

SECOND EDITION

# Applied Calculus of Variations for Engineers



Louis Komzsi k



CRC Press  
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**Louis Komzsik**

Siemens, Cypress  
California, USA



**CRC Press**

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To my daughter, Stella



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# *Contents*

|   |             |
|---|-------------|
| <b>Preface to the second edition</b>                            | <b>xi</b>   |
| <b>Preface to the first edition</b>                             | <b>xiii</b> |
| <b>Acknowledgments</b>  | <b>xv</b>   |
| <b>About the author</b>   | <b>xvii</b> |
| <b>List of notations</b>  | <b>xix</b>  |
| <b>I Mathematical foundation</b>                                | <b>1</b>    |
| <b>1 The foundations of calculus of variations</b>              | <b>3</b>    |
| 1.1 The fundamental problem and lemma of calculus of variations | 3           |
| 1.2 The Legendre test . . . . .                                 | 7           |
| 1.3 The Euler-Lagrange differential equation . . . . .          | 9           |
| 1.4 Application: minimal path problems . . . . .                | 11          |
| 1.4.1 Shortest curve between two points . . . . .               | 12          |
| 1.4.2 The brachistochrone problem . . . . .                     | 14          |
| 1.4.3 Fermat's principle . . . . .                              | 18          |
| 1.4.4 Particle moving in the gravitational field . . . . .      | 20          |
| 1.5 Open boundary variational problems . . . . .                | 21          |
| <b>2 Constrained variational problems</b>                       | <b>25</b>   |
| 2.1 Algebraic boundary conditions . . . . .                     | 25          |
| 2.2 Lagrange's solution . . . . .                               | 27          |
| 2.3 Application: iso-perimetric problems . . . . .              | 29          |
| 2.3.1 Maximal area under curve with given length . . . . .      | 29          |
| 2.3.2 Optimal shape of curve of given length under gravity .    | 31          |
| 2.4 Closed-loop integrals . . . . .                             | 35          |
| <b>3 Multivariate functionals</b>                               | <b>37</b>   |
| 3.1 Functionals with several functions . . . . .                | 37          |
| 3.2 Variational problems in parametric form . . . . .           | 38          |
| 3.3 Functionals with two independent variables . . . . .        | 39          |
| 3.4 Application: minimal surfaces . . . . .                     | 40          |
| 3.4.1 Minimal surfaces of revolution . . . . .                  | 43          |
| 3.5 Functionals with three independent variables . . . . .      | 44          |



|           |  |            |
|-----------|--|------------|
| <b>4</b>  | <b>Higher order derivatives</b>                                  | <b>49</b>  |
| 4.1       | The Euler-Poisson equation . . . . .                             | 49         |
| 4.2       | The Euler-Poisson system of equations . . . . .                  | 51         |
| 4.3       | Algebraic constraints on the derivative . . . . .                | 52         |
| 4.4       | Linearization of second order problems . . . . .                 | 54         |
| <b>5</b>  | <b>The inverse problem of calculus of variations</b>             | <b>57</b>  |
| 5.1       | The variational form of Poisson's equation . . . . .             | 58         |
| 5.2       | The variational form of eigenvalue problems . . . . .            | 59         |
| 5.2.1     | Orthogonal eigensolutions . . . . .                              | 61         |
| 5.3       | Sturm-Liouville problems . . . . .                               | 62         |
| 5.3.1     | Legendre's equation and polynomials . . . . .                    | 64         |
| <b>6</b>  | <b>Analytic solutions of variational problems</b>                | <b>69</b>  |
| 6.1       | Laplace transform solution . . . . .                             | 69         |
| 6.2       | Separation of variables . . . . .                                | 71         |
| 6.3       | Complete integral solutions . . . . .                            | 76         |
| 6.4       | Poisson's integral formula . . . . .                             | 80         |
| 6.5       | Method of gradients . . . . .                                    | 85         |
| <b>7</b>  | <b>Numerical methods of calculus of variations</b>               | <b>89</b>  |
| 7.1       | Euler's method . . . . .   | 89         |
| 7.2       | Ritz method . . . . .  | 91         |
| 7.2.1     | Application: solution of Poisson's equation . . . . .            | 95         |
| 7.3       | Galerkin's method . . . . .                                      | 96         |
| 7.4       | Kantorovich's method . . . . .                                   | 98         |
| 7.5       | Boundary integral method . . . . .                               | 103        |
| <b>II</b> | <b>Engineering applications</b>                                  | <b>109</b> |
| <b>8</b>  | <b>Differential geometry</b>                                     | <b>111</b> |
| 8.1       | The geodesic problem . . . . .                                   | 111        |
| 8.1.1     | Geodesics of a sphere . . . . .                                  | 113        |
| 8.2       | A system of differential equations for geodesic curves . . . . . | 114        |
| 8.2.1     | Geodesics of surfaces of revolution . . . . .                    | 116        |
| 8.3       | Geodesic curvature . . . . .                                     | 119        |
| 8.3.1     | Geodesic curvature of helix . . . . .                            | 121        |
| 8.4       | Generalization of the geodesic concept . . . . .                 | 122        |
| <b>9</b>  | <b>Computational geometry</b>                                    | <b>125</b> |
| 9.1       | Natural splines . . . . .  | 125        |
| 9.2       | B-spline approximation . . . . .                                 | 128        |
| 9.3       | B-splines with point constraints . . . . .                       | 133        |
| 9.4       | B-splines with tangent constraints . . . . .                     | 136        |
| 9.5       | Generalization to higher dimensions . . . . .                    | 139        |

|  |            |
|--|------------|
| <b>10 Variational equations of motion</b>                  | <b>143</b> |
| 10.1 Legendre’s dual transformation . . . . .              | 143        |
| 10.2 Hamilton’s principle for mechanical systems . . . . . | 144        |
| 10.2.1 Newton’s law of motion . . . . .                    | 145        |
| 10.3 Lagrange’s equations of motion . . . . .              | 146        |
| 10.4 Hamilton’s canonical equations . . . . .              | 147        |
| 10.4.1 Conservation of energy . . . . .                    | 149        |
| 10.5 Orbital motion . . . . .                              | 150        |
| 10.6 Variational foundation of fluid motion . . . . .      | 153        |
| <br>   |            |
| <b>11 Analytic mechanics</b>                               | <b>157</b> |
| 11.1 Elastic string vibrations . . . . .                   | 157        |
| 11.2 The elastic membrane . . . . .                        | 162        |
| 11.2.1 Circular membrane vibrations . . . . .              | 165        |
| 11.2.2 Non-zero boundary conditions . . . . .              | 167        |
| 11.3 Bending of a beam under its own weight . . . . .      | 169        |
| <br>   |            |
| <b>12 Computational mechanics</b>                          | <b>177</b> |
| 12.1 Three-dimensional elasticity . . . . .                | 177        |
| 12.2 Lagrangian formulation . . . . .                      | 180        |
| 12.3 Heat conduction . . . . .                             | 184        |
| 12.4 Fluid mechanics . . . . .                             | 186        |
| 12.5 The finite element method . . . . .                   | 189        |
| 12.5.1 Finite element meshing . . . . .                    | 189        |
| 12.5.2 Shape functions . . . . .                           | 191        |
| 12.5.3 Element matrix generation . . . . .                 | 195        |
| 12.5.4 Element matrix assembly and solution . . . . .      | 199        |
| <br>   |            |
| <b>Closing remarks</b>                                     | <b>205</b> |
| <br>   |            |
| <b>References</b>  | <b>207</b> |
| <br>   |            |
| <b>Index</b>   | <b>209</b> |
| <br>   |            |
| <b>List of Figures</b>                                     | <b>211</b> |
| <br>   |            |
| <b>List of Tables</b>                                      | <b>213</b> |



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## *Preface to the second edition*

Since the publication of the first edition six years ago several readers commented on the book and recommended improvements in a future edition. Academic readers requested expanding the mathematical foundation with a chapter dedicated to analytic solutions. Industrial readers desired a deeper exploration of the variational foundation of equations of motion for various engineering disciplines. Finally some reviewers suggested expanding the computational techniques section.

The modifications to the second edition reflect these requests. The major additions are in two new chapters (6 and 10) and two new Sections (7.5 and 12.5). Chapter 6 discusses a variety of analytic solutions to problems of calculus of variations and Chapter 10 derives various equations of motion from variational principles. Section 7.5 describes the boundary integral method and Section 12.5 contains a detailed description of the finite element method.

The typographical errors found in the first edition have been corrected and a strong effort was made to avoid introducing any in the new material. The new chapters and sections were indexed and more references were added to make this work as complete as possible.



---

## *Preface to the first edition*

The topic of this book has a long history. Its fundamentals were laid down by icons of mathematics like Euler and Lagrange. It was once heralded as the panacea for all engineering optimization problems by suggesting that all one needs to do was to apply the Euler-Lagrange equation form and solve the resulting differential equation.

This, as most all encompassing solutions, turned out to be not always true and the resulting differential equations are not necessarily easy to solve. On the other hand, many of the differential equations commonly used by engineers today are derived from a variational problem. Hence, it is important and useful for engineers to delve into this topic.

The book is organized into two parts: theoretical foundation and engineering applications. The first part starts with the statement of the fundamental variational problem and its solution via the Euler-Lagrange equation. This is followed by the gradual extension to variational problems subject to constraints, containing functions of multiple variables and functionals with higher order derivatives. It continues with the inverse problem of variational calculus, when the origin is in the differential equation form and the corresponding variational problem is sought. The first part concludes with the direct solution techniques of variational problems, such as the Ritz, Galerkin, and Kantorovich methods.

With the emphasis on applications, the second part starts with a detailed discussion of the geodesic concept of differential geometry and its extensions to higher order spaces. The computational geometry chapter covers the variational origin of natural splines and the variational formulation of B-splines under various constraints.

The final two chapters focus on analytic and computational mechanics. Topics of the first include the variational form and subsequent solution of several classical mechanical problems using Hamilton's principle. The last chapter discusses generalized coordinates and Lagrange's equations of motion. Some fundamental applications of elasticity, heat conduction, and fluid mechanics as well as their computational technology conclude the book.



---

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I appreciate the thorough review of the first edition by Dr. John Brauer, then at the Milwaukee School of Engineering, and I am still indebted to Professor Bajcsay Pál of the Technical University of Budapest for initiating my original interest in calculus of variations.

I am grateful for the courtesy of Sierra Nevada Corporation for the model in the cover art. The model depicts the re-entry configuration of the Dream Chaser spacecraft.

**Louis Komzsik**





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## *About the author*

Dr. Louis Komzsik is a graduate of the Technical University of Budapest, and the Eötvös Lóránd University, Hungary. He has been working in the industry for more than 40 years. He is currently the chief numerical analyst in the Office of Architecture and Technology at Siemens PLM Software.

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## *List of notations*

| Notation               | Meaning                                      |
|------------------------|--|
| $f(x)$                 | Function of one variable                     |
| $f(x, y), F(x, y)$     | Function of two variables                    |
| $r$                    | Radius of curvature                          |
| $g$                    | Acceleration of gravity                      |
| $\underline{r}$        | Vector in Cartesian coordinates              |
| $\underline{r}_x$      | First partial derivative with respect to $x$ |
| $\underline{r}_y$      | First partial derivative with respect to $y$ |
| $\dot{\underline{r}}$  | First parametric derivative                  |
| $\ddot{\underline{r}}$ | Second parametric derivative                 |
| $p$                    | Pressure                                     |
| $s$                    | Arc length                                   |
| $s(t)$                 | One-dimensional spline function              |
| $s(u, v)$              | Two-dimensional spline function              |
| $y'(x), f'(x)$         | First derivative                             |
| $y''(x), f''(x)$       | Second derivative                            |
| $\underline{n}$        | Normal vector                                |
| $\underline{t}$        | Tangent vector                               |
| $\underline{b}$        | Bi-normal vector                             |
| $\nabla$               | Gradient operator                            |
| $\Delta$               | Laplace operator                             |
| $\kappa$               | Curvature                                    |
| $\kappa_g$             | Geodesic curvature                           |
| $\kappa_n$             | Normal curvature                             |
| $\kappa_m$             | Mean curvature                               |
| $\delta I$             | Variation of integral functional             |
| $\Gamma_{ij}^k$        | Christoffel symbols                          |
| $\sigma$               | Stress                                       |
| $\epsilon$             | Strain                                       |
| $\nu$                  | Poisson's ratio                              |
| $\lambda$              | Eigenvalue                                   |
| $\rho$                 | Material density                             |
| $[B]$                  | Shape function derivative matrix             |

|              |                              |
|--------------|------------------------------|
| $B_{i,k}(t)$ | B-spline basis function      |
| $E, F, G$    | First fundamental quantities |
| $E$          | Young's modulus              |
| $[D]$        | Elasticity matrix            |
| $F$          | Active force                 |
| $G$          | Green's function             |
| $E_s$        | Strain energy                |
| $E_k$        | Kinetic energy               |
| $E_p$        | Potential energy             |
| $I()$        | Integral functional          |
| $I$          | Moment of inertia            |
| $[J]$        | Jacobian matrix              |
| $M$          | Momentum                     |
| $[K]$        | Stiffness matrix             |
| $[M]$        | Mass matrix                  |
| $[N]$        | Shape function matrix        |
| $Q$          | Heat source                  |
| $S$          | Surface area                 |
| $T$          | Surface (traction) force     |
| $[T]$        | Temperature matrix           |
| $V$          | Volume                       |
| $W_e$        | External work                |

## **Part I**

# **Mathematical foundation**



# 1

---

## *The foundations of calculus of variations*

The problem of the calculus of variations evolves from the analysis of functions. In the analysis of functions the focus is on the relation between two sets of numbers, the independent ( $x$ ) and the dependent ( $y$ ) set. The function  $f$  creates a one-to-one correspondence between these two sets, denoted as

$$y = f(x).$$

The generalization of this concept is based on allowing the two sets not to be restricted to being real numbers and to be functions themselves. The relationship between these sets is now called a functional. The topic of the calculus of variations is to find extrema of functionals, most commonly formulated in the form of an integral.

