*. A special case of Eq. (A.34) was encountered in Sec. 1.7 without any reference to Laplace transformations.

A.9 TABLE OF LAPLACE TRANSFORM PAIRS

$f(t)$	$F(s)$
$\delta(t)$ (Dirac delta function)	1
$u(t)$ (unit step function)	$\frac{1}{s}$
$t^n \quad n = 1, 2, \dots$	$\frac{n!}{s^{n+1}}$
$e^{-\omega t}$	$\frac{1}{s + \omega}$
$t e^{-\omega t}$	$\frac{1}{(s + \omega)^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
$1 - e^{-\omega t}$	$\frac{\omega}{s(s + \omega)}$
$1 - \cos \omega t$	$\frac{\omega^2}{s(s^2 + \omega^2)}$
$\omega t - \sin \omega t$	$\frac{\omega^3}{s^2(s^2 + \omega^2)}$
$\omega t \cos \omega t$	$\frac{\omega(s^2 - \omega^2)}{(s^2 + \omega^2)^2}$
$\omega t \sin \omega t$	$\frac{2\omega^2 s}{(s^2 + \omega^2)^2}$
$\frac{1}{(1 - \zeta^2)^{1/2} \omega} e^{-\zeta \omega t} \sin(1 - \zeta^2)^{1/2} \omega t$	$\frac{1}{s^2 + 2\zeta \omega s + \omega^2}$
$e^{-\zeta \omega t} \left[\cos(1 - \zeta^2)^{1/2} \omega t + \frac{\zeta}{(1 - \zeta^2)^{1/2}} \sin(1 - \zeta^2)^{1/2} \omega t \right]$	$\frac{s + 2\zeta \omega}{s^2 + 2\zeta \omega s + \omega^2}$

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B

ELEMENTS OF LINEAR ALGEBRA

The vibration of linear discrete systems is governed by sets of simultaneous linear ordinary differential equations. The solution of such sets of equations can be obtained most conveniently by means of a linear transformation rendering the set of equations independent. To find this linear transformation, it is necessary to solve a set of homogeneous algebraic equations containing a certain parameter. The problem of determining the values of the parameter such that the set of equations admits nontrivial solutions is known as the algebraic eigenvalue problem, a very important problem in linear algebra. This appendix is devoted to concepts from linear algebra of interest to the study of vibrations, such as vector spaces and matrices.

B.1 LINEAR VECTOR SPACES

In discussing vector spaces, it proves convenient to introduce the concept of a field. A *field* is defined as a set of scalars possessing certain algebraic properties. The real numbers constitute a field, and so do the complex numbers.

We consider a set of elements F such that for any two elements α and β in F it is possible to define another two unique elements belonging to F , the first denoted by $\alpha + \beta$ and called the *sum* of α and β , and the second denoted by $\alpha\beta$ and called the *product* of α and β . The set F is called a *field* if these two operations satisfy the five field postulates:

1. *Commutative laws.* For all α and β in F ,
(i) $\alpha + \beta = \beta + \alpha$, (ii) $\alpha\beta = \beta\alpha$.
2. *Associative laws.* For all α , β and γ in F ,
(i) $(\alpha + \beta) + \gamma = \alpha(\beta + \gamma)$, (ii) $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.
3. *Distributive laws.* For all α , β and γ in F ,
 $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.
4. *Identity elements.* There exist in F elements 0 and 1 called the zero and the unity elements, respectively, such that $0 \neq 1$, and for all α in F ,
(i) $\alpha + 0 = \alpha$, (ii) $1\alpha = \alpha$.
5. *Inverse elements.*
 - i. For every element α in F there exists a unique element $-\alpha$, called the additive inverse of α , such that $\alpha + (-\alpha) = 0$.

- ii. For element $\alpha \neq 0$ in F there exists a unique element α^{-1} , called the multiplicative inverse of α , such that $\alpha\alpha^{-1} = 1$.

Next, we define the concept of *linear vector space*, also referred to as *linear space* and *vector space*. Let L be a set of elements called *vectors* and F a field of *scalars*. Then, if L and F are such that two operations, namely, *vector addition* and *scalar multiplication*, are defined for L and F , the set of vectors together with the two operations are called a *linear vector space L over a field F* . For every two elements \mathbf{x} and \mathbf{y} in L , it satisfies the postulates:

1. *Commutativity.* $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
2. *Associativity.* $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$.
3. There exists a unique vector $\mathbf{0}$ in L such that $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$.
4. For every vector \mathbf{x} in L there exists a unique vector $-\mathbf{x}$ such that $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$.

Hence, the rules of vector addition are similar to those of ordinary algebra. Moreover, for any vector \mathbf{x} in L and any scalar α in F , there is a unique *scalar product* $\alpha\mathbf{x}$ which is also an element of L . The scalar multiplication must be such that, for all α and β in F and all \mathbf{x} and \mathbf{y} in L , it satisfies the postulates:

5. *Associativity.* $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$.
6. *Distributivity.* (i) $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$, (ii) $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$.
7. $1\mathbf{x} = \mathbf{x}$, where 1 is the unit scalar, and $0\mathbf{x} = \mathbf{0}$.

We have considerable interest in a vector space L possessing n elements of the field F , i.e., in a vector space of n -tuples. We write any two such vectors in L as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad (\text{B.1})$$

and refer to them as *n-vectors*. The set of all n -vectors is called the *vector space L^n* . Then, the addition of these two vectors is defined as

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \dots \\ x_n + y_n \end{bmatrix} \quad (\text{B.2})$$

Moreover, if α is a scalar in F , then the product of a scalar and a vector is defined as

$$\alpha\mathbf{x} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \dots \\ \alpha x_n \end{bmatrix} \quad (\text{B.3})$$

Let S be a subset of the vector space L . Then, S is a *subspace* of L if the following statements are true:

1. If \mathbf{x} and \mathbf{y} are in S , then $\mathbf{x} + \mathbf{y}$ is in S .
2. If \mathbf{x} is in S and α is in F , then $\alpha\mathbf{x}$ is in S .

B.2 LINEAR DEPENDENCE

We consider a set of vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ in a linear space L and a set of scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ in F . Then, the vector \mathbf{x} given by

$$\mathbf{x} = \alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \dots + \alpha_n\mathbf{x}_n \tag{B.4}$$

is said to be a *linear combination* of $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ with coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$. The vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are said to be *linearly independent* if the relation

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \dots + \alpha_n\mathbf{x}_n = \mathbf{0} \tag{B.5}$$

can be satisfied only for the trivial case, i.e., only when all the coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$ are identically zero. If relation (B.5) is satisfied and at least one of the coefficients $\alpha_1, \alpha_2, \dots, \alpha_n$ is different from zero, then the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are said to be *linearly dependent*, with the implication that one vector is a linear combination of the remaining $n - 1$ vectors.

The subspace S of L consisting of all the linear combinations of the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ is called a subspace *spanned* by the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. If $S = L$, then $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are said to *span* L .

Example B.1

Consider the two independent vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \tag{a}$$

in a three-dimensional space. The set of all linear combinations of \mathbf{x}_1 and \mathbf{x}_2 span a plane passing through the origin and the tips of \mathbf{x}_1 and \mathbf{x}_2 . The three vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 5 \\ 0 \\ 5 \end{bmatrix} \tag{b}$$

span the same plane, because \mathbf{x}_3 lies in the plane spanned by \mathbf{x}_1 and \mathbf{x}_2 . Hence, the three vectors are linearly dependent. Indeed, it can be easily verified that

$$\mathbf{x}_1 + 2\mathbf{x}_2 - \mathbf{x}_3 = \mathbf{0} \tag{c}$$

so that \mathbf{x}_3 is really a linear combination of \mathbf{x}_1 and \mathbf{x}_2 (see Fig. B.1).

On the other hand, the three vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_4 = \begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix} \tag{d}$$

are linearly independent because

$$\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2 + \alpha_4\mathbf{x}_4 \neq \mathbf{0} \tag{e}$$

for all cases other than the trivial one. The three vectors $\mathbf{x}_1, \mathbf{x}_2$ and \mathbf{x}_4 span a three-dimensional space.