---

### 1.1 The fundamental problem and lemma of calculus of variations

The fundamental problem of the calculus of variations is to find the extremum (maximum or minimum) of the functional

$$I(y) = \int_{x_0}^{x_1} f(x, y, y') dx,$$

where the solution satisfies the boundary conditions

$$y(x_0) = y_0$$

and

$$y(x_1) = y_1.$$

This problem may be generalized to the cases when higher derivatives or multiple functions are given and will be discussed in Chapters 3 and 4, respectively. These problems may also be extended with constraints, the topic of Chapter 2.

A solution process may be arrived at with the following logic. Let us assume that there exists such a solution  $y(x)$  for the above problem that satisfies



the boundary conditions and produces the extremum of the functional. Furthermore, we assume that it is twice differentiable. In order to prove that this function results in an extremum, we need to prove that any alternative function does not attain the extremum.

We introduce an alternative solution function of the form

$$Y(x) = y(x) + \epsilon\eta(x),$$

where  $\eta(x)$  is an arbitrary auxiliary function of  $x$ , that is also twice differentiable and vanishes at the boundary:

$$\eta(x_0) = \eta(x_1) = 0.$$

In consequence the following is also true:

$$Y(x_0) = y(x_0) = y_0$$

and

$$Y(x_1) = y(x_1) = y_1.$$

A typical relationship between these functions is shown in Figure 1.1 where the function is represented by the solid line and the alternative function by the dotted line. The dashed line represents the arbitrary auxiliary function.

Since the alternative function  $Y(x)$  also satisfies the boundary conditions of the functional, we may substitute into the variational problem:

$$I(\epsilon) = \int_{x_0}^{x_1} f(x, Y, Y') dx.$$

where

$$Y'(x) = y'(x) + \epsilon\eta'(x).$$

The new functional in terms of  $\epsilon$  is identical with the original in the case when  $\epsilon = 0$  and has its extremum when

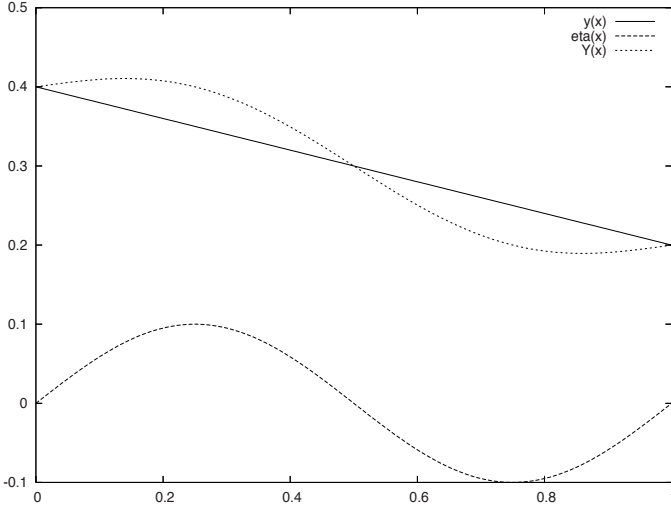
$$\left. \frac{\partial I(\epsilon)}{\partial \epsilon} \right|_{\epsilon=0} = 0.$$

Executing the derivation and taking the derivative into the integral, since the limits are fixed, with the chain rule we obtain

$$\frac{\partial I(\epsilon)}{\partial \epsilon} = \int_{x_0}^{x_1} \left( \frac{\partial f}{\partial Y} \frac{dY}{d\epsilon} + \frac{\partial f}{\partial Y'} \frac{dY'}{d\epsilon} \right) dx.$$

Clearly

$$\frac{dY}{d\epsilon} = \eta(x),$$



**FIGURE 1.1** Alternative solutions example

and

$$\frac{dY'}{d\epsilon} = \eta'(x),$$

resulting in

$$\frac{\partial I(\epsilon)}{\partial \epsilon} = \int_{x_0}^{x_1} \left( \frac{\partial f}{\partial Y'} \eta(x) + \frac{\partial f}{\partial Y'^{\prime}} \eta'(x) \right) dx.$$

Integrating the second term by parts yields

$$\int_{x_0}^{x_1} \left( \frac{\partial f}{\partial Y'^{\prime}} \eta'(x) \right) dx = \frac{\partial f}{\partial Y'^{\prime}} \eta(x) \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \left( \frac{d}{dx} \frac{\partial f}{\partial Y'^{\prime}} \right) \eta(x) dx.$$

Due to the boundary conditions, the first term vanishes. With substitution and factoring the auxiliary function, the problem becomes

$$\frac{\partial I(\epsilon)}{\partial \epsilon} = \int_{x_0}^{x_1} \left( \frac{\partial f}{\partial Y'} - \frac{d}{dx} \frac{\partial f}{\partial Y'^{\prime}} \right) \eta(x) dx.$$

The extremum is achieved when  $\epsilon = 0$  as stated above, hence

$$\frac{\partial I(\epsilon)}{\partial \epsilon} \Big|_{\epsilon=0} = \int_{x_0}^{x_1} \left( \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \eta(x) dx.$$

Let us now consider the following integral:

$$\int_{x_0}^{x_1} \eta(x)F(x)dx,$$

where  $x_0 \leq x \leq x_1$  and  $F(x)$  is continuous, while  $\eta(x)$  is continuously differentiable, satisfying

$$\eta(x_0) = \eta(x_1) = 0.$$

The fundamental lemma of calculus of variations states that if for all such  $\eta(x)$

$$\int_{x_0}^{x_1} \eta(x)F(x)dx = 0,$$

then

$$F(x) = 0$$

in the whole interval.

The following proof by contradiction is from [18]. Let us assume that there exists at least one such location  $x_0 \leq \zeta \leq x_1$  where  $F(x)$  is not zero, for example,

$$F(\zeta) > 0.$$

By the condition of continuity of  $F(x)$  there must be a neighborhood of

$$\zeta - h \leq \zeta \leq \zeta + h$$

where  $F(x) > 0$ . In this case, however, the integral becomes

$$\int_{x_0}^{x_1} \eta(x)F(x)dx > 0,$$

for the right choice of  $\eta(x)$ , which contradicts the original assumption. Hence the statement of the lemma must be true.

Applying the lemma to this case results in the **Euler-Lagrange differential equation** specifying the extremum

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0.$$

## 1.2 The Legendre test

The Euler-Lagrange differential equation just introduced represents a necessary, but not sufficient, condition for the solution of the fundamental variational problem.

The alternative functional of

$$I(\epsilon) = \int_{x_0}^{x_1} f(x, Y, Y') dx,$$

may be expanded as

$$I(\epsilon) = \int_{x_0}^{x_1} f(x, y + \epsilon\eta(x), y' + \epsilon\eta'(x)) dx.$$

Assuming that the  $f$  function has continuous partial derivatives, the mean-value theorem is applicable:

$$\begin{aligned} f(x, y + \epsilon\eta(x), y' + \epsilon\eta'(x)) &= f(x, y, y') + \\ &\epsilon\left(\eta(x) \frac{\partial f(x, y, y')}{\partial y} + \eta'(x) \frac{\partial f(x, y, y')}{\partial y'}\right) + O(\epsilon^2). \end{aligned}$$

By substituting we obtain

$$\begin{aligned} I(\epsilon) &= \int_{x_0}^{x_1} f(x, y, y') dx + \\ &\epsilon \int_{x_0}^{x_1} \left(\eta(x) \frac{\partial f(x, y, y')}{\partial y} + \eta'(x) \frac{\partial f(x, y, y')}{\partial y'}\right) dx + O(\epsilon^2). \end{aligned}$$

With the introduction of

$$\delta I_1 = \epsilon \int_{x_0}^{x_1} \left(\eta(x) \frac{\partial f(x, y, y')}{\partial y} + \eta'(x) \frac{\partial f(x, y, y')}{\partial y'}\right) dx,$$

we can write

$$I(\epsilon) = I(0) + \delta I_1 + O(\epsilon^2),$$

where  $\delta I_1$  is called the first variation. The vanishing of the first variation is a necessary, but not sufficient, condition to have an extremum. To establish a sufficient condition, assuming that the function is thrice continuously differentiable, we further expand as

$$I(\epsilon) = I(0) + \delta I_1 + \delta I_2 + O(\epsilon^3).$$

Here the newly introduced second variation is

$$\begin{aligned} \delta I_2 = & \frac{\epsilon^2}{2} \int_{x_0}^{x_1} (\eta^2(x) \frac{\partial^2 f(x, y, y')}{\partial y^2} + \\ & 2\eta(x)\eta'(x) \frac{\partial^2 f(x, y, y')}{\partial y \partial y'} + \\ & \eta'^2(x) \frac{\partial^2 f(x, y, y')}{\partial y'^2}) dx. \end{aligned}$$

We now possess all the components to test for the existence of the extremum (maximum or minimum). The **Legendre test** in [7] states that if independently of the choice of the auxiliary  $\eta(x)$  function

- the Euler-Lagrange equation is satisfied,
- the first variation vanishes ( $\delta I_1 = 0$ ), and
- the second variation does not vanish ( $\delta I_2 \neq 0$ )

over the interval of integration, then the functional has an extremum. This test manifests the necessary conditions for the existence of the extremum. Specifically, the extremum will be a maximum if the second variation is negative, and conversely a minimum if it is positive. Certain similarities to the extremum evaluation of regular functions by the teaching of classical calculus are obvious.

We finally introduce the variation of the function as

$$\delta y = Y(x) - y(x) = \epsilon \eta(x),$$

and the variation of the derivative as

$$\delta y' = Y'(x) - y'(x) = \epsilon \eta'(x).$$

Based on these variations, we distinguish between the following cases:

- strong extremum occurs when  $\delta y$  is small, however,  $\delta y'$  is large, while
- weak extremum occurs when both  $\delta y$  and  $\delta y'$  are small.

On a final note: the above considerations did not ever state the finding or presence of an absolute extremum; only the local extremum in the interval of the integrand is obtained.

### 1.3 The Euler-Lagrange differential equation

Let us expand the derivative in the second term of the Euler-Lagrange differential equation as follows:

$$\frac{d}{dx} \frac{\partial f}{\partial y'} = \frac{\partial^2 f}{\partial x \partial y'} + \frac{\partial^2 f}{\partial y \partial y'} y' + \frac{\partial^2 f}{\partial y'^2} y''.$$

This demonstrates that the Euler-Lagrange equation is usually of second order.

$$\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - \frac{\partial^2 f}{\partial y \partial y'} y' - \frac{\partial^2 f}{\partial y'^2} y'' = 0.$$

The above form is also called the extended form. Consider the case when the multiplier of the second derivative term vanishes:

$$\frac{\partial^2 f}{\partial y'^2} = 0.$$

In this case  $f$  must be a linear function of  $y'$ , in the form of

$$f(x, y, y') = p(x, y) + q(x, y)y'.$$

For this form, the other derivatives of the equation are computed as

$$\frac{\partial f}{\partial y} = \frac{\partial p}{\partial y} + \frac{\partial q}{\partial y} y',$$

$$\frac{\partial f}{\partial y'} = q,$$

$$\frac{\partial^2 f}{\partial x \partial y'} = \frac{\partial q}{\partial x},$$

and

$$\frac{\partial^2 f}{\partial y \partial y'} = \frac{\partial q}{\partial y}.$$

Substituting results in the Euler-Lagrange differential equation of the form

$$\frac{\partial p}{\partial y} - \frac{\partial q}{\partial x} = 0,$$

or

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}.$$

In order to have a solution, this must be an identity, in which case there must be a function of two variables

$$u(x, y)$$

whose total differential is of the form

$$du = p(x, y)dx + q(x, y)dy = f(x, y, y')dx.$$

The functional may be evaluated as

$$I(y) = \int_{x_0}^{x_1} f(x, y, y')dx = \int_{x_0}^{x_1} du = u(x_1, y_1) - u(x_0, y_0).$$

It follows from this that the necessary and sufficient condition for the solution of the Euler-Lagrange differential equation is that the integrand of the functional be the total differential with respect to  $x$  of a certain function of both  $x$  and  $y$ .

Considering furthermore, that the Euler-Lagrange differential equation is linear with respect to  $f$ , it also follows that a term added to  $f$  will not change the necessity and sufficiency of that condition.

Another special case may be worthy of consideration. Let us assume that the integrand does not explicitly contain the  $x$  term. Then by executing the differentiations

$$\begin{aligned} & \frac{d}{dx}(y' \frac{\partial f}{\partial y'} - f) = \\ & y' \frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} y' = \\ & y' \left( \frac{d}{dx} \frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} \right) - \frac{\partial f}{\partial x}. \end{aligned}$$

With the last term vanishing in this case, the differential equation simplifies to

$$\frac{d}{dx}(y' \frac{\partial f}{\partial y'} - f) = 0.$$

Its consequence is the expression also known as Beltrami's formula:

$$y' \frac{\partial f}{\partial y'} - f = c_1, \tag{1.1}$$

where the right-hand side term is an integration constant. The classical problem of the brachistochrone, discussed in the next section, belongs to this class.