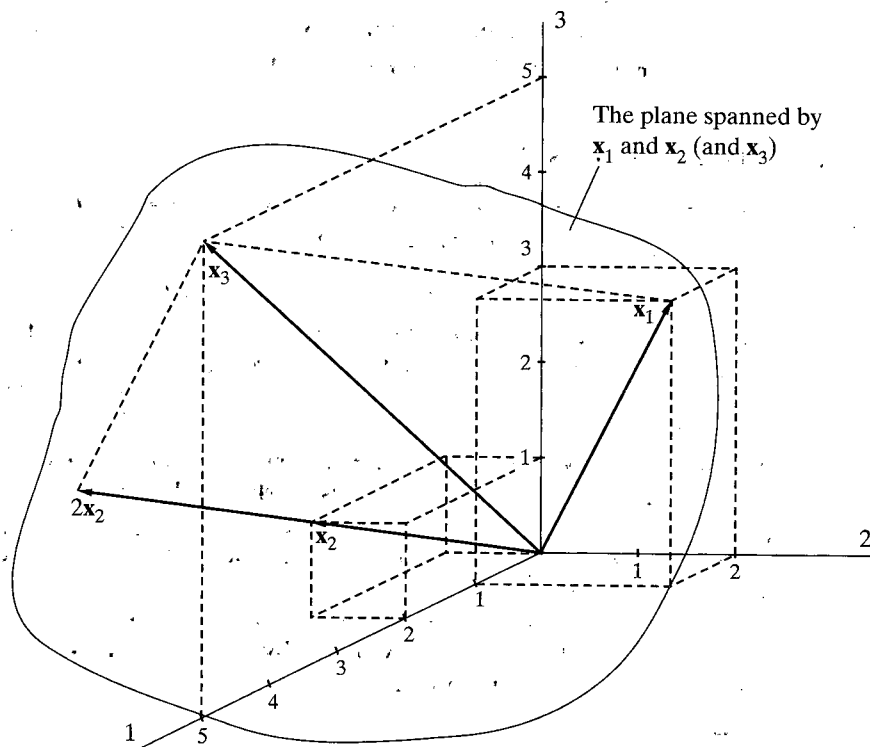


Figure B.1 Linearly dependent vectors

B.3 BASES AND DIMENSION OF A VECTOR SPACE

A vector space L over F is said to be finite-dimensional if there exists a finite set of vectors x_1, x_2, \dots, x_n that span L , i.e., such that every vector in L is a linear combination of x_1, x_2, \dots, x_n .

Let L be a vector space over F . A set of vectors x_1, x_2, \dots, x_n that span L is called a *generating system* of L . If x_1, x_2, \dots, x_n are linearly independent and span L , then the generating system is called a *basis* for L . If L is a finite-dimensional vector space, any two bases for L contain the same number of vectors. The basis can be regarded as the generalization of the concept of a coordinate system.

Let L be a finite-dimensional vector space over F . The *dimension* of L is defined as the number of vectors in any basis for L . This integer is denoted by $\dim L$. The vector space L^n is spanned by n linearly independent vectors, so that $\dim L^n = n$.

Consider an arbitrary n -vector x in L^n with components x_1, x_2, \dots, x_n and introduce a set of n -vectors given by

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \quad e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad (\text{B.6})$$

The vector \mathbf{x} can be written in terms of the vectors \mathbf{e}_i ($i = 1, 2, \dots, n$) as the linear combination

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \dots + x_n\mathbf{e}_n = \sum_{i=1}^n x_i\mathbf{e}_i \quad (\text{B.7})$$

It follows that L^n is spanned by the set of vectors \mathbf{e}_i ($i = 1, 2, \dots, n$), so that the vectors \mathbf{e}_i constitute a generating system of L^n . The set of vectors \mathbf{e}_i can be verified as being linearly independent and they are generally referred to as the *standard basis* for L^n .

Example B.2

The vectors \mathbf{x}_1 , \mathbf{x}_2 and \mathbf{x}_4 of Example B.1 form a basis for a three-dimensional vector space. Any vector \mathbf{x} in L^3 can be written as a unique linear combination of \mathbf{x}_1 , \mathbf{x}_2 and \mathbf{x}_4 . For example, it can be verified that the vector

$$\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \quad (\text{a})$$

can be represented in the form

$$\mathbf{x} = 2\mathbf{x}_1 + 3\mathbf{x}_2 - \mathbf{x}_4 \quad (\text{b})$$

The same vector \mathbf{x} can be also represented in the terms of the standard basis \mathbf{e}_1 , \mathbf{e}_2 , \mathbf{e}_3 for L^3 . Indeed, it is easy to see that

$$\mathbf{x} = 3\mathbf{e}_1 + 0\mathbf{e}_2 + 4\mathbf{e}_3 \quad (\text{c})$$

B.4 INNER PRODUCTS AND ORTHOGONAL VECTORS

Various concepts encountered in two- and three-dimensional spaces, such as the length of a vector and orthogonality, can be generalized to n -dimensional spaces. This requires the introduction of additional definitions.

Let L^n be an n -dimensional vector space defined over the field F of scalars. If to each pair of vectors \mathbf{x} and \mathbf{y} in L^n is assigned a unique scalar in F , called the *inner product* of \mathbf{x} and \mathbf{y} , then L^n is said to be an *inner product space*. The vectors \mathbf{x} and \mathbf{y} can be complex, in which case $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ denote their complex conjugates. The inner product is denoted by (\mathbf{x}, \mathbf{y}) and must satisfy the following postulates:

1. $(\mathbf{x}, \mathbf{x}) \geq 0$ for all \mathbf{x} in L^n and $(\mathbf{x}, \mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$.
2. $(\mathbf{x}, \mathbf{y}) = \overline{(\mathbf{y}, \mathbf{x})}$.
3. $(\lambda\mathbf{x}, \mathbf{y}) = \lambda(\mathbf{x}, \mathbf{y})$ and $(\mathbf{x}, \lambda\mathbf{y}) = \bar{\lambda}(\mathbf{x}, \mathbf{y})$ for all λ in F .
4. $(\mathbf{x}, \mathbf{y} + \mathbf{z}) = (\mathbf{x}, \mathbf{y}) + (\mathbf{x}, \mathbf{z})$ for all \mathbf{x}, \mathbf{y} and \mathbf{z} in L^n .

The most common definition of the *complex inner product* is

$$(\mathbf{x}, \mathbf{y}) = x_1\bar{y}_1 + x_2\bar{y}_2 + \dots + x_n\bar{y}_n \quad (\text{B.8})$$

which represents a complex number. When \mathbf{x} and \mathbf{y} are real vectors, Eq. (B.8) reduces to

$$(\mathbf{x}, \mathbf{y}) = x_1y_1 + x_2y_2 + \dots + x_ny_n \quad (\text{B.9})$$

which defines the *real inner product*, a real number. A finite-dimensional inner product space defined over the real scalar field is called a *Euclidean space*.

It is often desirable to have a measure of the size of a vector. Such a measure is called the *norm*. It is designated by the symbol $\|\mathbf{x}\|$ and is required to possess the following properties:

1. $\|\mathbf{x}\| \geq 0$ and $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$
2. $\|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|$ for any scalar λ
3. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$

where property 3 is known as the triangle inequality. Note that $|\lambda|$ denotes the absolute value, or modulus of λ .

A commonly used norm is the *quadratic norm*

$$\|\mathbf{x}\| = (\mathbf{x}, \mathbf{x})^{1/2} \quad (\text{B.10})$$

which defines the *length* of the vector \mathbf{x} . In the case of real vector spaces, Eq. (B.10) reduces to

$$\|\mathbf{x}\| = \left(\sum_{i=1}^n x_i^2 \right)^{1/2} \quad (\text{B.11})$$

which defines the *Euclidean norm*. Equation (B.11) can be recognized as the extension to n dimensions of the ordinary concept of length of a vector in two and three dimensions.

A vector whose norm is equal to unity, $\|\mathbf{x}\| = (\mathbf{x}, \mathbf{x})^{1/2} = 1$, is called a *unit vector*. Any nonzero vector can be *normalized* so as to form a unit vector by simply dividing the vector by its norm

$$\hat{\mathbf{x}} = \frac{\mathbf{x}}{\|\mathbf{x}\|} \quad (\text{B.12})$$

It is easy to verify that the vectors \mathbf{e}_i defined by Eqs. (B.6) are unit vectors.