Finally, it is also often the case that the integrand does not contain the  $y$  term explicitly. Then

$$\frac{\partial f}{\partial y} = 0$$

and the differential equation has the simpler

$$\frac{d}{dx} \frac{\partial f}{\partial y'} = 0$$

form. As above, the result is

$$\frac{\partial f}{\partial y'} = c_2$$

where  $c_2$  is another integration constant. The geodesic problems, also the subject of Chapter 8, represent this type of Euler-Lagrange equations.

We can surmise that the Euler-Lagrange differential equation's general solution is of the form

$$y = y(x, c_1, c_2),$$

where the  $c_1, c_2$  are constants of integration, and are solved from the boundary conditions

$$y_0 = y(x_0, c_1, c_2)$$

and

$$y_1 = y(x_1, c_1, c_2).$$

## 1.4 Application: minimal path problems

This section deals with several classical problems to illustrate the methodology. The problem of finding the minimal path between two points in space will be addressed in different senses.

The first problem is simple geometry, the shortest geometric distance between the points. The second one is the well-known classical problem of the brachistochrone, originally posed and solved by Bernoulli. This is the path of the shortest time required to move from one point to the other under the force of gravity.

The third problem considers a minimal path in an optical sense and leads to Snell's law of reflection in optics. The fourth example finds the path of minimal kinetic energy of a particle moving under the force of gravity.

All four problems will be presented in two-dimensional space, although they may also be posed and solved in three dimensions with some more algebraic difficulty but without any additional instructional benefit.



### 1.4.1 Shortest curve between two points

First we consider the rather trivial variational problem of finding the solution of the shortest curve between two points,  $P_0, P_1$ , in the plane. The form of the problem using the arc length expression is

$$\int_{P_0}^{P_1} ds = \int_{x_0}^{x_1} \sqrt{1 + y'^2} dx = \textit{extremum}.$$

The obvious boundary conditions are the curve going through its endpoints:

$$y(x_0) = y_0,$$

and

$$y(x_1) = y_1.$$

It is common knowledge that the solution in Euclidean geometry is a straight line from point  $(x_0, y_0)$  to point  $(x_1, y_1)$ . The solution function is of the form

$$y(x) = y_0 + m(x - x_0),$$

with slope

$$m = \frac{y_1 - y_0}{x_1 - x_0}.$$

To evaluate the integral, we compute the derivative as

$$y' = m$$

and the function becomes

$$f(x, y, y') = \sqrt{1 + m^2}.$$

Since the integrand is constant, the integral is trivial

$$I(y) = \sqrt{1 + m^2} \int_{x_0}^{x_1} dx = \sqrt{1 + m^2}(x_1 - x_0).$$

The square of the functional is

$$I^2(y) = (1 + m^2)(x_1 - x_0)^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2.$$

This is the square of the distance between the two points in the plane, hence the extremum is the distance between the two points along the straight line. Despite the simplicity of the example, the connection of a geometric problem to a variational formulation of a functional is clearly visible. This will be the most powerful justification for the use of this technique.

Let us now solve the

$$\int_{x_0}^{x_1} \sqrt{1 + y'^2} dx = \textit{extremum}$$

problem via its Euler-Lagrange equation form. Note that the form of the integrand dictates the use of the extended form.

$$\frac{\partial f}{\partial y} = 0,$$

$$\frac{\partial^2 f}{\partial x \partial y'} = 0,$$

$$\frac{\partial^2 f}{\partial y \partial y'} = 0,$$

and

$$\frac{\partial^2 f}{\partial y'^2} = \frac{1}{(1 + y'^2)^{3/2}}.$$

Substituting into the extended form gives

$$\frac{1}{(1 + y'^2)^{3/2}} y'' = 0,$$

which simplifies into

$$y'' = 0.$$

Integrating twice, one obtains

$$y(x) = c_0 + c_1 x,$$

clearly the equation of a line. Substituting into the boundary conditions we obtain two equations,

$$y_0 = c_0 + c_1 x_0,$$

and

$$y_1 = c_0 + c_1 x_1.$$

The solution of the resulting linear system of equations is

$$c_0 = y_0 - c_1 x_0,$$

and

$$c_1 = \frac{y_1 - y_0}{x_1 - x_0}.$$

It is easy to reconcile that

$$y(x) = y_0 - \frac{y_1 - y_0}{x_1 - x_0} x_0 + \frac{y_1 - y_0}{x_1 - x_0} x$$

is identical to

$$y(x) = y_0 + m(x - x_0).$$

The noticeable difference between the two solutions of this problem is that using the Euler-Lagrange equation required no a priori assumption on the shape of the curve and the geometric know-how was not used. This is the case in most practical engineering applications and this is the reason for the utmost importance of the Euler-Lagrange equation.

### 1.4.2 The brachistochrone problem

The problem of the brachistochrone may be the first problem of variational calculus, already solved by Johann Bernoulli in the late 1600s. The name stands for the shortest time in Greek, indicating the origin of the problem.

The problem is elementary in a physical sense. Its goal is to find the shortest path of a particle moving in a vertical plane from a higher point to a lower point under the (only) force of gravity. The sought solution is the function  $y(x)$  with boundary conditions  $y(x_0) = y_0$  and  $y(x_1) = y_1$  where

$$P_0 = (x_0, y_0)$$

and

$$P_1 = (x_1, y_1)$$

are the starting and terminal points, respectively. Based on elementary physics considerations, the problem represents an exchange of potential energy with kinetic energy.

A moving body's kinetic energy is related to its velocity and its mass. The higher the velocity and the mass, the bigger the kinetic energy. A body can gain kinetic energy using its potential energy, and conversely, can use its kinetic energy to build up potential energy. At any point during the movement, the total energy is at equilibrium. This is the principle of Hamilton's that will be discussed in more detail in Chapter 10.

The potential energy of the particle at any  $x, y$  point during the motion is

$$E_p = mg(y_0 - y),$$

where  $m$  is the mass of the particle and  $g$  is the acceleration of gravity [11]. The kinetic energy is

$$E_k = \frac{1}{2}mv^2$$

assuming that the particle at the  $(x, y)$  point has velocity  $v$ . They are in balance as

$$E_k = E_p,$$

resulting in an expression of the velocity as

$$v = \sqrt{2g(y_0 - y)}.$$

The velocity by definition is

$$v = \frac{ds}{dt},$$

where  $s$  is the arc length of the yet unknown curve. The time required to run the length of the curve is

$$t = \int_{P_0}^{P_1} dt = \int_{P_0}^{P_1} \frac{1}{v} ds.$$

Using the arc length expression from calculus, we get

$$t = \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}}{v} dx.$$

Substituting the velocity expression yields

$$t = \frac{1}{\sqrt{2g}} \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{y_0 - y}} dx.$$

Since we are looking for the minimal time, this is a variational problem of

$$I(y) = \frac{1}{\sqrt{2g}} \int_{x_0}^{x_1} \frac{\sqrt{1 + y'^2}}{\sqrt{y_0 - y}} dx = \text{extremum}.$$

The integrand does not contain the independent variable, hence we can apply Beltrami's formula of Equation (1.1). This results in the form of

$$\frac{y'^2}{\sqrt{(y_0 - y)(1 + y'^2)}} - \frac{\sqrt{1 + y'^2}}{\sqrt{y_0 - y}} = c_0.$$

Creating a common denominator on the left-hand side produces

$$\frac{y'^2 \sqrt{y_0 - y} - \sqrt{1 + y'^2} \sqrt{(y_0 - y)(1 + y'^2)}}{\sqrt{(y_0 - y)(1 + y'^2)} \sqrt{y_0 - y}} = c_0.$$

Grouping the numerator simplifies to

$$\frac{-\sqrt{y_0 - y}}{\sqrt{(y_0 - y)(1 + y'^2)} \sqrt{y_0 - y}} = c_0.$$

Canceling and squaring results in the solution for  $y'$  as

$$y'^2 = \frac{1 - c_0^2(y_0 - y)}{c_0^2(y_0 - y)}.$$

Since

$$y'^2 = \left(\frac{dy}{dx}\right)^2,$$

the differential equation may be separated as

$$dx = \frac{\sqrt{y_0 - y}}{\sqrt{2c_1 - (y_0 - y)}} dy.$$

Here the new constant is introduced for simplicity as

$$c_1 = \frac{1}{2c_0^2}.$$

Finally  $x$  may be expressed directly by integrating

$$x = \int \frac{\sqrt{y_0 - y}}{\sqrt{2c_1 - (y_0 - y)}} dy + c_2.$$

The usual trigonometric substitution of

$$y_0 - y = 2c_1 \sin^2\left(\frac{t}{2}\right)$$

yields the integral of

$$x = 2c_1 \int \sin^2\left(\frac{t}{2}\right) dt = c_1(t - \sin(t)) + c_2,$$

where  $c_2$  is another constant of integration. Reorganizing yields

$$y = y_0 - c_1(1 - \cos(t)).$$

The final solution of the brachistochrone problem therefore is a cycloid. Figure 1.2 depicts the problem of the point moving from  $(0, 1)$  until it reaches the  $x$  axis.

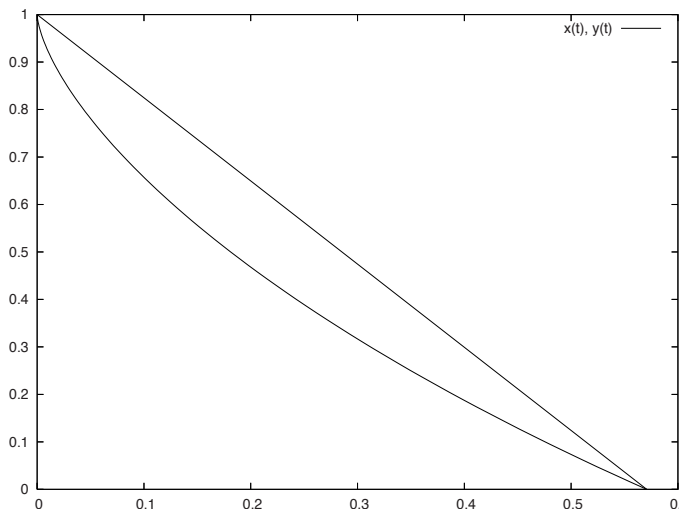
The resulting curve seems somewhat counter-intuitive, especially in view of the earlier example of the shortest geometric distance between two points in the plane and demonstrated by the straight line chord between the two points. The shortest time, however, when the speed obtained during the traversal of the interval depends on the path taken, is an entirely different matter.

The constants of the integration may be solved by substituting the boundary points. From the  $x$  equation above at  $t = 0$  we easily find

$$x = c_2 = x_0.$$

Substituting the endpoint location into the  $y$  equation, we obtain

$$y = y_0 - c_1(1 - \cos(t)) = y_1,$$



**FIGURE 1.2** Solution of the brachistochrone problem

which is inconclusive, since the time of reaching the endpoint is not known. For a simple conclusion of this discussion, let us assume that the particle reaches the endpoint at time  $t = \Pi/2$ . Then

$$c_1 = y_0 - y_1,$$

and the final solution is

$$x = x_0 + (y_0 - y_1)(t - \sin(t))$$

and

$$y = y_1(1 - \cos(t)).$$

For the case shown in Figure 1.2, the point moving from  $(0, 1)$  until it reaches the  $x$  axis, the solution curve is

$$x = (t - \sin(t))$$

and

$$y = \cos(t).$$

Another intriguing characteristic of the brachistochrone particle is that when two particles are let go from two different points of the curve they will

reach the terminal point of the curve at the same time. This is also counter-intuitive, since clearly they have different geometric distances to cover; however, since they are acting under the gravity and the slope of the curve is different at the two locations, the particle starting from a higher location gathers much bigger speed than the particle starting at a lower location.

This so-called tautochrone behavior may be proven by calculation of the time of the particles using the formula developed earlier. Evaluation of this integral between points  $(x_0, y_0)$  and  $(x_1, y_1)$  as well as between  $(x_2, y_2)$  and  $(x_1, y_1)$  (where  $(x_2, y_2)$  lies on the solution curve anywhere between the starting and terminal point) will result in the same time.

Hence the brachistochrone problem may also be posed with a specified terminal point and a variable starting point, leading to the class of variational problems with open boundary, subject of Section 1.5.

### 1.4.3 Fermat's principle

Fermat's principle states that light traveling through an inhomogeneous medium chooses the path of minimal optical length. The path's optimal length depends on the speed of light in the medium, which is defined as a continuous function of

$$c = c(y),$$

where  $y$  is the vertical component of the path. Here

$$c(y) = \frac{ds}{dt},$$

as the derivative of the length of the path and its inverse will be in the variational form. The time required to cover the distance between two points is

$$t = \int \frac{1}{c(y)} ds.$$

The problem is now posed as a variational problem of

$$I(y) = \int_{(x_1, y_1)}^{(x_2, y_2)} \frac{ds}{c(y)}.$$

Substituting the arc length

$$\int_{x_1}^{x_2} \frac{\sqrt{1 + y'^2}}{c(y)} dx = \text{extremum},$$

with boundary conditions given at the two points  $P_1, P_2$ .

$$y(x_1) = y_1; y(x_2) = y_2.$$

The functional does not contain the  $x$  term explicitly, allowing the use of Beltrami's formula of Equation (1.1) and resulting in the simplified form of

$$y' \frac{\partial f}{\partial y'} - f = k_1$$

where  $k_1$  is a constant of integration and its notation is chosen to distinguish from the speed of light value  $c$ . Substituting  $f$ , differentiating and simplifying yields

$$\frac{1}{c(y)\sqrt{1+y'^2}} = -k_1.$$

Reordering and separating results

$$\int dx = \pm k_1 \int \frac{c(y)}{\sqrt{1-k_1^2 c^2(y)}} dy.$$

Depending on the particular model of the speed of light in the medium, the result varies. In the case of the inhomogeneous optical medium consisting of two homogeneous media in which the speed of light is piecewise constant, the result is the well-known Snell's law describing the scenario of the breaking path of light at the water's surface.