When the vectors \mathbf{x} and \mathbf{y} are real, the inner product is sometimes referred to as the *dot product*. We recall from ordinary vector analysis that the dot product of two vectors in the two- and three-dimensional space can be used to define the cosine of the angle between the two vectors. This concept can be generalized to the n -dimensional space by writing

$$\cos \theta = \frac{(\mathbf{x}, \mathbf{y})}{\|\mathbf{x}\|\|\mathbf{y}\|} = (\hat{\mathbf{x}}, \hat{\mathbf{y}}) \quad (\text{B.13})$$

Any two vectors \mathbf{x} and \mathbf{y} in L^n are said to be *orthogonal* if and only if

$$(\mathbf{x}, \mathbf{y}) = 0 \quad (\text{B.14})$$

which represents a generalization of the ordinary concept of perpendicularity. If each pair of vectors in a given set are mutually orthogonal, then the set is said to be an *orthogonal set*. If, in addition, the vectors have unit norms, the vectors are said to be *orthonormal*. Any set of mutually orthogonal nonzero vectors in L^n is

linearly independent. To show this, we assume that the orthogonal set of vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ satisfies a relation of the type (B.5) and form the inner products

$$\begin{aligned} 0 &= (\mathbf{x}_i, \mathbf{0}) = (\mathbf{x}_i, \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 + \dots + \alpha_n \mathbf{x}_n) \\ &= \alpha_1 (\mathbf{x}_i, \mathbf{x}_1) + \alpha_2 (\mathbf{x}_i, \mathbf{x}_2) + \dots + \alpha_n (\mathbf{x}_i, \mathbf{x}_n) \\ &= \alpha_i (\mathbf{x}_i, \mathbf{x}_i), \quad i = 1, 2, \dots, n \end{aligned} \tag{B.15}$$

Because $(\mathbf{x}_i, \mathbf{x}_i) \neq 0$, it follows that Eqs. (B.15) can be satisfied if and only if all the coefficients α_i are identically zero, so that the set of vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ must be linearly independent. Owing to the independence property, orthogonal vectors, and in particular orthonormal vectors, are convenient choices for basis vectors. A classical example of an orthonormal set of vectors used as a basis are the unit vectors \mathbf{e}_i , which explains why these vectors are referred to as a standard basis for L^n .

B.5 THE GRAM-SCHMIDT ORTHOGONALIZATION PROCESS

Orthogonal vectors are by definition independent, but independent vectors are not necessarily orthogonal. A set of independent vectors, however, can be rendered orthogonal. In computational work, it is often desirable to work with a set of orthogonal vectors, so that the procedure for rendering independent vectors orthogonal is of special interest. The procedure is known as the *Gram-Schmidt orthogonalization process*.

We consider the set of independent vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ and denote the desired orthogonal vectors by $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$. These latter vectors can be normalized by dividing each of the vectors by its norm, so that the orthonormal vectors $\hat{\mathbf{y}}_1, \hat{\mathbf{y}}_2, \dots, \hat{\mathbf{y}}_n$ are given by

$$\hat{\mathbf{y}}_i = \mathbf{y}_i / \|\mathbf{y}_i\|, \quad i = 1, 2, \dots, n \tag{B.16}$$

The first vector of the desired orthonormal set is simply

$$\hat{\mathbf{y}}_1 = \hat{\mathbf{x}}_1 = \mathbf{x}_1 / \|\mathbf{x}_1\| \tag{B.17}$$

The second vector, \mathbf{y}_2 , must be orthogonal to $\hat{\mathbf{y}}_1$. A vector \mathbf{y}_2 satisfying this condition can be taken in the form

$$\mathbf{y}_2 = \mathbf{x}_2 - (\mathbf{x}_2, \hat{\mathbf{y}}_1) \hat{\mathbf{y}}_1 \tag{B.18}$$

Indeed, we have

$$(\mathbf{y}_2, \hat{\mathbf{y}}_1) = (\mathbf{x}_2, \hat{\mathbf{y}}_1) - (\mathbf{x}_2, \hat{\mathbf{y}}_1) = 0 \tag{B.19}$$

Of course, the vector \mathbf{y}_2 can be normalized by using the second of Eqs. (B.16) to obtain $\hat{\mathbf{y}}_2$. The third vector, \mathbf{y}_3 , can be written in the form

$$\mathbf{y}_3 = \mathbf{x}_3 - (\mathbf{x}_3, \hat{\mathbf{y}}_1) \hat{\mathbf{y}}_1 - (\mathbf{x}_3, \hat{\mathbf{y}}_2) \hat{\mathbf{y}}_2 \tag{B.20}$$

which is orthonormal to $\hat{\mathbf{y}}_1$ and $\hat{\mathbf{y}}_2$, as it satisfies

$$\begin{aligned} (\mathbf{y}_3, \hat{\mathbf{y}}_1) &= (\mathbf{x}_3, \hat{\mathbf{y}}_1) - (\mathbf{x}_3, \hat{\mathbf{y}}_1) = 0 \\ (\mathbf{y}_3, \hat{\mathbf{y}}_2) &= (\mathbf{x}_3, \hat{\mathbf{y}}_2) - (\mathbf{x}_3, \hat{\mathbf{y}}_2) = 0 \end{aligned} \tag{B.21}$$

The vector y_3 can be normalized to obtain \hat{y}_3 . Generalizing, we can write

$$y_i = x_i - \sum_{j=1}^{i-1} (x_i, \hat{y}_j) \hat{y}_j, \quad i = 1, 2, \dots, n \quad (\text{B.22})$$

which can be used to compute \hat{y}_i . Clearly, \hat{y}_i is orthonormal to $\hat{y}_1, \hat{y}_2, \dots, \hat{y}_{i-1}$ ($i = 1, 2, \dots, n$). The process is concluded with the computation of \hat{y}_n .

The Gram-Schmidt process described above can often yield computed vectors that are far from being orthogonal (Ref. 3, p. 148). An orthogonalization process mathematically equivalent but computationally superior to the Gram-Schmidt process is the *modified Gram-Schmidt process*. In the ordinary Gram-Schmidt process, an orthonormal basis \hat{y}_i ($i = 1, 2, \dots, n$) is computed in successive steps without altering the original vectors x_i ($i = 1, 2, \dots, n$). In the modified Gram-Schmidt process, however, upon computing y_i the vectors $x_{i+1}, x_{i+2}, \dots, x_n$ are also changed by insisting that they be orthogonal to \hat{y}_i , as well as to $\hat{y}_1, \hat{y}_2, \dots, \hat{y}_{i-1}$. The first step is as given by Eq. (B.17), but in addition the vectors x_i ($i = 2, 3, \dots, n$) are modified by writing

$$x_i^{(1)} = x_i - (\hat{y}_1, x_i) \hat{y}_1, \quad i = 2, 3, \dots, n \quad (\text{B.23})$$

Forming the inner product $(\hat{y}_1, x_i^{(1)})$, we conclude that $x_i^{(1)}$ ($i = 2, 3, \dots, n$) are all orthogonal to \hat{y}_1 . The next step consists of normalizing $x_2^{(1)}$ to produce \hat{y}_2 as well as of modifying $x_i^{(1)}$ ($i = 3, 4, \dots, n$) by writing

$$x_i^{(2)} = x_i^{(1)} - (\hat{y}_2, x_i^{(1)}) \hat{y}_2, \quad i = 3, 4, \dots, n \quad (\text{B.24})$$

It is not difficult to verify that $x_i^{(2)}$ ($i = 3, 4, \dots, n$) are all orthogonal to both \hat{y}_1 and \hat{y}_2 . Of course, \hat{y}_3 is obtained by normalizing $x_3^{(2)}$. Generalizing, the j th step consists of computing $x_j^{(j-1)}$ and normalizing it to produce \hat{y}_j , or

$$\hat{y}_j = x_j^{(j-1)} / \|x_j^{(j-1)}\| \quad (\text{B.25})$$

and then computing the modified vectors

$$x_i^{(j)} = x_i^{(j-1)} - (\hat{y}_j, x_i^{(j-1)}) \hat{y}_j, \quad i = j + 1, j + 2, \dots, n \quad (\text{B.26})$$

which renders $x_i^{(j)}$ orthogonal to \hat{y}_j as well as to $\hat{y}_1, \hat{y}_2, \dots, \hat{y}_{j-1}$, or

$$(\hat{y}_k, x_i^{(j)}) = 0, \quad k = 1, 2, \dots, j \quad (\text{B.27})$$

The process is completed with the computation and normalization of $x_n^{(n-1)}$, yielding \hat{y}_n .

When the vectors x_i ($i = 1, 2, \dots, n$) are independent, the ordinary and the modified Gram-Schmidt processes yield the same results. When the vectors x_i are nearly dependent, however, the ordinary Gram-Schmidt process fails to yield orthonormal vectors, but the modified Gram-Schmidt process does yield vectors that are nearly orthonormal (Ref. 3, p. 149).