Assume the speed of light is  $c_1$  between points  $P_1$  and  $P_0$  and  $c_2$  between points  $P_0$  and  $P_2$ , both constant in their respective medium. The boundary between the two media is represented by

$$P_0(x, y_0),$$

where the notation signifies the fact that the  $x$  location of the light ray is not known yet. The known  $y_0$  location specifies the distance of the points in the two separate media from the boundary.

Then the time to run the full path between  $P_1$  and  $P_2$  is simply

$$t = \frac{\sqrt{(x-x_1)^2 + (y_0-y_1)^2}}{c_1} + \frac{\sqrt{(x_2-x)^2 + (y_2-y_0)^2}}{c_2}.$$

The minimum of this is simply obtained by classical calculus as

$$\frac{dt}{dx} = 0,$$

or

$$\frac{x-x_1}{c_1\sqrt{(x-x_1)^2 + (y_0-y_1)^2}} - \frac{x_2-x}{c_2\sqrt{(x_2-x)^2 + (y_2-y_0)^2}} = 0.$$

The solution of this equation yields the  $x$  location of the ray crossing the boundary, and produces the well-known Snell's law of



$$\frac{\sin\phi_1}{c_1} = \frac{\sin\phi_2}{c_2},$$

where the angles are measured with respect to the normal of the boundary between the two media. The preceding work generalizes to multiples of homogeneous media, which is a practical application in lens systems of optical machinery.

#### 1.4.4 Particle moving in the gravitational field

The motion of a particle moving in the gravitational field of the Earth is computed based on the principle of least action. The principle, a sub-case of Hamilton's principle, has been known for several hundred years, and was first proven by Euler. The principle states that a particle under the influence of a gravitational field moves on a path along which the kinetic energy is minimal. As such, it is a variational problem of

$$I = 2 \int E_k dt = \text{extremum}.$$

Here the multiplier is introduced for computational convenience. The kinetic energy is expressed as

$$E_k = \frac{1}{2}mv^2,$$

where  $v$  is the velocity of the particle. Substituting

$$v dt = ds$$

and

$$ds = \sqrt{1 + (y')^2} dx$$

the functional may be written as

$$I = m \int v \sqrt{1 + (y')^2} dx.$$

Since the gravitational field induces the particle's motion, its speed is related to its height known from elementary physics as

$$v^2 = u^2 - 2gy,$$

where  $u$  is an initial speed with yet undefined direction. Substituting into the functional yields

$$I = m \int \sqrt{u^2 - 2gy} \sqrt{1 + (y')^2} dx = \text{extremum}.$$

Since the functional does not contain  $x$  explicitly, we can use Beltrami's formula of Equation (1.1), resulting in the Euler-Lagrange equation of

$$\frac{\sqrt{u^2 - 2gy}}{\sqrt{1 + y'^2}} = c_1,$$

where  $c_1$  is an arbitrary constant. Expressing  $y'$ , separating and integrating yields

$$c_1^2(u^2 - 2gy - c_1^2) = g^2(x - c_2)^2,$$

with  $c_2$  being another constant of integration. Reordering yields the well-known parabolic trajectory of

$$y = \frac{u^2 - c_1^2}{2g} - \frac{g}{2c_1^2}(x - c_2)^2.$$

The resolution of the constants may be by giving boundary conditions of the initial location and velocity of the particle. The constant  $c_1$  is related to the latter and the constant  $c_2$  is related to the location. Assuming the origin as initial location and an angle  $\alpha$  of the initial velocity  $u$  with respect to the horizontal axis, the formula may be simplified into

$$y = x \tan(\alpha) - \frac{gx^2}{2u^2 \cos^2(\alpha)}.$$

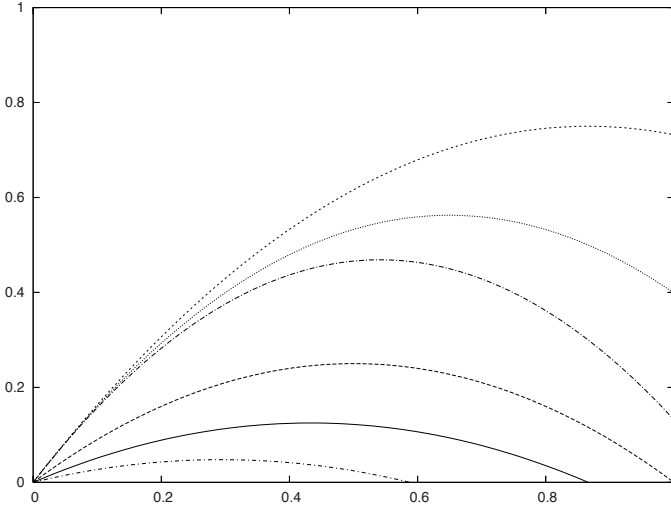
Figure 1.3 demonstrates the path of the particle. The upper three curves show the path with a 60-degree angle of the initial velocity and with different magnitudes. The lower three curves demonstrate the paths obtained by the same magnitude (10 units), but different angles of the initial velocity. For visualization purposes the gravity constant was chosen to be 10 units as well.

A similarity between the four problems of this section is apparent. This recognition is a very powerful aspect of variational calculus. There are many instances in engineering applications when one physical problem may be solved in an analogous form using another principle. The common variational formulation of both problems is the key to such recognition in most cases.

## 1.5 Open boundary variational problems

Let us consider the variational problem of Section 1.1:

$$I(y) = \int_{x_0}^{x_1} f(x, y, y') dx,$$



**FIGURE 1.3** Trajectory of particle

with boundary condition

$$y(x_0) = y_0.$$

Let the boundary condition at the upper end be undefined. We introduce an auxiliary function  $\eta(x)$  that in this case only satisfies

$$\eta(x_0) = 0.$$

The extremum in this case is obtained from the same concept as earlier:

$$\frac{\partial I(\epsilon)}{\partial \epsilon} \Big|_{\epsilon=0} = \int_{x_0}^{x_1} \left( \frac{\partial f}{\partial y} \eta(x) + \frac{d}{dx} \frac{\partial f}{\partial y'} \eta'(x) \right) dx = 0,$$

while recognizing the fact that  $x_1$  is undefined. Integrating by parts and considering the one-sided boundary condition posed on the auxiliary function yields

$$\frac{\partial I(\epsilon)}{\partial \epsilon} \Big|_{\epsilon=0} = \frac{\partial f}{\partial y'} \Big|_{x=x_1} \eta(x_1) + \int_{x_0}^{x_1} \left( \left( \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \eta'(x) \right) dx = 0.$$

The extremum is obtained when the Euler-Lagrange equation of

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0$$

along with the given boundary condition of

$$y(x_0) = y_0$$

is satisfied, in addition to obeying the constraint of

$$\frac{\partial f}{\partial y'} \Big|_{x=x_1} = 0.$$

Similar arguments may be applied when the starting point is open. This problem is the predecessor of the more generic constrained variational problems, the topic of the next chapter.



# 2

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## *Constrained variational problems*

The boundary values applied in the prior discussion may also be considered as constraints. The subject of this chapter is to generalize the constraint concept in two senses. The first is to allow more difficult, algebraic boundary conditions, and the second is to allow constraints imposed on the interior of the domain as well.

---

### 2.1 Algebraic boundary conditions

There is the possibility of defining the boundary condition at one end of the integral of the variational problem with an algebraic constraint. Let the

$$\int_{x_0}^{x_1} f(x, y, y') dx = \textit{extremum}$$

variational problem subject to the customary boundary condition

$$y(x_0) = y_0,$$

on the lower end, and on the upper end an algebraic condition of the following form is given:

$$g(x, y) = 0.$$

We again consider an alternative solution of the form

$$Y(x) = y(x) + \epsilon\eta(x).$$

The given boundary condition in this case is

$$\eta(x_0) = 0.$$

Then, following [9], the intersection of the alternative solution and the algebraic curve is

$$X_1 = X_1(\epsilon)$$

and

$$Y_1 = Y_1(\epsilon).$$

The notation is to distinguish from the fixed boundary condition values given via  $x_1, y_1$ . Therefore the algebraic condition is

$$g(X_1, Y_1) = 0.$$

This must be true for any  $\epsilon$ , hence applying the chain rule yields

$$\frac{dg}{d\epsilon} = \frac{\partial g}{\partial X_1} \frac{dX_1}{d\epsilon} + \frac{\partial g}{\partial Y_1} \frac{dY_1}{d\epsilon} = 0. \quad (2.1)$$

Since

$$Y_1 = y(X_1) + \epsilon\eta(X_1),$$

we expand the last derivative of the second term of Equation (2.1) as

$$\frac{dY_1}{d\epsilon} = \frac{dy}{dx}\Big|_{x=X_1} \frac{dX_1}{d\epsilon} + \eta(X_1) + \epsilon \frac{d\eta}{dx}\Big|_{x=X_1} \frac{dX_1}{d\epsilon}.$$

Substituting into Equation (2.1) results in

$$\frac{dg}{d\epsilon} = \frac{\partial g}{\partial X_1} \frac{dX_1}{d\epsilon} + \frac{\partial g}{\partial Y_1} \left( \frac{dy}{dx}\Big|_{x=X_1} \frac{dX_1}{d\epsilon} + \eta(X_1) + \epsilon \frac{d\eta}{dx}\Big|_{x=X_1} \frac{dX_1}{d\epsilon} \right) = 0.$$

Since  $(X_1, Y_1)$  becomes  $(x_1, y_1)$  when  $\epsilon = 0$ ,

$$\frac{dX_1}{d\epsilon}\Big|_{\epsilon=0} = - \frac{\eta(x_1) \frac{\partial g}{\partial y}\Big|_{y=y_1}}{\frac{\partial g}{\partial x}\Big|_{x=x_1} + \frac{\partial g}{\partial y}\Big|_{y=y_1} \frac{dy}{dx}\Big|_{x=x_1}}. \quad (2.2)$$

We now consider the variational problem of

$$I(\epsilon) = \int_{x_0}^{X_1} f(x, Y, Y') dx.$$

The derivative of this is

$$\frac{\partial I(\epsilon)}{\partial \epsilon} = \frac{dX_1}{d\epsilon} f\Big|_{x=X_1} + \int_{x_0}^{X_1} \left( \frac{\partial f}{\partial Y} \eta + \frac{\partial f}{\partial Y'} \eta' \right) dx.$$

Integrating by parts and taking  $\epsilon = 0$  yields

$$\frac{\partial I(\epsilon)}{\partial \epsilon}\Big|_{\epsilon=0} = \frac{dX_1}{d\epsilon}\Big|_{\epsilon=0} f\Big|_{x=x_1} + \frac{\partial f}{\partial y'}\Big|_{x=x_1} \eta(x_1) + \int_{x_0}^{x_1} \left( \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \eta dx.$$

Substituting the first expression with Equation (2.2) results in

$$\left( \frac{\partial f}{\partial y'}\Big|_{x=x_1} - \frac{\frac{\partial g}{\partial y}\Big|_{y=y_1} f\Big|_{x=x_1}}{\frac{\partial g}{\partial x}\Big|_{x=x_1} + \frac{\partial g}{\partial y}\Big|_{y=y_1} \frac{dy}{dx}\Big|_{x=x_1}} \right) \eta(x_1) + \int_{x_0}^{x_1} \left( \frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \eta dx = 0.$$

Due to the fundamental lemma of calculus of variations, to find the constrained variational problem's extremum the Euler-Lagrange differential equation of

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0,$$

with the given boundary condition

$$y(x_0) = y_0,$$

and the algebraic constraint condition of the form

$$\frac{\partial f}{\partial y'} \Big|_{x=x_1} = \frac{\frac{\partial g}{\partial y} \Big|_{y=y_1} f \Big|_{x=x_1}}{\frac{\partial g}{\partial x} \Big|_{x=x_1} + \frac{\partial g}{\partial y} \Big|_{y=y_1} \frac{dy}{dx} \Big|_{x=x_1}}$$

all need to be satisfied.