Example B.3

Consider the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad (\text{a})$$

and obtain an orthonormal basis in terms of these vectors by the modified Gram-Schmidt process. Use Euclidean norms for the vectors.

From Eq. (B.17), we obtain the first normalized vector

$$\hat{\mathbf{y}}_1 = \hat{\mathbf{x}}_1 = \mathbf{x}_1 / \|\mathbf{x}_1\| = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (\text{b})$$

so that, from Eqs. (B.23), we can write

$$\begin{aligned} \mathbf{x}_2^{(1)} &= \mathbf{x}_2 - (\hat{\mathbf{y}}_1, \mathbf{x}_2)\hat{\mathbf{y}}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right) \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \\ \mathbf{x}_3^{(1)} &= \mathbf{x}_3 - (\hat{\mathbf{y}}_1, \mathbf{x}_3)\hat{\mathbf{y}}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \end{aligned} \quad (\text{c})$$

The second vector of the orthonormal set is obtained by simply normalizing $\mathbf{x}_2^{(1)}$, or

$$\hat{\mathbf{y}}_2 = \mathbf{x}_2^{(1)} / \|\mathbf{x}_2^{(1)}\| = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad (\text{d})$$

Finally, from Eqs. (B.24), we can write

$$\begin{aligned} \mathbf{x}_3^{(2)} &= \mathbf{x}_3^{(1)} - (\hat{\mathbf{y}}_2, \mathbf{x}_3^{(1)})\hat{\mathbf{y}}_2 \\ &= \frac{2}{3} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} - \left(\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \frac{2}{3} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \end{aligned} \quad (\text{e})$$

so that

$$\hat{\mathbf{y}}_3 = \mathbf{x}_3^{(2)} / \|\mathbf{x}_3^{(2)}\| = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad (\text{f})$$

It can be verified easily that the vectors $\hat{\mathbf{y}}_1$, $\hat{\mathbf{y}}_2$ and $\hat{\mathbf{y}}_3$ are orthonormal.

B.6 MATRICES

A *matrix* is a rectangular array of scalars of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad (\text{B.28})$$

The scalars a_{ij} ($i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$), called the *elements of A*, belong to a given field F . The field is assumed to be either the real field R or the complex

field C . Because the matrix A has m rows and n columns it is referred to as an $m \times n$ matrix. It is customary to say that the *dimensions* of A are $m \times n$. The position of the element a_{ij} in the matrix A is in the i th row and j th column, so that i is referred to as the row index and j as the column index.

If $m = n$, the matrix A reduces to a *square matrix of order n* . The elements a_{ii} in the square matrix are called the *main diagonal elements* of A . The remaining elements are referred to as the *off-diagonal elements* of A . In the special case in which all the off-diagonal elements of A are zero, A is said to be a *diagonal matrix*. If A is diagonal and if all its diagonal elements are unity, $a_{ii} = 1$, then the matrix is called *unit matrix*, or *identity matrix*, and denoted by I . Introducing the Kronecker delta symbol δ_{ij} defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad (\text{B.29})$$

the identity matrix can be regarded as a matrix with every element equal to the Kronecker delta and can be written in the form $I = [\delta_{ij}]$. Similarly, a diagonal matrix D can be written in terms of the Kronecker delta in the form $D = [a_{ij}\delta_{ij}]$.

A square matrix A is said to be *upper (lower) triangular* if $a_{ij} = 0$ for $i > j$ ($i < j$). If the diagonal elements of an upper (lower) triangular matrix are unity, then the matrix is referred to as *unit upper (lower) triangular*. A square matrix A is said to be *upper (lower) Hessenberg* if $a_{ij} = 0$ for $i > j + 1$ ($i < j - 1$). If A is upper and lower Hessenberg simultaneously, then it is said to be *tridiagonal*. Clearly, a tridiagonal matrix has nonzero elements only on the main diagonal and on the diagonals immediately above and below the main diagonal.

A matrix obtained from A by interchanging all its rows and columns is referred to as the *transpose* of A and is denoted by A^T . Hence,

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix} \quad (\text{B.30})$$

It is obvious that if A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix.

Next, we consider a square matrix A . If the elements of A are such that $a_{ij} = a_{ji}$, then the matrix A is said to be *symmetric*. Otherwise, the matrix is *nonsymmetric*. Hence, a matrix is symmetric if $A = A^T$. On the other hand, if the elements of A are such that $a_{ij} = -a_{ji}$ for $i \neq j$ and $a_{ii} = 0$, then the matrix A is said to be *skew symmetric*. It follows that A is skew symmetric if $A = -A^T$.

A matrix consisting of one column and n rows is called a *column matrix* and denoted by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} \quad (\text{B.31})$$

The column matrix \mathbf{x} can clearly be identified with a vector in L^n and is also known as a *column vector*. The transpose of the column matrix \mathbf{x} is the *row matrix*

$$\mathbf{x}^T = [x_1 \ x_2 \ \dots \ x_n] \quad (\text{B.32})$$

and is also called a *row vector*.

A matrix with all its elements equal to zero is called the *zero matrix* or the *null matrix* and is denoted by 0 , $\mathbf{0}$, or $\mathbf{0}^T$, depending on whether it is a rectangular, a column, or a row matrix, respectively.

B.7 BASIC MATRIX OPERATIONS

Two matrices A and B are said to be equal if and only if they have the same number of rows and columns and $a_{ij} = b_{ij}$ for all pairs of subscripts i and j .

If A and B are two $m \times n$ matrices, then the sum of A and B is defined as a matrix C whose elements are

$$c_{ij} = a_{ij} + b_{ij}, \quad i = 1, 2, \dots, m; \quad j = 1, 2, \dots, n \quad (\text{B.33})$$

Clearly, $C = A + B$ is also an $m \times n$ matrix. Matrix addition is *commutative* and *associative*. Indeed, if A , B and C are arbitrary $m \times n$ matrices, then

$$A + B = B + A, \quad (A + B) + C = A + (B + C) \quad (\text{B.34})$$

The *product of a matrix A and a scalar α* implies that every element of A is multiplied by α . Hence, if A is an $m \times n$ matrix, then the statement $C = \alpha A$ implies

$$c_{ij} = \alpha a_{ij}, \quad i = 1, 2, \dots, m; \quad j = 1, 2, \dots, n \quad (\text{B.35})$$

Next we define the *product of two matrices*. If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product $C = AB$ of the two matrices is an $m \times p$ matrix with the elements

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj} = \sum_{k=1}^n a_{ik}b_{kj} \quad (\text{B.36})$$

It is clear from the above that the matrix product is defined only if the number of columns of A is equal to the number of rows of B . In this case, the matrices A and B are said to be *conformable* in the order stated. The matrix product AB can be described as *B premultiplied by A* or *A postmultiplied by B* . It can also be described as *B multiplied on the left by A* or *A multiplied on the right by B* . Matrix multiplication is in general *not commutative*

$$AB \neq BA \quad (\text{B.37})$$

or, stated differently, matrices A and B *do not commute*. In fact, unless $m = p$, the matrix product BA is not even defined. One notable exception is the case in which one of the matrices is the identity matrix because then

$$AI = IA = A \quad (\text{B.38})$$

Clearly, the order of I must be such that the product is defined.

The matrix product

$$AB = 0 \quad (\text{B.39})$$

does not imply that either A or B is a null matrix, or both A and B are null matrices. Indeed, an illustration of this statement is provided by

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{B.40})$$

The matrix product satisfies *associative laws*. If A , B and C are $m \times n$, $n \times p$ and $p \times q$ matrices, respectively, then it can be verified that

$$D = (AB)C = A(BC) \quad (\text{B.41})$$

is an $m \times q$ matrix whose elements are given by

$$d_{ij} = \sum_{l=1}^p \sum_{k=1}^n a_{ik} b_{kl} c_{lj} = \sum_{k=1}^n \sum_{l=1}^p a_{ik} b_{kl} c_{lj} \quad (\text{B.42})$$

The matrix product satisfies *distributive laws*. If A and B are $m \times n$ matrices, C is an $n \times m$ matrix and D is an $p \times q$ matrix, then it can be shown that

$$C(A+B) = CA + CB, \quad (A+B)D = AD + BD \quad (\text{B.43})$$

If A is an $m \times n$ matrix and B is an $n \times p$ matrix, so that the product $C = AB$ is given by Eq. (B.36), then

$$C^T = (AB)^T = B^T A^T \quad (\text{B.44})$$

To show this, we recognize that to any element a_{ik} in A corresponds the element a_{ki} in A^T , and to any element b_{kj} in B corresponds the element b_{jk} in B^T . Then the product

$$\sum_{k=1}^n b_{jk} a_{ki} = c_{ji} \quad (\text{B.45})$$

establishes the validity of Eq. (B.44). In words, *the transpose of a product of two matrices is equal to the product of the transposed matrices in reversed order*. As a corollary, it can be verified that if

$$C = A_1 A_2 \dots A_{s-1} A_s \quad (\text{B.46})$$

then

$$C^T = A_s^T A_{s-1}^T \dots A_2^T A_1^T \quad (\text{B.47})$$

We have considerable interest in the concepts of inner product and orthogonality of vectors, so that we will find it convenient to recast some of the relations in Sec. B.4 in terms of matrix notation. The *inner product* of two n -vectors \mathbf{x} and \mathbf{y} , Eq. (B.9), can be expressed as

$$\mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x} = c \quad (\text{B.48})$$

and it represents a scalar. When the inner product is zero, or

$$\mathbf{x}^T \mathbf{y} = \mathbf{y}^T \mathbf{x} = 0 \quad (\text{B.49})$$

the vectors \mathbf{x} and \mathbf{y} are said to be *orthogonal*.

Less frequently, we encounter the *outer product* of two vectors \mathbf{x} and \mathbf{y} , defined as

$$\mathbf{xy}^T = C \tag{B.50}$$

and it represents a matrix C . If \mathbf{x} is an m -vector and \mathbf{y} an n -vector, then the matrix C is $m \times n$. Clearly, the outer product is not symmetric in \mathbf{x} and \mathbf{y} , because

$$\mathbf{yx}^T = C^T \neq \mathbf{xy}^T \tag{B.51}$$

Example B.4

Calculate the matrix product AB , where

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 7 \\ -1 & 4 & 2 \end{bmatrix} \tag{a}$$

What can be said about the matrix product BA ?

The matrix product AB is formed as follows:

$$AB = \begin{bmatrix} 2(1) - 3(-1) & 2(3) - 3(4) & 2(7) - 3(2) \\ 1(1) + 5(-1) & 1(3) + 5(4) & 1(7) + 5(2) \end{bmatrix} = \begin{bmatrix} 5 & -6 & 8 \\ -4 & 23 & 17 \end{bmatrix} \tag{b}$$

The matrix product BA is not defined, because B is a 2×3 matrix and A a 2×2 matrix and hence the matrices are not conformable in that order.

Example B.5

Calculate the matrix products AB and CA , where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 & 1 & -2 \\ -1 & 5 & 7 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 3 \\ 7 & 4 \\ -2 & 2 \end{bmatrix} \tag{a}$$

The matrix products AB and CA are as follows:

$$AB = \begin{bmatrix} -1 & 5 & 7 & 3 \\ 2 & 4 & 1 & -2 \end{bmatrix}, \quad CA = \begin{bmatrix} 3 & 1 \\ 4 & 7 \\ 2 & -2 \end{bmatrix} \tag{b}$$

We observe that the matrix AB is obtained from B by interchanging the rows. Similarly, CA is obtained from C by interchanging the columns. Hence, the effect of premultiplying a matrix by A is to permute its rows and the effect of postmultiplying a matrix by A is to permute its columns. For this reason, A is called a *permutation matrix*. In general, a permutation matrix is a matrix obtained by interchanging rows or columns of the identity matrix.

B.8 DETERMINANTS

If A is any square matrix of order n with elements in the field F , then it is possible to associate with A a number in F called the *determinant of A* , and denoted by $\det A$ or $|A|$. The determinant of A is said to be of order n and can be exhibited in the form

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} \tag{B.52}$$

Determinants have many interesting and useful properties. We examine only those properties pertinent to our study.

Unlike the matrix A , which represents a given array of numbers, the determinant of A represents a single number with a unique value that can be calculated by following rules for the expansion of determinants. The expansion rules can be most conveniently discussed by introducing the concept of minor determinants. The *minor determinant* $|M_{rs}|$ corresponding to the element a_{rs} is the determinant obtained from $|A|$ by striking out the r th row and s th column. Clearly, the order of $|M_{rs}|$ is $n - 1$. The signed minor determinant corresponding to the element a_{rs} is called the *cofactor* of a_{rs} and is given by

$$\det A_{rs} = |A_{rs}| = (-1)^{r+s} |M_{rs}| \quad (\text{B.53})$$

The value of the determinant of A can be obtained by expanding in terms of cofactors by the r th row as follows:

$$|A| = \sum_{s=1}^n a_{rs} |A_{rs}| \quad (\text{B.54})$$

The determinant can also be expanded by the s th column in the form

$$|A| = \sum_{r=1}^n a_{rs} |A_{rs}| \quad (\text{B.55})$$

The value of the determinant is unique, regardless of whether it is expanded by a row or a column, and regardless of which row or column. The expansion by cofactors is known as a *Laplace expansion*. The cofactors $|A_{rs}|$ are determinants of order $n - 1$. If $n = 2$, then these cofactors are simply scalars. If $n > 2$, then these cofactors can be expanded in terms of their own cofactors, and the process repeated until the minor determinants are of order 2. As an illustration, let $n = 3$ and expand $\det A$ by the first row as follows:

$$\begin{aligned} \det A &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}|A_{11}| + a_{12}|A_{12}| + a_{13}|A_{13}| \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ &\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned} \quad (\text{B.56})$$

Because the value of $\det A$ is the same, regardless of whether the determinant is expanded by a row or a column, it follows that

$$\det A = \det A^T \quad (\text{B.57})$$

or the determinant of a matrix is equal to the determinant of the transposed matrix. It is easy to verify that the determinant of a triangular matrix is equal to the product of the main diagonal elements. It follows immediately that the determinant of a diagonal matrix is equal to the product of the diagonal elements, and the determinant of the identity matrix is equal to 1.

If $\det A = 0$, then the matrix A is said to be *singular*, and if $\det A \neq 0$, the matrix is *nonsingular*. Clearly, a matrix with an entire row or an entire column equal to zero is singular. The evaluation of determinants can be greatly simplified by invoking certain properties of determinants. In fact, it is often possible to establish that a determinant is zero without actually expanding it. From Eq. (B.54), one can deduce the following properties:

1. If two rows (or two columns) are interchanged, then the determinant changes sign.
2. If all the elements of one row (or of one column) are multiplied by a scalar α , then the determinant is multiplied by α .
3. The value of a determinant does not change if one row (or one column) multiplied by a scalar α is added or subtracted from another row (or another column).
4. If every element in one row (or one column) is the sum of two terms, then the determinant is equal to the sum of two determinants, each of the two determinants being obtained by splitting every sum so that one term is in one determinant and the remaining term is in the other determinant.

The above properties permit us to make two observations. Property 2 implies that $\det(\alpha A) = \alpha^n \det A$, where α is a scalar and n is the order of the matrix. On the other hand, property 3 implies that a determinant with two proportional rows, or two proportional columns, is equal to zero. Property 3 can be used in general to simplify the evaluation of a determinant, and in particular to show that its value is zero, if indeed this is the case.

Example B.6

Calculate the value of the determinant

$$|A| = \begin{vmatrix} 3 & 2 & -1 \\ 1 & 5 & 2 \\ 3 & -1 & 2 \end{vmatrix} \quad (a)$$

Expanding by the first row, we obtain

$$\begin{aligned} A &= 3 \begin{vmatrix} 5 & 2 \\ -1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 5 \\ 3 & -1 \end{vmatrix} \\ &= 3[5(2) - 2(-1)] - 2[1(2) - 2(3)] - [1(-1) - 5(3)] = 60 \end{aligned} \quad (b)$$

On the other hand, subtracting three times the second row from the first and third rows, we can write

$$A = \begin{vmatrix} 0 & -13 & -7 \\ 1 & 5 & 2 \\ 0 & -16 & -4 \end{vmatrix} \quad (c)$$

Next, expanding by the first column, we obtain

$$A = - \begin{vmatrix} -13 & -7 \\ -16 & -4 \end{vmatrix} = [(-13)(-4) - (-7)(-16)] = 60 \quad (d)$$

which is the same value as that given by Eq. (b).

B.9 INVERSE OF A MATRIX

If A and B are two $n \times n$ matrices such that

$$AB = BA = I \quad (\text{B.58})$$

then B is said to be the *inverse* of A and is denoted by

$$B = A^{-1} \quad (\text{B.59})$$

Note that at the same time A is the inverse of B , $A = B^{-1}$.