## 2.2 Lagrange's solution

We now further generalize the variational problem and impose both boundary conditions as well as an algebraic condition on the whole domain as follows:

$$I(y) = \int_{x_0}^{x_1} f(x, y, y') dx = \textit{extremum},$$

with

$$y(x_0) = y_0, y(x_1) = y_1,$$

while

$$J(y) = \int_{x_0}^{x_1} g(x, y, y') dx = \textit{constant}.$$

Following the earlier established pattern, we introduce an alternative solution function, at this time, however, with two auxiliary functions as

$$Y(x) = y(x) + \epsilon_1 \eta_1(x) + \epsilon_2 \eta_2(x).$$

Here the two auxiliary functions are arbitrary and both satisfy the boundary conditions:

$$\eta_1(x_0) = \eta_1(x_1) = \eta_2(x_0) = \eta_2(x_1) = 0.$$



Substituting these into the integrals gives

$$I(Y) = \int_{x_0}^{x_1} f(x, Y, Y') dx,$$

and

$$J(Y) = \int_{x_0}^{x_1} g(x, Y, Y') dx.$$

Lagrange's ingenious solution is to tie the two integrals together with a yet unknown multiplier (now called the Lagrange multiplier) as follows:

$$I(\epsilon_1, \epsilon_2) = I(Y) + \lambda J(Y) = \int_{x_0}^{x_1} h(x, Y, Y') dx,$$

where

$$h(x, y, y') = f(x, y, y') + \lambda g(x, y, y').$$

The condition to solve this variational problem is

$$\frac{\partial I}{\partial \epsilon_i} = 0$$

when

$$\epsilon_i = 0; i = 1, 2.$$

The derivatives are of the form

$$\frac{\partial I}{\partial \epsilon_i} = \int_{x_0}^{x_1} \left( \frac{\partial h}{\partial Y} \eta_i + \frac{\partial h}{\partial Y'} \eta'_i \right) dx.$$

The extremum is obtained when

$$\frac{\partial I}{\partial \epsilon_i} \Big|_{\epsilon_i=0, i=1,2} = \int_{x_0}^{x_1} \left( \frac{\partial h}{\partial Y} \eta_i + \frac{\partial h}{\partial Y'} \eta'_i \right) dx = 0.$$

Considering the boundary conditions and integrating by parts yields

$$\int_{x_0}^{x_1} \left( \frac{\partial h}{\partial y} - \frac{d}{dx} \frac{\partial h}{\partial y'} \right) \eta_i dx = 0,$$

which, due to the fundamental lemma of calculus of variations, results in the relevant Euler-Lagrange differential equation

$$\frac{\partial h}{\partial y} - \frac{d}{dx} \frac{\partial h}{\partial y'} = 0.$$

This equation contains three undefined coefficients: the two coefficients of integration satisfying the boundary conditions and the Lagrange multiplier, enforcing the constraint.

## 2.3 Application: iso-perimetric problems

Iso-perimetric problems use a given perimeter of a certain object as the constraint of some variational problem. The perimeter may be a curve in the two-dimensional case, as in the example of the next section. It may also be the surface of a certain body, in the three-dimensional case.

### 2.3.1 Maximal area under curve with given length

This problem is conceptually very simple, but useful to illuminate the process just established. It is also a very practical problem with more difficult geometries involved. Here we focus on the simple case of finding the curve of given length between two points in the plane. Without restricting the generality of the discussion, we'll position the two points on the  $x$  axis in order to simplify the arithmetic.

The given points are  $(x_0, 0)$  and  $(x_1, 0)$  with  $x_0 < x_1$ . The area under any curve going from the start point to the endpoint in the upper half-plane is

$$I(y) = \int_{x_0}^{x_1} y dx.$$

The constraint of the given length  $L$  is presented by the equation

$$J(y) = \int_{x_0}^{x_1} \sqrt{1 + y'^2} dx = L.$$

The Lagrange multiplier method brings the function

$$h(x, y, y') = y(x) + \lambda \sqrt{1 + y'^2}.$$

The constrained variational problem is

$$I(y) = \int_{x_0}^{x_1} h(x, y, y') dx$$

whose Euler-Lagrange equation becomes

$$1 - \lambda \frac{d}{dx} \frac{y'}{\sqrt{1 + y'^2}} = 0.$$

Integration yields

$$\frac{\lambda y'}{\sqrt{1 + y'^2}} = x - c_1.$$

First we separate the variables

$$dy = \pm \frac{x - c_1}{\sqrt{\lambda^2 - (x - c_1)^2}} dx,$$

and integrate again to produce

$$y(x) = \pm \sqrt{\lambda^2 - (x - c_1)^2} + c_2.$$

It is easy to reorder this into

$$(x - c_1)^2 + (y - c_2)^2 = \lambda^2,$$

which is the equation of a circle. Since the two given points are on the  $x$  axis, the center of the circle must lie on the perpendicular bisector of the chord, which implies that

$$c_1 = \frac{x_0 + x_1}{2}.$$

To solve for the value of the Lagrange multiplier and the other constant, we consider that the circular arc between the two points is the given length:

$$L = \lambda\theta,$$

where  $\theta$  is the angle of the arc. The angle is related to the remaining constant as

$$2\Pi - \theta = \text{atan}\left(\frac{x_1 - x_0}{2c_2}\right).$$

The two equations may be simultaneously satisfied with

$$\theta = \Pi,$$

resulting in the shape being a semi-circle. This yields the solutions of

$$c_2 = 0$$

and

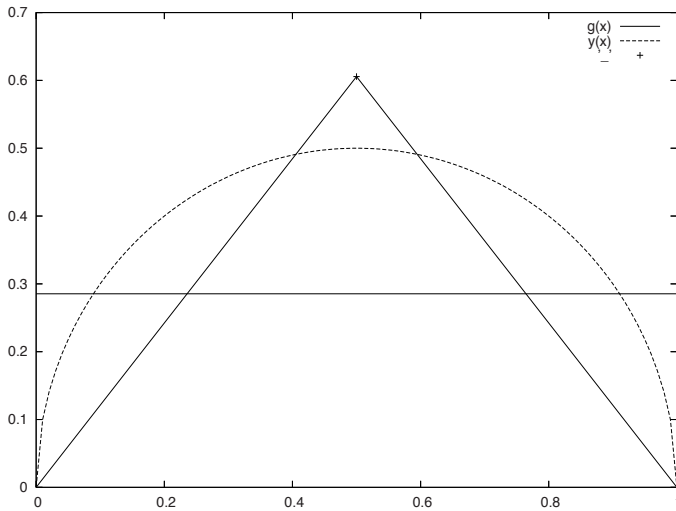
$$\lambda = \frac{L}{\pi}.$$

The final solution function in implicit form is

$$\left(x - \frac{x_0 + x_1}{2}\right)^2 + y^2 = \left(\frac{L}{\pi}\right)^2,$$

or explicitly

$$y(x) = \sqrt{\left(\frac{L}{\pi}\right)^2 - \left(x - \frac{x_0 + x_1}{2}\right)^2}.$$



**FIGURE 2.1** Maximum area under curves

It is simple to verify that the solution produces the extremum of the original variational problem.

Figure 2.1 visibly demonstrates the phenomenon with three curves of equal length ( $\pi/2$ ) over the same interval. None of the solid curves denoted by  $g(x)$ , the triangle, or the rectangle cover as much area as the semi-circle  $y(x)$  marked by the dashed lines.

### 2.3.2 Optimal shape of curve of given length under gravity

Another constrained variational problem, whose final result is often used in engineering, is the rope hanging under its weight. The practical importance of the problem regarding power lines and suspended cables is well-known. Here we derive the solution of this problem from a variational origin.

A body in a force field is in static equilibrium when its potential energy has a stationary value. Furthermore, if the stationary value is a minimum, then the body is in stable equilibrium. This is also known as principle of minimum potential energy.

Assume a body of a homogeneous cable with a given weight per unit length of  $\rho = \text{constant}$ , and suspension point locations of

$$P_0 = (x_0, y_0),$$

and

$$P_1 = (x_1, y_1).$$

These constitute the boundary conditions. A constraint is also given on the length of the curve:  $L$ . The potential energy of the cable is

$$E_p = \int_{P_0}^{P_1} \rho y ds$$

where  $y$  is the height of the infinitesimal arc segment above the horizontal base line and  $\rho ds$  is its weight. Using the arc length formula we obtain

$$E_p = \rho \int_{x_0}^{x_1} y \sqrt{1 + y'^2} dx.$$

The principle of minimal potential energy dictates that the equilibrium position of the cable is the solution of the variational problem of

$$I(y) = \rho \int_{x_0}^{x_1} y \sqrt{1 + y'^2} dx = \text{extremum},$$

under boundary conditions

$$y(x_0) = y_0; y(x_1) = y_1$$

and constraint of

$$\int_{x_0}^{x_1} \sqrt{1 + y'^2} dx = L.$$

Introducing the Lagrange multiplier and the constrained function

$$h(y) = \rho y \sqrt{1 + y'^2} + \lambda \sqrt{1 + y'^2},$$

the Euler-Lagrange differential equation of the problem after the appropriate differentiations becomes

$$\rho \sqrt{1 + y'^2} - \frac{d}{dx} \frac{(\rho y + \lambda) y'}{\sqrt{1 + y'^2}} = 0.$$

Some algebraic activity, which does not add anything to the discussion, and hence is not detailed, yields

$$(\rho y + \lambda) \left( \frac{y'^2}{\sqrt{1 + y'^2}} - \sqrt{1 + y'^2} \right) = c_1,$$

where the right-hand side is a constant of the integration. Another integration results in the solution of the so-called catenary curve

$$y = -\frac{\lambda}{\rho} - \frac{c_1}{\rho} \cosh\left(\frac{\rho(x - c_2)}{c_1}\right),$$

with  $c_2$  being another constant of integration. The constants of integration may be determined by the boundary conditions albeit the calculation, due to the presence of the hyperbolic function, is rather tedious. Let us consider the specific case of the suspension points being at the same height and symmetric with respect to the origin. This is a typical engineering scenario for the span of suspension cables. This results in the following boundary conditions:

$$P_0 = (x_0, y_0) = (-s, h)$$

and

$$P_1 = (x_1, y_1) = (s, h).$$

Without the loss of the generality, we can consider unit weight ( $\rho = 1$ ) and by substituting above boundary conditions we obtain

$$h + \lambda = c_1 \cosh\left(\frac{-s + c_2}{c_1}\right) = c_1 \cosh\left(\frac{s + c_2}{c_1}\right).$$

This implies that

$$c_2 = 0.$$

The value of the second coefficient is solved by adhering to the length constraint. Integrating the constraint equation yields

$$L = 2c_1 \sinh\left(\frac{s}{c_1}\right)$$

whose only unknown is the integration constant  $c_1$ . This problem is not solvable by analytic means; however, it can be solved by an iterative procedure numerically by considering the unknown coefficient as a variable:

$$c_1 = x,$$

and intersecting the curve

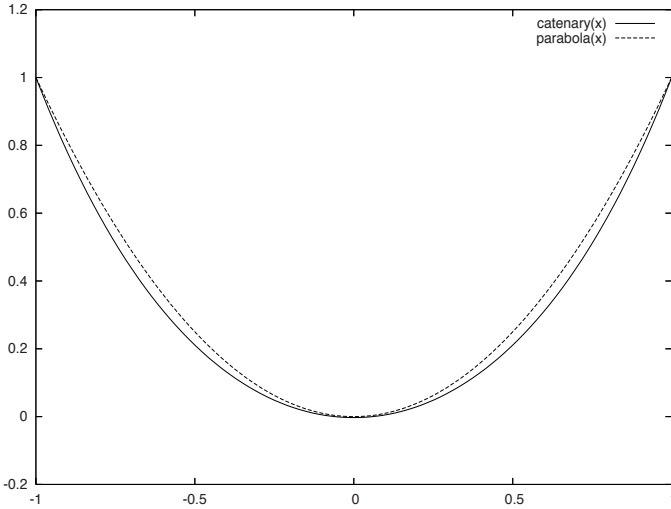
$$y = x \sinh\left(\frac{s}{x}\right)$$

and the horizontal line

$$y = \frac{L}{2}.$$

The minimal cable length must exceed the width of the span, hence we expect the cable to have some slack. Then, for example, using

$$L = 3s$$



**FIGURE 2.2** The catenary curve

will result in an approximate solution of

$$c_1 = 0.6175.$$

Clearly, depending on the length of the cable between similarly posted suspension locations, different catenary curves may be obtained.

The Lagrange multiplier may finally be resolved by the expression

$$\lambda = c_1 \cosh\left(\frac{s}{c_1}\right) - h.$$

Assuming a cable suspended with a unit half-span ( $s = 1$ ) and from unit height ( $h = 1$ ) and length of three times the half-span ( $L = 3$ ), the value of the Lagrange multiplier becomes

$$\lambda = 0.6175 \cosh\left(\frac{1}{0.6175}\right) - 1 = 0.6204.$$

The final catenary solution curve, shown with a solid line in Figure 2.2, is represented by

$$y = 0.6175 \cosh\left(\frac{1}{0.6175}\right) - 0.6204.$$

For comparison purposes, the figure also shows a parabola with dashed lines, representing an approximation of the catenary and obeying the same boundary conditions.

## 2.4 Closed-loop integrals

As a final topic in this chapter, we briefly view variational problems posed in terms of closed-loop integrals, such as

$$I = \oint f(x, y, y') dx = \text{extremum},$$

subject to the constraint of

$$J = \oint g(x, y, y') dx.$$

Note that there are no boundary points of the path given since it is a closed loop. The substitution of

$$x = a \cos(t), \quad y = a \sin(t),$$

changes the problem to the conventional form of

$$I = \int_{t_0}^{t_1} F(x, y, \dot{x}, \dot{y}) dt,$$

subject to

$$J = \int_{t_0}^{t_1} G(x, y, \dot{x}, \dot{y}) dt.$$

The arbitrary  $t_0$  and the specific  $t_1 = t_0 + 2\pi$  boundary points clearly cover a complete loop.





# 3

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## Multivariate functionals

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### 3.1 Functionals with several functions

The variational problem of multiple dependent variables is posed as

$$I(y_1, y_2, \dots, y_n) = \int_{x_0}^{x_1} f(x, y_1, y_2, \dots, y_n, y'_1, y'_2, \dots, y'_n) dx$$

with a pair of boundary conditions given for all functions:

$$y_i(x_0) = y_{i,0}$$

and

$$y_i(x_1) = y_{i,1}$$

for each  $i = 1, 2, \dots, n$ . The alternative solutions are:

$$Y_i(x) = y_i(x) + \epsilon_i \eta_i(x); i = 1, \dots, n$$

with all the arbitrary auxiliary functions obeying the conditions:

$$\eta_i(x_0) = \eta_i(x_1) = 0.$$

The variational problem becomes

$$I(\epsilon_1, \dots, \epsilon_n) = \int_{x_0}^{x_1} f(x, \dots, y_i + \epsilon_i \eta_i, \dots, y'_i + \epsilon_i \eta'_i, \dots) dx,$$

whose derivative with respect to the auxiliary variables is

$$\frac{\partial I}{\partial \epsilon_i} = \int_{x_0}^{x_1} \frac{\partial f}{\partial \epsilon_i} dx = 0.$$

Applying the chain rule we get

$$\frac{\partial f}{\partial \epsilon_i} = \frac{\partial f}{\partial Y_i} \frac{\partial Y_i}{\partial \epsilon_i} + \frac{\partial f}{\partial Y'_i} \frac{\partial Y'_i}{\partial \epsilon_i} = \frac{\partial f}{\partial Y_i} \eta_i + \frac{\partial f}{\partial Y'_i} \eta'_i.$$

Substituting into the variational equation yields, for  $i = 1, 2, \dots, n$ :

$$I(\epsilon_i) = \int_{x_0}^{x_1} \left( \frac{\partial f}{\partial Y_i} \eta_i + \frac{\partial f}{\partial Y'_i} \eta'_i \right) dx.$$

Integrating by parts and exploiting the alternative function form results in

$$I(\epsilon_i) = \int_{x_0}^{x_1} \eta_i \left( \frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial y'_i} \right) dx.$$

To reach the extremum, based on the fundamental lemma, we need the solution of a set of  $n$  Euler-Lagrange equations of the form

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial y'_i} = 0; i = 1, \dots, n.$$

### 3.2 Variational problems in parametric form

Most of the discussion heretofore was focused on functions in explicit form. The concepts also apply to problems posed in parametric form. The explicit form variational problem of

$$I(y) = \int_{x_0}^{x_1} f(x, y, y') dx$$

may be reformulated with the substitutions

$$x = u(t), \quad y = v(t).$$

The parametric variational problem becomes of the form

$$I(x, y) = \int_{t_0}^{t_1} f(x, y, \frac{\dot{y}}{\dot{x}}) \dot{x} dt,$$

or

$$I(x, y) = \int_{t_0}^{t_1} F(t, x, y, \dot{x}, \dot{y}) dt.$$

The Euler-Lagrange differential equation system for this case becomes

$$\frac{\partial F}{\partial x} - \frac{d}{dt} \frac{\partial F}{\partial \dot{x}} = 0,$$

and

$$\frac{\partial F}{\partial y} - \frac{d}{dt} \frac{\partial F}{\partial \dot{y}} = 0.$$

It is proven in [9] that an explicit variational problem is invariant under parameterization. In other words, independently of the algebraic form of the parameterization, the same explicit solution will be obtained.

Parametrically given problems may be considered as functionals with several functions. As an example, we consider the following twice differentiable

functions

$$x = x(t), y = y(t), z = z(t).$$

The variational problem in this case is presented as

$$I(x, y, z) = \int_{t_0}^{t_1} f(t, x, y, z, \dot{x}, \dot{y}, \dot{z}) dx.$$

Here the independent variable  $t$  is the parameter, and there are three dependent variables :  $x, y, z$ . Applying the steps just explained for this specific case results in the system of Euler-Lagrange equations

$$\begin{aligned} \frac{\partial f}{\partial x} - \frac{d}{dt} \frac{\partial f}{\partial \dot{x}} &= 0, \\ \frac{\partial f}{\partial y} - \frac{d}{dt} \frac{\partial f}{\partial \dot{y}} &= 0, \end{aligned}$$

and

$$\frac{\partial f}{\partial z} - \frac{d}{dt} \frac{\partial f}{\partial \dot{z}} = 0.$$

The most practical applications of this case are variational problems in three-dimensional space, presented in parametric form. This is usual in many geometry problems and will be exploited in Chapters 8 and 9.

### 3.3 Functionals with two independent variables

All our discussions so far were confined to a single integral of the functional. The next step of generalization is to allow a functional with multiple independent variables. The simplest case is that of two independent variables, and this will be the vehicle to introduce the process. The problem is of the form

$$I(z) = \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y, z, z_x, z_y) dx dy = \text{extremum}.$$

Here the derivatives are

$$z_x = \frac{\partial z}{\partial x}$$

and

$$z_y = \frac{\partial z}{\partial y}.$$

The alternative solution is also a function of two variables

$$Z(x, y) = z(x, y) + \epsilon \eta(x, y).$$

The now familiar process emerges as

$$I(\epsilon) = \int_{y_0}^{y_1} \int_{x_0}^{x_1} f(x, y, Z, Z_x, Z_y) dx dy = \text{extremum}.$$

The extremum is obtained via the derivative

$$\frac{\partial I}{\partial \epsilon} = \int_{y_0}^{y_1} \int_{x_0}^{x_1} \frac{\partial f}{\partial \epsilon} dx dy.$$

Differentiating and substituting yields

$$\frac{\partial I}{\partial \epsilon} = \int_{y_0}^{y_1} \int_{x_0}^{x_1} \left( \frac{\partial f}{\partial Z} \eta + \frac{\partial f}{\partial Z_x} \eta_x + \frac{\partial f}{\partial Z_y} \eta_y \right) dx dy.$$

The extremum is reached when  $\epsilon = 0$ :

$$\left. \frac{\partial I}{\partial \epsilon} \right|_{\epsilon=0} = \int_{y_0}^{y_1} \int_{x_0}^{x_1} \left( \frac{\partial f}{\partial z} \eta + \frac{\partial f}{\partial z_x} \eta_x + \frac{\partial f}{\partial z_y} \eta_y \right) dx dy = 0.$$

Applying Green's identity for the second and third terms produces

$$\int_{y_0}^{y_1} \int_{x_0}^{x_1} \left( \frac{\partial f}{\partial z} - \frac{\partial}{\partial x} \frac{\partial f}{\partial z_x} - \frac{\partial}{\partial y} \frac{\partial f}{\partial z_y} \right) \eta dx dy + \int_{\partial D} \left( \frac{\partial f}{\partial z_x} \frac{dy}{ds} - \frac{\partial f}{\partial z_y} \frac{dx}{ds} \right) \eta ds = 0.$$

Here  $\partial D$  is the boundary of the domain of the problem and the second integral vanishes by the definition of the auxiliary function. Due to the fundamental lemma of calculus of variations, the Euler-Lagrange differential equation becomes

$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial x} \frac{\partial f}{\partial z_x} - \frac{\partial}{\partial y} \frac{\partial f}{\partial z_y} = 0.$$

### 3.4 Application: minimal surfaces

Minimal surfaces occur in intriguing applications. For example, soap films spanned over various types of wire loops intrinsically attain such shapes, no matter how difficult the boundary curve is. Various biological cell interactions also manifest similar phenomena.

From a differential geometry point of view a minimal surface is a surface for which the mean curvature of the form

$$\kappa_m = \frac{\kappa_1 + \kappa_2}{2}$$

vanishes, where  $\kappa_1$  and  $\kappa_2$  are the principal curvatures. A subset of minimal surfaces are the surfaces of minimal area, and surfaces of minimal area passing

through a closed space curve are minimal surfaces. Finding minimal surfaces is called the problem of Plateau.

We seek the surface of minimal area with equation

$$z = f(x, y), (x, y) \in D,$$

with a closed-loop boundary curve

$$g(x, y, z) = 0; (x, y) \in \partial D.$$

The boundary condition represents a three-dimensional curve defined over the perimeter of the domain. The curve may be piecewise differentiable, but continuous and forms a closed loop, a Jordan curve.

The corresponding variational problem is

$$I(z) = \int \int_D \sqrt{1 + \frac{\partial z^2}{\partial x} + \frac{\partial z^2}{\partial y}} \, dxdy = \text{extremum.}$$

subject to the constraint of the boundary condition above. The Euler-Lagrange equation for this case is of the form

$$-\frac{\partial}{\partial x} \frac{\frac{\partial z}{\partial x}}{\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}} - \frac{\partial}{\partial y} \frac{\frac{\partial z}{\partial y}}{\sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}} = 0.$$

After considerable algebraic work, this equation becomes

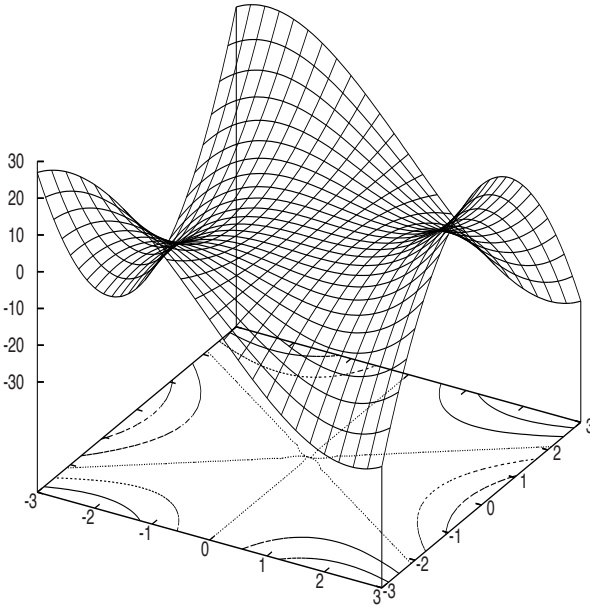
$$\left(1 + \left(\frac{\partial z}{\partial y}\right)^2\right) \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} \frac{\partial^2 z}{\partial x \partial y} + \left(1 + \left(\frac{\partial z}{\partial x}\right)^2\right) \frac{\partial^2 z}{\partial y^2} = 0.$$

This is the differential equation of minimal surfaces, originally obtained by Lagrange himself. The equation is mainly of verification value as this is one of the most relevant examples for the need of a numerical solution. Most of the problems of finding minimal surfaces are solved by Ritz type methods, the subject of Chapter 7.

The simplest solutions for such problems are the so-called saddle surfaces, such as, for example, shown in Figure 3.1, whose equation is

$$z = x^3 - 2xy^2.$$

It is easy to verify that this satisfies the equation. The figure also shows the level curves of the surface projected to the  $x - y$  plane. The straight lines on the plane correspond to geodesic paths, a subject of detailed discussion in Chapter 8. It is apparent that the  $x = 0$  planar cross-section of the surface is the  $z = 0$  line in the  $x - y$  plane, as indicated by the algebra. The intersection with the  $y = 0$  plane produces the  $z = x^3$  curve, again in full adherence to



**FIGURE 3.1** Saddle surface

the equation.

When a minimal surface is sought in a parametric form

$$\underline{r} = x(u, v)\underline{i} + y(u, v)\underline{j} + z(u, v)\underline{k}.$$

the variational problem becomes

$$I(\underline{r}) = \int \int_D \sqrt{EF - G^2} dA,$$

where the so-called first fundamental quantities are defined as

$$E(u, v) = (\underline{r}'_u)^2,$$

$$F(u, v) = \underline{r}'_u \underline{r}'_v,$$

and

$$G(u, v) = (\underline{r}'_v)^2.$$

The solution may be obtained from the differential equation

$$\frac{\partial}{\partial u} \frac{F \underline{r}'_u - G \underline{r}'_v}{\sqrt{EF - G^2}} + \frac{\partial}{\partial v} \frac{E \underline{r}'_v - G \underline{r}'_u}{\sqrt{EF - G^2}} = 0.$$

Finding minimal surfaces for special boundary arrangements arising from revolving curves is discussed in the next section.

### 3.4.1 Minimal surfaces of revolution

The problem has obvious relevance in mechanical engineering and computer-aided manufacturing (CAM). Let us now consider two points

$$P_0 = (x_0, y_0), P_1 = (x_1, y_1),$$

and find the function  $y(x)$  going through the points that generates an object of revolution  $z = f(x, y)$  when rotated around the  $x$  axis with minimal surface area. The surface of that object of revolution is

$$S = 2\pi \int_{x_0}^{x_1} y \sqrt{1 + y'^2} dx.$$

The corresponding variational problem is

$$I(y) = 2\pi \int_{x_0}^{x_1} y \sqrt{1 + y'^2} dx = \text{extremum},$$

with the boundary conditions of

$$y(x_0) = y_0, y(x_1) = y_1.$$

The Beltrami formula of Equation (1.1) produces

$$y \sqrt{1 + y'^2} - \frac{yy'^2}{\sqrt{1 + y'^2}} = c_1.$$

Reordering and another integration yields

$$x = c_1 \int \frac{1}{\sqrt{y^2 - c_1^2}} dy.$$

Hyperbolic substitution enables the integration as

$$x = c_1 \cosh^{-1} \left( \frac{y}{c_1} \right) + c_2.$$

Finally the solution curve generating the minimal surface of revolution between the two points is

$$y = c_1 \cosh \left( \frac{x - c_2}{c_1} \right),$$

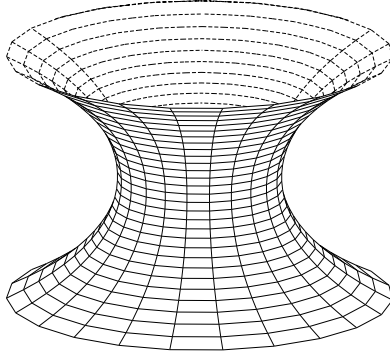
where the integration constants are resolved with the boundary conditions as

$$y_0 = c_1 \cosh \left( \frac{x_0 - c_2}{c_1} \right),$$



and

$$y_1 = c_1 \cosh\left(\frac{x_1 - c_2}{c_1}\right).$$



**FIGURE 3.2** Catenoid surface

An example of such a surface of revolution, the catenoid, is shown in Figure 3.2 where the meridian curves are catenary curves.