Next, we derive a formula for the inverse of a matrix. To this end, we consider Eq. (B.53) and introduce the *adjugate* of A in the form

$$\text{adj } A = [(-1)^{r+s} |M_{rs}|]^T \quad (\text{B.60})$$

where $(-1)^{r+s} |M_{rs}|$ is the cofactor corresponding to a_{rs} . Then we can write

$$A \text{ adj } A = \left[\sum_{s=1}^n (-1)^{r+s} a_{ps} |M_{rs}| \right] \quad (\text{B.61})$$

Recalling Eq. (B.54), we conclude that every element of $A \text{ adj } A$ can be regarded as a determinantal expansion. When $p = r$, the element is simply equal to $\det A$. On the other hand, when $p \neq r$ the result is zero. This can be explained by recognizing that the determinant corresponding to $p \neq r$ is obtained from the matrix A by replacing the r th row by the p th row and keeping the p th row intact. Because the corresponding determinant has two identical rows, its value is zero. In view of this, Eq. (B.61) can be rewritten as

$$A \text{ adj } A = (\det A) I \quad (\text{B.62})$$

where I is the identity matrix of order N . Multiplying Eq. (B.62) on the left by A^{-1} and dividing through by $\det A$, we obtain

$$A^{-1} = \frac{\text{adj } A}{\det A} \quad (\text{B.63})$$

If $\det A = 0$, then no matrix B exists such that Eq. (B.58) is satisfied. To show this, we invoke the following theorem (Ref. 2, p. 134): *If A and B are $n \times n$ matrices, then*

$$\det AB = \det A \det B \quad (\text{B.64})$$

But, from Eq. (B.58), we conclude that $\det AB = \det I$ if $B = A^{-1}$ exists, so that $\det A \neq 0$. Hence, if $\det A = 0$, Eq. (B.58) cannot be satisfied, so that $B = A^{-1}$ does not exist. Recalling that when $\det A = 0$ the matrix is singular, it follows that *an $n \times n$ matrix A has an inverse if and only if A is nonsingular.*

To calculate the inverse of a matrix of large order by means of formula (B.63) it is necessary to evaluate a large number of determinants. For example, if A is of order n , then the calculation of $\det A$ requires the evaluation of $n!/2$ determinants of order 2. Hence, as n increases, it is necessary to carry out an increasingly large number of multiplications with a progressive loss of accuracy, so that the use of the

formula (B.63) is not recommended. In this text we study more efficient and more accurate methods for the calculation of the inverse of a matrix (see Sec. 6.1).

Next we consider the product of matrices given by Eq. (B.46). Multiplying both sides of Eq. (B.46) on the right by $A_s^{-1}, A_{s-1}^{-1}, \dots, A_1^{-1}$, in sequence, and then on the left by C^{-1} , we obtain

$$C^{-1} = A_s^{-1}A_{s-1}^{-1} \dots A_2^{-1}A_1^{-1} \tag{B.65}$$

or *the inverse of a product of matrices is equal to the product of the inverse matrices in reversed order*. Of course, Eq. (B.65) implies that all the inverse matrices in question exist.

Example B.7

Calculate the inverse of the matrix

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 5 & 2 \\ 3 & -1 & 2 \end{bmatrix} \tag{a}$$

First, we evaluate the minor determinants

$$\begin{aligned} |M_{11}| &= \begin{vmatrix} 5 & 2 \\ -1 & 2 \end{vmatrix} = 12, & |M_{12}| &= \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = -4, & |M_{13}| &= \begin{vmatrix} 1 & 5 \\ 3 & -1 \end{vmatrix} = -16 \\ |M_{21}| &= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3, & |M_{22}| &= \begin{vmatrix} 3 & -1 \\ 3 & 2 \end{vmatrix} = 9, & |M_{23}| &= \begin{vmatrix} 3 & 2 \\ 3 & -1 \end{vmatrix} = -9 \\ |M_{31}| &= \begin{vmatrix} 2 & -1 \\ 5 & 2 \end{vmatrix} = 9, & |M_{32}| &= \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 7, & |M_{33}| &= \begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} = 13 \end{aligned} \tag{b}$$

Using Eq. (B.60), we obtain the adjugate matrix

$$\text{adj } A = \begin{bmatrix} 12 & -(-3) & 9 \\ -(-4) & 9 & -(-7) \\ (-16) & -(-9) & 13 \end{bmatrix} = \begin{bmatrix} 12 & 3 & 9 \\ 4 & 9 & 7 \\ -16 & 9 & 13 \end{bmatrix} \tag{c}$$

Recalling from Example B.6 that $\det A = 60$ and using Eq. (B.63), we obtain

$$A^{-1} = \frac{1}{60} \begin{bmatrix} 12 & 3 & 9 \\ 4 & 9 & 7 \\ -16 & 9 & 13 \end{bmatrix} \tag{d}$$

B.10 PARTITIONED MATRICES

On occasion it is convenient to partition matrices into submatrices. Then, under proper circumstances, certain matrix operations can be performed by treating the submatrices as if they were single elements. As an example, we consider a 3×4 matrix A and partition it as follows:

$$A = \left[\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array} \right] = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right] \tag{B.66}$$

where

$$\begin{aligned} A_{11} &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, & A_{12} &= \begin{bmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{bmatrix} \\ A_{21} &= \begin{bmatrix} a_{31} & a_{32} \end{bmatrix}, & A_{22} &= \begin{bmatrix} a_{33} & a_{34} \end{bmatrix} \end{aligned} \quad (\text{B.67})$$

are the submatrices of A . Next, we consider a 4×4 matrix B partitioned in the form

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \quad (\text{B.68})$$

where

$$\begin{aligned} B_{11} &= \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, & B_{12} &= \begin{bmatrix} b_{13} & b_{14} \\ b_{23} & b_{24} \end{bmatrix} \\ B_{21} &= \begin{bmatrix} b_{31} & b_{32} \\ b_{41} & b_{42} \end{bmatrix}, & B_{22} &= \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix} \end{aligned} \quad (\text{B.69})$$

It is not difficult to verify that the matrix product AB can be obtained by treating the submatrices A_{ik} and B_{kj} as if they were ordinary matrices. Indeed, the elements of the product $C = AB$ are

$$C_{ij} = \sum_{k=1}^2 A_{ik} B_{kj}, \quad i, j = 1, 2 \quad (\text{B.70})$$

It should be pointed out, however, that products such as (B.70) are possible only if the matrix A_{ik} has as many columns as the matrix B_{kj} has rows, which is clearly true in the particular case at hand.

If the off-diagonal submatrices of a square matrix are null matrices, then the matrix is said to be *block-diagonal*. For block-diagonal matrices, the determinant of the matrix is equal to the product of the determinants of the submatrices on the main diagonal. For example, if B_{12} and B_{21} in Eq. (B.68) are null matrices, then

$$\det B = \det B_{11} \det B_{22} \quad (\text{B.71})$$

Actually the above statement is true even if the matrix is only *block-triangular*, i.e., if only the submatrices above (or below) the main diagonal are null matrices.

B.11 SYSTEMS OF LINEAR ALGEBRAIC EQUATIONS

We consider a system of m nonhomogeneous linear equations in n unknowns x_1, x_2, \dots, x_n of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= c_2 \\ \dots & \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= c_m \end{aligned} \quad (\text{B.72})$$

The system of equations can be written as the compact matrix equation

$$Ax = c \tag{B.73}$$

where $A = [a_{ij}]$ is an $m \times n$ matrix known as the matrix of the coefficients, $x = [x_1 \ x_2 \ \dots \ x_n]^T$ is the n -vector of the unknowns and $c = [c_1 \ c_2 \ \dots \ c_m]^T$ is the m -vector of "nonhomogeneous terms." Our interest is in the conditions under which Eq. (B.73) has a solution.

The matrix A can be partitioned into n column vectors of dimension m in the form

$$A = [a_1 \ a_2 \ \dots \ a_n] \tag{B.74}$$

where $a_1 = [a_{11} \ a_{21} \ \dots \ a_{m1}]^T$, etc., are the column vectors. In view of this, the matrix product Ax can be looked upon as a linear combination of the columns of A , so that Eq. (B.73) can be written as

$$x_1 a_1 + x_2 a_2 + \dots + x_n a_n = c \tag{B.75}$$

Equation (B.75) implies that the set of all products Ax is the same as the set of linear combinations of the columns of A . The subspace of L^m spanned by the columns of A is called the *column space* of A and denoted by $\mathcal{R}(A)$. If y is an m -vector, then y^T is a row vector with m components. Now, if A is partitioned into m row vectors, then the product $y^T A$ is linear combination of the rows of A whose coefficients are the components of y . Hence, the *row space* of A , written $\mathcal{R}(A^T)$, is the subspace of L^n spanned by the row vectors of A .

Next, we define the *rank* of a matrix A , denoted by $\text{rank } A$, as the dimension of the linear space spanned by its columns. Because the latter is simply the dimension of $\mathcal{R}(A)$, we have

$$\text{rank } A = \dim \mathcal{R}(A) \tag{B.76}$$

It would appear that $\text{rank } A$ should have been more properly referred to as the *column rank* of A , which would have naturally called for the introduction of a *row rank* of A as the dimension of $\mathcal{R}(A^T)$. It turns out, however, that the column rank and row rank of any matrix A are equal (Ref. 4, p. 93), so that no such distinction is necessary. In view of the definition of the dimension of a linear space, it follows that *the rank of a matrix A is equal to the maximum number of linearly independent columns of A , and it is also equal to the maximum number of linearly independent rows of A , where the two numbers must be the same.*

There is one more vector space associated with any $m \times n$ matrix A , the *nullspace* of A . It is denoted by $\mathcal{N}(A)$ and defined as the space of all the solutions $x \neq 0$ satisfying the homogeneous equation $Ax = 0$. The dimension of the nullspace \mathcal{N} is called the *nullity* of A , $\dim \mathcal{N} = \text{null } A$.

Let us return now to Eq. (B.73) and introduce the *augmented matrix* of the system defined by

$$B = [A, c] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & c_1 \\ a_{21} & a_{22} & \dots & a_{2n} & c_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} & c_m \end{bmatrix} \tag{B.77}$$

Then, Eq. (B.73) has a solution \mathbf{x} if and only if the rank of the augmented matrix B is equal to the rank of A . If a solution \mathbf{x} exists, then \mathbf{c} is a linear combination of the columns of A and hence lies in $\mathcal{R}(A)$. It follows that $\mathcal{R}(B) = \mathcal{R}(A)$, and $\text{rank } B = \text{rank } A$.

The rank of an arbitrary matrix can be connected with the order of its nonsingular square submatrices. Indeed, *the rank of any matrix A is equal to the order of the square submatrix of A of greatest order whose determinant does not vanish.* It follows that:

- (a) If $m \geq n$, then the largest possible rank of A is n . If $\text{rank } A = \text{rank } B = n$, then Eq. (B.73) has a unique solution.
- (b) If $m < n$, then the largest possible rank of A is m . If $\text{rank } A = \text{rank } B = m$, then Eq. (B.73) has an infinity of solutions. A unique solution can be chosen in the form of the solution with the minimum norm

$$\mathbf{x} = A^T(AA^T)^{-1}\mathbf{c} \quad (\text{B.78})$$

where AA^T is an $m \times m$ matrix of rank m and is therefore nonsingular.

The case in which the number of equations is equal to the number of unknowns is of particular interest. If A is a square matrix of order n , then the following statements are equivalent:

1. The rank of A is n , $\text{rank } A = n$.
2. The system $A\mathbf{x} = \mathbf{c}$ has a unique solution for arbitrary vectors \mathbf{c} .
3. The system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$, which implies that $\text{null } A = \mathbf{0}$.

The implication of statements 1 and 2 is that the matrix A is nonsingular, so that A possesses an inverse. Considering the case in which the matrix A in Eq. (B.73) is square and premultiplying both sides of the equation by A^{-1} , we obtain

$$\mathbf{x} = A^{-1}\mathbf{c} \quad (\text{B.79})$$

Hence, when A is nonsingular the solution of Eq. (B.73) can be produced by simply calculating the inverse of A . We have shown in Sec. B.9 that A^{-1} can be obtained by dividing the adjugate of A by the determinant of A ; this method for solving sets of simultaneous equations is generally known as *Cramer's rule*. This approach is mainly of academic interest, and in computational work the procedure is seldom used, especially for large order matrices A . Indeed, the procedure involves the evaluation of a large number of determinants, which is time-consuming and leads to loss of accuracy. In Sec. 6.1, we discuss a more efficient method for deriving the solution of Eq. (B.73), namely, the Gaussian elimination.

Next, we turn our attention to the homogeneous system $A\mathbf{x} = \mathbf{0}$. As pointed out earlier, the matrix product $A\mathbf{x}$ represents a linear combination of the column vectors of A . Because this linear combination must be equal to zero, it follows from Sec. B.2 that the columns of A are not independent. Hence, the rank of A must be less than n , so that $\det A = 0$. This conclusion can be stated in a more formal manner by means of the well-known theorem of linear algebra: *If A is an $n \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x} \neq \mathbf{0}$ if and only if $\det A = 0$.*

To determine the nullity of A , we determine first the null space of A by solving the equation

$$Ax = \mathbf{0} \quad (b)$$

which can be written in the explicit form

$$\begin{aligned} 2x_1 - x_2 + 4x_3 + 3x_4 &= 0 \\ x_1 + 5x_2 - 2x_3 + 4x_4 &= 0 \\ 5x_1 + 3x_2 + 6x_3 + 10x_4 &= 0 \\ -x_1 + 6x_2 - 6x_3 + x_4 &= 0 \end{aligned} \quad (c)$$

The above four equations can be reduced to the two equations

$$\begin{aligned} x_1 + (18/11)x_3 + (19/11)x_4 &= 0 \\ x_2 - (8/11)x_3 + (5/11)x_4 &= 0 \end{aligned} \quad (d)$$

while the remaining two equations are identically zero. It can be verified that every solution of Eqs. (d) can be written in the form

$$\mathbf{x} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 \quad (e)$$

where

$$\mathbf{u}_1 = [-18 \ 8 \ 11 \ 0]^T, \quad \mathbf{u}_2 = [-19 \ -5 \ 0 \ 11]^T \quad (f)$$

The vectors \mathbf{u}_1 and \mathbf{u}_2 are clearly independent and they span the null space. Hence, they form a basis for the space. The dimension of the null space $\mathcal{N}(A)$ is two, $\dim \mathcal{N} = 2$, so that the nullity of A is two, $\text{null } A = 2$.

B.12 LINEAR TRANSFORMATIONS

As shown in Sec. B.3, any n -vector \mathbf{x} in L^n with components x_1, x_2, \dots, x_n can be expressed as the linear combination

$$\mathbf{x} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + \dots + x_n \mathbf{e}_n = \sum_{j=1}^n x_j \mathbf{e}_j \quad (B.84)$$

where \mathbf{e}_i ($i = 1, 2, \dots, n$) are the standard unit vectors. Moreover, the scalars x_1, x_2, \dots, x_n are called the *coordinates* of the vector \mathbf{x} with respect to the basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ (Fig. B.2). Next, we consider an $n \times n$ matrix A and write

$$\mathbf{x}' = A\mathbf{x} \quad (B.85)$$

The resulting vector \mathbf{x}' is another vector in L^n , so that Eq. (B.85) represents a *linear transformation* on the vector space L^n which maps the vector \mathbf{x} into a vector \mathbf{x}' .

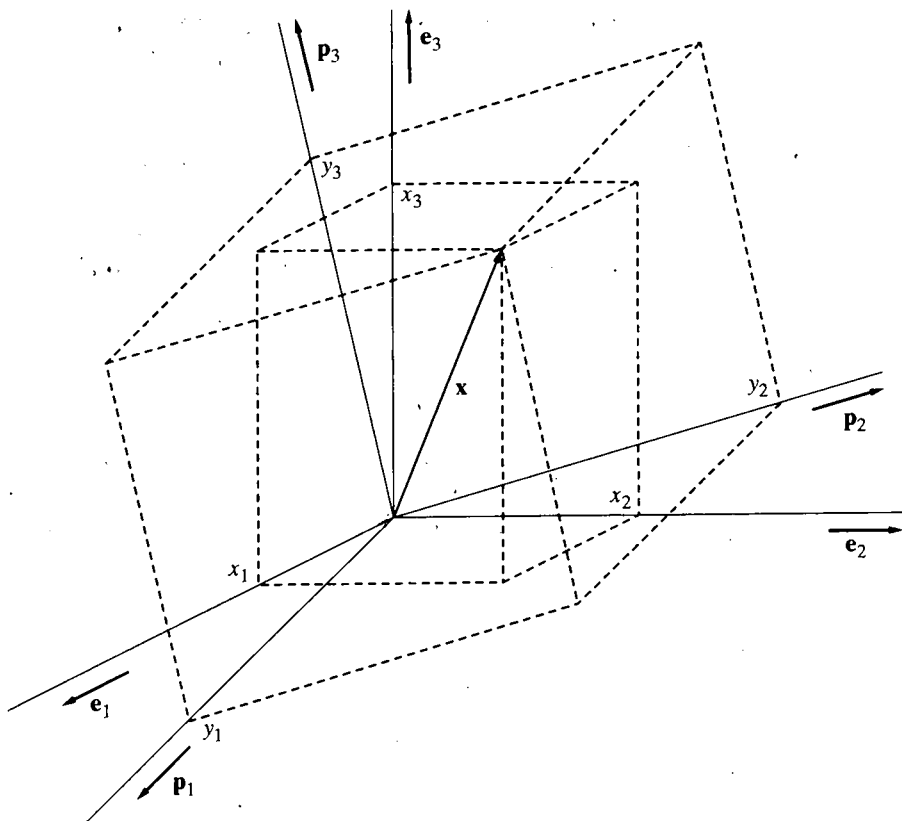


Figure B.2 Decomposition of a three-dimensional vector \mathbf{x} in terms of the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and an arbitrary basis $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$