### 3.5 Functionals with three independent variables

The generalization to functionals with multiple independent variables is rather straightforward from the last section. The case of three independent variables, however, has such enormous engineering importance that it is worthy of a special section. The problem is of the form

$$I(u(x, y, z)) = \int \int \int_D f(x, y, z, u, u_x, u_y, u_z) dx dy dz = \text{extremum}.$$

The solution function  $u(x, y, z)$  may be some engineering quantity describing a physical phenomenon acting on a three-dimensional body. Here the domain is generalized as well to

$$x_0 \leq x \leq x_1, \quad y_0 \leq y \leq y_1, \quad z_0 \leq z \leq z_1.$$

The alternative solution is also a function of three variables

$$U(x, y, z) = u(x, y, z) + \epsilon \eta(x, y, z).$$

As usual

$$I(\epsilon) = \int \int \int_D f(x, y, z, U, U_x, U_y, U_z) dx dy dz.$$

The extremum is reached when:

$$\frac{\partial I}{\partial \epsilon} \Big|_{\epsilon=0} = \int \int \int_D \left( \frac{\partial f}{\partial u} \eta + \frac{\partial f}{\partial u_x} \eta_x + \frac{\partial f}{\partial u_y} \eta_y + \frac{\partial f}{\partial u_z} \eta_z \right) dx dy dz = 0.$$

Applying Green's identity for the last three terms and a considerable amount of algebra produces the Euler-Lagrange differential equation for this case

$$\frac{\partial f}{\partial u} - \frac{\partial}{\partial x} \frac{\partial f}{\partial u_x} - \frac{\partial}{\partial y} \frac{\partial f}{\partial u_y} - \frac{\partial}{\partial z} \frac{\partial f}{\partial u_z} = 0.$$

An even more practical three-variable case, important in engineering dynamics, is when the Euclidean spatial coordinates are extended with time. Let us consider the variational problem of one temporal and two spatial dimensions as

$$I(u) = \int_{t_0}^{t_1} \int \int_D f(x, y, t, u, u_x, u_y, u_t) dx dy dt = \text{extremum}.$$

Here again

$$u_x = \frac{\partial u}{\partial x}; \quad u_y = \frac{\partial u}{\partial y},$$

and

$$u_t = \frac{\partial u}{\partial t}.$$

We introduce the alternative solution as

$$U(x, y, t) = u(x, y, t) + \epsilon \eta(x, y, t),$$

with the temporal boundary conditions of

$$\eta(x, y, t_0) = \eta(x, y, t_1) = 0.$$

As above

$$I(\epsilon) = \int_{t_0}^{t_1} \int \int_D f(x, y, t, U, U_x, U_y, U_t) dx dy dt,$$

and the extremum is reached when:

$$\frac{\partial I}{\partial \epsilon} \Big|_{\epsilon=0} = \int_{t_0}^{t_1} \int \int_D \left( \frac{\partial f}{\partial u} \eta + \frac{\partial f}{\partial u_x} \eta_x + \frac{\partial f}{\partial u_y} \eta_y + \frac{\partial f}{\partial u_t} \eta_t \right) dx dy dt = 0. \quad (3.1)$$

The last member of the integral may be written as

$$\int_{t_0}^{t_1} \int \int_D \frac{\partial f}{\partial u_t} \eta_t dx dy dt = \int \int_D \int_{t_0}^{t_1} \frac{\partial f}{\partial u_t} \eta_t dt dx dy.$$

Integrating by parts yields

$$\int \int_D \left( \frac{\partial f}{\partial u_t} \eta \Big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \eta \frac{\partial}{\partial t} \left( \frac{\partial f}{\partial u_t} \right) dt \right) dx dy.$$

Due to the temporal boundary condition the first term vanishes and

$$- \int_{t_0}^{t_1} \int \int_D \eta \frac{\partial}{\partial t} \left( \frac{\partial f}{\partial u_t} \right) dx dy dt$$

remains. The second and third terms of Equation (3.1) may be rewritten by Green's identity as follows:

$$\begin{aligned} & \int_{t_0}^{t_1} \int \int_D \left( \frac{\partial f}{\partial u_x} \eta_x + \frac{\partial f}{\partial u_y} \eta_y \right) dx dy dt = \\ & - \int_{t_0}^{t_1} \int \int_D \eta \left( \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial u_x} \right) + \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial u_y} \right) \right) dx dy dt + \\ & \int_{t_0}^{t_1} \int_{\partial D} \eta \left( \frac{\partial f}{\partial u_x} \frac{dy}{ds} + \frac{\partial f}{\partial u_y} \frac{dx}{ds} \right) ds dt. \end{aligned}$$

With these changes, Equation (3.1) becomes

$$\begin{aligned} \frac{\partial I}{\partial \epsilon} \Big|_{\epsilon=0} = & \int_{t_0}^{t_1} \left( \int \int_D \eta \left( \frac{\partial f}{\partial u} - \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial u_x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial u_y} \right) - \frac{\partial}{\partial t} \left( \frac{\partial f}{\partial u_t} \right) \right) dx dy + \right. \\ & \left. \int_{\partial D} \eta \left( \frac{\partial f}{\partial u_x} \frac{dy}{ds} - \frac{\partial}{\partial u_y} \frac{dx}{ds} \right) ds \right) dt = 0. \end{aligned}$$

Since the auxiliary function  $\eta$  is arbitrary, by the fundamental lemma of calculus of variations the first integral is only zero when

$$\frac{\partial f}{\partial u} - \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial u_x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial u_y} \right) - \frac{\partial}{\partial t} \left( \frac{\partial f}{\partial u_t} \right) = 0,$$

in the interior of the domain  $D$ . This is the Euler-Lagrange differential equation of the problem. Since the boundary conditions of the auxiliary function

were only temporal, the second integral is only zero when

$$\frac{\partial f}{\partial u_x} \frac{dy}{ds} - \frac{\partial}{\partial u_y} \frac{dx}{ds} = 0,$$

on the boundary  $\partial D$ . This is the constraint of the variational problem. This result will be utilized in Chapter 11 to solve the elastic membrane problem.

The case of functions with four independent variables such as

$$u(x, y, z, t)$$

will also be discussed in Chapter 12 in connection with elasticity problems in solids.

The generalization of the process to even more independent variables is algebraically straightforward. Generalization to more spatial coordinates is not very frequent, although in some manufacturing applications five-dimensional hyper-spaces do occur.



# 4

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## *Higher order derivatives*

The fundamental problem of the calculus of variations involved the first derivative of the unknown function. In this chapter we will allow the presence of higher order derivatives.

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### **4.1 The Euler-Poisson equation**

First let us consider the variational problem of a functional with a single function, but containing its higher derivatives:

$$I(y) = \int_{x_0}^{x_1} f(x, y, y', \dots, y^{(m)}) dx.$$

Accordingly, boundary conditions for all derivatives will also be given as

$$\begin{aligned} y(x_0) &= y_0, y(x_1) = y_1, \\ y'(x_0) &= y'_0, y'(x_1) = y'_1, \\ y''(x_0) &= y''_0, y''(x_1) = y''_1, \end{aligned}$$

and so on until

$$y^{(m-1)}(x_0) = y_0^{(m-1)}, y^{(m-1)}(x_1) = y_1^{(m-1)}.$$

As in the past chapters, we introduce an alternative solution of

$$Y(x) = y(x) + \epsilon \eta(x),$$

where the arbitrary auxiliary function  $\eta(x)$  is continuously differentiable on the interval  $x_0 \leq x \leq x_1$  and satisfies

$$\eta(x_0) = 0, \eta(x_1) = 0.$$

The variational problem in terms of the alternative solution is

$$I(\epsilon) = \int_{x_0}^{x_1} f(x, Y, Y', \dots, Y^{(m)}) dx.$$

The differentiation with respect to  $\epsilon$  follows

$$\frac{dI}{d\epsilon} = \int_{x_0}^{x_1} \frac{d}{d\epsilon} f(x, Y, Y', \dots, Y^{(m)}) dx,$$

and by using the chain rule the integrand is reshaped as

$$\frac{\partial f}{\partial Y} \frac{dY}{d\epsilon} + \frac{\partial f}{\partial Y'} \frac{dY'}{d\epsilon} + \frac{\partial f}{\partial Y''} \frac{dY''}{d\epsilon} + \dots + \frac{\partial f}{\partial Y^{(m)}} \frac{dY^{(m)}}{d\epsilon}.$$

Substituting the alternative solution and its derivatives with respect to  $\epsilon$  the integrand yields

$$\frac{\partial f}{\partial Y} \eta + \frac{\partial f}{\partial Y'} \eta' + \frac{\partial f}{\partial Y''} \eta'' + \dots + \frac{\partial f}{\partial Y^{(m)}} \eta^{(m)}.$$

Hence the functional becomes

$$\frac{dI}{d\epsilon} = \int_{x_0}^{x_1} \left( \frac{\partial f}{\partial Y} \eta + \frac{\partial f}{\partial Y'} \eta' + \frac{\partial f}{\partial Y''} \eta'' + \dots + \frac{\partial f}{\partial Y^{(m)}} \eta^{(m)} \right) dx.$$

Integrating by terms results in

$$\frac{dI}{d\epsilon} = \int_{x_0}^{x_1} \frac{\partial f}{\partial Y} \eta dx + \int_{x_0}^{x_1} \frac{\partial f}{\partial Y'} \eta' dx + \int_{x_0}^{x_1} \frac{\partial f}{\partial Y''} \eta'' dx + \dots + \int_{x_0}^{x_1} \frac{\partial f}{\partial Y^{(m)}} \eta^{(m)} dx,$$

and integrating by parts produces

$$\begin{aligned} \frac{dI}{d\epsilon} &= \int_{x_0}^{x_1} \eta \frac{\partial f}{\partial Y} dx - \int_{x_0}^{x_1} \eta \frac{d}{dx} \frac{\partial f}{\partial Y'} dx + \int_{x_0}^{x_1} \eta \frac{d^2}{dx^2} \frac{\partial f}{\partial Y''} dx - \\ &\quad \dots (-1)^m \int_{x_0}^{x_1} \eta \frac{d^{(m)}}{dx^{(m)}} \frac{\partial f}{\partial Y^{(m)}} dx. \end{aligned}$$

Factoring the auxiliary function and combining the terms again simplifies to

$$\frac{dI}{d\epsilon} = \int_{x_0}^{x_1} \eta \left( \frac{\partial f}{\partial Y} - \frac{d}{dx} \frac{\partial f}{\partial Y'} + \frac{d^2}{dx^2} \frac{\partial f}{\partial Y''} - \dots (-1)^m \frac{d^{(m)}}{dx^{(m)}} \frac{\partial f}{\partial Y^{(m)}} \right) dx.$$

Finally the extremum at  $\epsilon = 0$  and the fundamental lemma produces the **Euler-Poisson equation**

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} - \dots (-1)^m \frac{d^{(m)}}{dx^{(m)}} \frac{\partial f}{\partial y^{(m)}} = 0.$$

The Euler-Poisson equation is an ordinary differential equation of order  $2m$  and requires the aforementioned  $2m$  boundary conditions, where  $m$  is the highest order derivative contained in the functional.

For example, the simple  $m = 2$  functional

$$I(y) = \int_{x_0}^{x_1} (y^2 - (y'')^2) dx$$

results in the derivatives

$$\frac{\partial f}{\partial y''} = -2y'',$$

and

$$\frac{\partial f}{\partial y} = 2y.$$

The corresponding Euler-Poisson equation derivative term is

$$\frac{d^2}{dx^2} \frac{\partial f}{\partial y''} = \frac{d^2}{dx^2} (-2y'') = -2 \frac{d^4}{dx^4} y,$$

and the equation, after cancellation by  $-2$ , becomes

$$\frac{d^4}{dx^4} y - y = 0.$$

Clearly the solution of this may be achieved by classical calculus tools with four boundary conditions. Application problems exploiting this will be addressed in Chapters 9 (the natural spline) and 11 (the bending beam).

## 4.2 The Euler-Poisson system of equations

In the case of a functional with multiple functions along with their higher order derivatives, the problem gets more difficult. Assuming  $p$  functions in the functional, the problem is posed in the form of

$$I(y_1, \dots, y_p) = \int_{x_0}^{x_1} f(x, y_1, y_1', \dots, y_1^{(m_1)}, \dots, y_p, y_p', \dots, y_p^{(m_p)}) dx.$$

Note that the highest order of the derivative of the various functions is not necessarily the same. This is a rather straightforward generalization of the case of the last section, leading to a system of Euler-Poisson equations as follows:

$$\begin{aligned} \frac{\partial f}{\partial y_1} - \frac{d}{dx} \frac{\partial f}{\partial y_1'} + \frac{d^2}{dx^2} \frac{\partial f}{\partial y_1''} - \dots (-1)^{m_1} \frac{d^{(m_1)}}{dx^{(m_1)}} \frac{\partial f}{\partial y_1^{(m_1)}} &= 0, \\ \dots, \\ \frac{\partial f}{\partial y_p} - \frac{d}{dx} \frac{\partial f}{\partial y_p'} + \frac{d^2}{dx^2} \frac{\partial f}{\partial y_p''} - \dots (-1)^{m_p} \frac{d^{(m_p)}}{dx^{(m_p)}} \frac{\partial f}{\partial y_p^{(m_p)}} &= 0. \end{aligned}$$

This is a set of  $p$  ordinary differential equations that may or may not be coupled, hence resulting in a varying level of ease of the solution.