Our interest lies in expressing the vector \mathbf{x} in terms of any arbitrary basis $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ for L^n , rather than the standard basis, as follows:

$$\mathbf{x} = y_1\mathbf{p}_1 + y_2\mathbf{p}_2 + \dots + y_n\mathbf{p}_n = \sum_{i=1}^n y_i\mathbf{p}_i = P\mathbf{y} \quad (\text{B.86})$$

where

$$P = [\mathbf{p}_1 \ \mathbf{p}_2 \ \dots \ \mathbf{p}_n] \quad (\text{B.87})$$

is an $n \times n$ matrix of basis vectors and

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \quad (\text{B.88})$$

is an n -vector whose components y_1, y_2, \dots, y_n are the coordinates of \mathbf{x} with respect to the basis $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ (Fig. B.2). By the definition of a basis, the vectors $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ are linearly independent, so that the matrix P is nonsingular.

Similarly, denoting by y'_1, y'_2, \dots, y'_n the coordinates of \mathbf{x}' with respect to the basis $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$, we can use the analogy with Eq. (B.86) and write

$$\mathbf{x}' = P\mathbf{y}' \quad (\text{B.89})$$

Then, inserting Eqs. (B.86) and (B.89) into Eq. (B.85), we have

$$P\mathbf{y}' = A(P\mathbf{y}) \quad (\text{B.90})$$

so that, premultiplying both sides of Eq. (B.90) by P^{-1} , we obtain

$$\mathbf{y}' = B\mathbf{y} \quad (\text{B.91})$$

where

$$B = P^{-1}AP \quad (\text{B.92})$$

The matrix B represents the same linear transformation as A , but in a different coordinate system. Two square matrices A and B related by an equation of the type (B.92) are said to be *similar* and Eq. (B.92) itself represents a *similarity transformation*.

A similarity transformation of particular interest is the orthonormal transformation. A matrix P is said to be *orthonormal* if it satisfies

$$P^T P = I \quad (\text{B.93})$$

from which it follows that an orthonormal matrix also satisfies

$$P^{-1} = P^T \quad (\text{B.94})$$

Introducing Eq. (B.94) into Eq. (B.92), we obtain

$$B = P^T A P \quad (\text{B.95})$$

Equation (B.95) represents an *orthonormal transformation*, a very important special type of similarity transformation, particularly when the matrix A is symmetric.

B.13 THE ALGEBRAIC EIGENVALUE PROBLEM

The equations for the free vibration of discrete systems can be written in the state form

$$\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t) \quad (\text{B.96})$$

where $\mathbf{x}(t)$ is the n -dimensional state vector and A is an $n \times n$ matrix of coefficients. Equation (B.96) represents a set of homogeneous ordinary differential equations and has the solution

$$\mathbf{x}(t) = e^{\lambda t} \mathbf{x} \quad (\text{B.97})$$

in which λ is a constant scalar and \mathbf{x} a constant vector. Inserting Eq. (B.97) into Eq. (B.96) and dividing through by $e^{\lambda t}$, we obtain

$$A\mathbf{x} = \lambda\mathbf{x} \quad (\text{B.98})$$

Equation (B.98) represents the *algebraic eigenvalue problem* and can be stated as follows: *Determine the values of the parameter λ so that Eq. (B.98) admits nontrivial solutions.* The values of λ are known as *eigenvalues* and are the roots of the *characteristic equation*

$$\det(A - \lambda I) = 0 \quad (\text{B.99})$$

Similarity transformations are often used in numerical algorithms for the algebraic eigenvalue problem. To examine the reason why, we consider Eq. (B.92) and write the characteristic determinant

$$\begin{aligned} \det(B - \lambda I) &= \det(P^{-1}AP - \lambda I) = \det(P^{-1}(A - \lambda I)P) \\ &= \det P^{-1} \det(A - \lambda I) \det P \end{aligned} \quad (\text{B.100})$$

But,

$$\det(P^{-1}P) = \det P^{-1} \det P = 1 \quad (\text{B.101})$$

so that

$$\det(B - \lambda I) = \det(A - \lambda I) \quad (\text{B.102})$$

Because matrices A and B possess the same characteristic determinant, they possess the same eigenvalues. It follows that *eigenvalues do not change under similarity transformations.* Of course the similarity transformations generally used in numerical algorithms for the solution of the eigenvalue problem are orthogonal transformations.

The characteristic determinant can be expressed in the form of the *characteristic polynomial*

$$\begin{aligned} \det(A - \lambda I) &= (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n) = (-1)^n \prod_{i=1}^n (\lambda - \lambda_i) \\ &= (-1)^n (\lambda^n + c_1 \lambda^{n-1} + \dots + c_{n-1} \lambda + c_n) \end{aligned} \quad (\text{B.103})$$

where \prod is the product symbol. Because the eigenvalues do not change under similarity transformations, it follows that *the coefficients c_i ($i = 1, 2, \dots, n$) of the polynomial are invariant.* Two of the coefficients have special significance, namely, c_1 , and c_n . It can be verified that

$$c_1 = - \int_{i=1}^n \lambda_i = - \int_{i=1}^n a_{ii} = -\text{tr} A \quad (\text{B.104})$$

in which $\text{tr} A$ denotes the *trace* of the matrix A , defined as the sum of the diagonal elements of A . Hence, *the trace of A is invariant under similarity transformations.* Similarly, it can be shown that

$$c_n = \prod_{i=1}^n (-\lambda_i) = (-1)^n \prod_{i=1}^n \lambda_i = (-1)^n \det A \quad (\text{B.105})$$

from which we conclude that *the determinant of A is invariant under similarity transformations.*

B.14 MATRIX NORMS

As with vectors, it is useful to assign a single number to a matrix, thus providing a measure of the magnitude of the matrix in some sense. Such a measure is provided by the norm. The *norm of a square matrix* A is a nonnegative number $\|A\|$ satisfying the conditions

1. $\|A\| \geq 0$, $\|A\| = 0$ if and only if $A = 0$.
2. $\|kA\| = |k|\|A\|$ for any complex scalar k .
3. $\|A + B\| \leq \|A\| + \|B\|$.
4. $\|AB\| \leq \|A\| \cdot \|B\|$.

Corresponding to any vector norm, one can associate with any matrix A a nonnegative quantity defined by $\max \|Ax\|/\|x\|$, $\|x\| \neq 0$. This quantity is a function of the matrix A and it satisfies the conditions of a matrix norm. It is called the matrix norm *subordinate* to the vector norm. Because

$$\|A\| = \max_{\|x\| \neq 0} \frac{\|Ax\|}{\|x\|}, \quad (B.106)$$

we have

$$\|Ax\| \leq \|A\| \cdot \|x\| \quad (B.107)$$

where inequality (B.104) is true for $\|x\| \neq 0$ or for $\|x\| = 0$. Matrix and vector norms satisfying an inequality of the type (B.104) for all A and x are said to be *compatible*. Hence, a vector norm and its subordinate matrix norm are always compatible.

A matrix norm of particular importance is the *Euclidean norm*, denoted by $\|A\|_E$ and defined as

$$\|A\|_E = \left(\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2 \right)^{1/2} \quad (B.108)$$

The Euclidean norm has the advantage that it is easy to compute. Moreover, it has the important property that its value is invariant under orthogonal transformations (Ref. 3, p. 287).

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