### 4.3 Algebraic constraints on the derivative

It is also common in engineering applications to impose boundary conditions on some of the derivatives (Neumann boundary conditions). These result in algebraic constraints posed on the derivative, such as

$$I(y) = \int_{x_0}^{x_1} f(x, y, y') dx$$

subject to

$$g(x, y, y') = 0.$$

In order to be able to solve such problems, we need to introduce a Lagrange multiplier as a function of the independent variable as

$$h(x, y, y', \lambda) = f(x, y, y') + \lambda(x)g(x, y, y').$$

The use of this approach means that the functional now contains two unknown functions and the variational problem becomes

$$I(y, \lambda) = \int_{x_0}^{x_1} h(x, y, y', \lambda) dx,$$

with the original boundary conditions, but without a constraint. The solution for this unconstrained, two function case is obtained by a system of two Euler-Lagrange equations.

Derivative constraints may also be applied to the case of higher order derivatives. The second order problem of

$$I(y) = \int_{x_0}^{x_1} f(x, y, y', y'') dx$$

may be subject to a constraint

$$g(x, y, y', y'') = 0.$$

In order to be able to solve such problems, we also introduce a Lagrange multiplier function as

$$h(x, y, y', y'', \lambda) = f(x, y, y', y'') + \lambda(x)g(x, y, y', y'').$$

The result is a variational problem of two functions with higher order derivatives as

$$I(y, \lambda) = \int_{x_0}^{x_1} h(x, y, y', y'', \lambda) dx.$$

Hence the solution may be obtained by the application of a system of two Euler-Poisson equations.

Finally, derivative constraints may also be applied to a variational problem originally exhibiting multiple functions, such as

$$I(y, z) = \int_{x_0}^{x_1} f(x, y, y', z, z') dx$$

subject to

$$g(x, y, y', z, z') = 0.$$

Here the new functional is

$$h(x, y, y', z, z', \lambda) = f(x, y, y', z, z') + \lambda(x)g(x, y, y', z, z').$$

Following above, this problem translates into the unconstrained form of

$$I(y, z, \lambda) = \int_{x_0}^{x_1} f(x, y, y', z, z', \lambda) dx$$

that may be solved by a system of three Euler-Lagrange differential equations

$$\frac{\partial h}{\partial y} - \frac{d}{dx} \frac{\partial h}{\partial y'} = 0,$$

$$\frac{\partial h}{\partial z} - \frac{d}{dx} \frac{\partial h}{\partial z'} = 0,$$

and

$$\frac{\partial h}{\partial \lambda} - \frac{d}{dx} \frac{\partial h}{\partial \lambda'} = 0.$$

For example, the variational problem of

$$I(y, z) = \int_{x_0}^{x_1} (y^2 - z^2) dx = \text{extremum},$$

under the derivative constraint of

$$y' - y + z = 0,$$

results in

$$h(x, y, y', z, z', \lambda) = y^2 - z^2 + \lambda(x)(y' - y + z).$$

The solution is obtained from the following three equations

$$2y - \lambda + \lambda' = 0,$$

$$-2z + \lambda = 0,$$

and

$$y' - y + z = 0.$$

The elimination of the Lagrange multiplier results in the system of

$$y - z + z' = 0,$$

and

$$y' - y + z = 0,$$

whose solution follows from classical calculus.

#### 4.4 Linearization of second order problems

It is very common in engineering practice that the highest derivative of interest is of second order. Accelerations in engineering analysis of motion, curvature in description of space curves, and other important application concepts are tied to the second derivative.

This specific case of quadratic problems may be reverted to a linear problem involving two functions. Consider

$$I(y) = \int_{x_0}^{x_1} f(x, y, y', y'') dx = \text{extremum}.$$

with the following boundary conditions given

$$y(x_0), y(x_1), y'(x_0), y'(x_1).$$

By introducing a new function

$$z(x) = y'(x),$$

we can reformulate the unconstrained second order variational problem as a variational problem of the first order with multiple functions in the integrand

$$I(y, z) = \int_{x_0}^{x_1} f(x, y, z, z') dx = \text{extremum},$$

but subject to a constraint involving the derivative

$$g(x, y, z) = z - y' = 0.$$

Using a Lagrange multiplier function in the form of

$$h(x, y, z, z', \lambda) = f(x, y, z, z') + \lambda(x)(z - y'),$$

and following the process laid out in the last section we can produce a system of three Euler-Lagrange differential equations.

$$\begin{aligned}\frac{\partial h}{\partial y} - \frac{d}{dx} \frac{\partial h}{\partial y'} &= \frac{\partial f}{\partial y} - \frac{d\lambda}{dx} = 0, \\ \frac{\partial h}{\partial z} - \frac{d}{dx} \frac{\partial h}{\partial z'} &= \frac{\partial f}{\partial z} + \lambda - \frac{d}{dx} \frac{\partial f}{\partial z'} = 0,\end{aligned}$$

and

$$\frac{\partial h}{\partial \lambda} - \frac{d}{dx} \frac{\partial h}{\partial \lambda'} = z - y'.$$

This may, of course, be turned into the Euler-Poisson equation by expressing

$$\lambda = \frac{d}{dx} \frac{\partial f}{\partial z'} - \frac{\partial f}{\partial z}$$

from the middle equation and differentiating as

$$\frac{d\lambda}{dx} = \frac{d^2}{dx^2} \frac{\partial f}{\partial z'} - \frac{d}{dx} \frac{\partial f}{\partial z}.$$

Substituting this and the third equation into the first yields the Euler-Poisson equation we could have achieved, had we approached the original quadratic problem directly:

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} = 0.$$

Depending on the particular application circumstance, the linear system of Euler-Lagrange equations may be more conveniently solved than the quadratic Euler-Poisson equation.



# 5

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## *The inverse problem of calculus of variations*

It is often the case that the engineer starts from a differential equation with certain boundary conditions, which is difficult to solve. Executing the inverse of the Euler-Lagrange process and obtaining the variational formulation of the boundary value problem may also be advantageous.

It is not necessarily easy, or may not even be possible to reconstruct the variational problem from a differential equation. For differential equations, partial or ordinary, containing a linear, self-adjoint, positive operator, the task may be accomplished. Such an operator exhibits

$$(Au, v) = (u, Av),$$

where the parenthesis expression denotes a scalar product in the function space of the solution of the differential equation. Positive definiteness of the operator means

$$(Au, u) \geq 0,$$

with zero attained only for the trivial ( $u = 0$ ) solution. Let us consider the differential equation of

$$Au = f,$$

where the operator obeys the above conditions and  $f$  is a known function. If the differential equation has a solution, it corresponds to the minimum value of the functional

$$I(u) = (Au, u) + 2(u, f).$$

This may be proven by simply applying the appropriate Euler-Lagrange equation to this functional.

## 5.1 The variational form of Poisson's equation

We demonstrate the inverse process through the example of Poisson's equation, a topic of much interest for engineers:

$$\Delta u(x) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y).$$

Here the left-hand side is the well-known Laplace operator. We impose Dirichlet type boundary conditions on the boundary of the domain of interest.

$$u(x, y) = 0; (x, y) \in \partial D,$$

where  $D$  is the domain of solution and  $\partial D$  is its boundary. According to the above proposition, we need to compute

$$(Au, u) = \int \int_D u \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dx dy.$$

Applying some vector calculus results in

$$(Au, u) = \int_{\partial D} \left( u \frac{\partial u}{\partial y} dx - u \frac{\partial u}{\partial x} dy \right) + \int \int_D \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 dx dy.$$

Due to the boundary conditions, the first term vanishes and we obtain

$$(Au, u) = \int \int_D \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 dx dy.$$

The right-hand side term of the differential equation is processed as

$$(u, f) = \int \int_D u f(x, y) dx dy.$$

The variational formulation of Poisson's equation finally is

$$\int \int_D \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + 2uf \right) dx dy = \int \int_D F(x, y, u, u_x, u_y) dx dy = \text{extremum}.$$

To prove this, we will apply the Euler-Lagrange equation developed in Section 3.3. The terms for this particular case are:

$$\frac{\partial F}{\partial u} = 2f,$$

$$\frac{\partial}{\partial x} \frac{\partial F}{\partial u_x} = \frac{\partial}{\partial x} 2u_x = 2 \frac{\partial^2 u}{\partial x^2},$$

and

$$\frac{\partial}{\partial y} \frac{\partial F}{\partial u_y} = \frac{\partial}{\partial y} 2u_y = 2 \frac{\partial^2 u}{\partial y^2}.$$

The resulting equation of

$$2f - 2\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial y^2} = 0$$

is clearly equivalent with Poisson's equation.

## 5.2 The variational form of eigenvalue problems

Eigenvalue problems of various kinds may also be formulated as variational problems [12]. We consider the equation of the form

$$\Delta u(x) - \lambda u(x) = 0, \quad (5.1)$$

where the unknown function  $u(x)$  defined on domain  $D$  is the eigensolution and  $\lambda$  is the eigenvalue. The boundary condition is imposed as

$$u(x, y) = 0,$$

on the perimeter  $\partial D$  of the domain  $D$ . The corresponding variational problem is of the form

$$I = \int \int_D \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 \right) dx dy = \text{extremum}, \quad (5.2)$$

under the condition of

$$g(x, y) = \int \int_D u^2(x, y) dx dy = 1.$$

This relation is proven as follows. Following the Lagrange solution of constrained variational problems introduced in Section 2.2, we can write

$$h(x, y) = u(x, y) + \lambda g(x, y),$$

and

$$I = \int \int_D \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + \lambda u^2(x, y) \right) dx dy.$$

Note that the  $\lambda$  is still only in the role of the Lagrange multiplier, although its name hints at its final meaning as well. Introducing

$$U(x, y) = u(x, y) + \epsilon \eta(x, y)$$

the variational form becomes

$$I(\epsilon) = \int \int_D \left( \left( \frac{\partial u}{\partial x} + \epsilon \frac{\partial \eta}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} + \epsilon \frac{\partial \eta}{\partial y} \right)^2 + \lambda (u + \epsilon \eta)^2 \right) dx dy.$$



The extremum is reached when

$$\frac{dI(\epsilon)}{d\epsilon}\Big|_{\epsilon=0} = 0,$$

which gives rise to the equation

$$2 \int \int_D \left( \frac{\partial u}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial \eta}{\partial y} + \lambda u \eta \right) dx dy = 0. \quad (5.3)$$

Green's identity in its original three-dimensional form was exploited on several occasions earlier; here we apply it for the special vector field

$$\eta \nabla u$$

in a two-dimensional domain. The result is

$$\int \int_D (\nabla \eta \cdot \nabla u) dA = \int_{\partial D} \eta (\nabla u \cdot \underline{n}) ds - \int \int_D \eta \nabla^2 u dA.$$

Since the tangent of the circumference is in the direction of

$$dx \underline{i} + dy \underline{j},$$

the unit normal may be computed as

$$\underline{n} = \frac{dy \underline{i} - dx \underline{j}}{\sqrt{dx^2 + dy^2}}.$$

Finally utilizing the arc length formula of

$$ds = \sqrt{dx^2 + dy^2},$$

the line integral over the circumference of the domain becomes

$$\int_{\partial D} \eta \left( \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx \right).$$

Applying the above for the first two terms of Equation (5.3) results in

$$\begin{aligned} & \int \int_D \left( \frac{\partial u}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial \eta}{\partial y} \right) dx dy = \\ & \int \int_{\partial D} \left( \eta \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) \cdot \underline{n} \right) ds - \int \int_D \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \eta dx dy = \\ & \int_{\partial D} \frac{\partial u}{\partial x} \eta dy - \frac{\partial u}{\partial y} \eta dx - \int \int_D \Delta u \eta dx dy. \end{aligned}$$

The integral over the boundary vanishes due to the assumption on  $\eta$ , and substituting the remainder part into Equation (5.3) we obtain

$$-2 \int \int_D (\Delta u - \lambda u) \eta dx dy = 0.$$

Since  $\eta(x, y)$  is arbitrarily chosen, in order to satisfy this equation

$$\Delta u - \lambda u = 0$$

must be satisfied. Thus we have established that Equation (5.2) is indeed the variational form of Equation (5.1) and the Lagrange multiplier is the eigenvalue.

### 5.2.1 Orthogonal eigensolutions

The eigenvalue problem has an infinite sequence of eigenvalues and for each eigenvalue there exists a corresponding eigensolution that is unique apart from a constant factor. Hence the variational form should also provide means for the solution of multiple pairs.

Let us denote the series of eigenpairs as

$$(\lambda_1, u_1), (\lambda_2, u_2), \dots (\lambda_n, u_n).$$

Assuming that we have already found the first pair satisfying

$$\Delta u_1 - \lambda_1 u_1 = 0,$$

we seek the second solution  $u_2, \lambda_2 \neq \lambda_1$  following the process laid out in the last section. Then for any arbitrary auxiliary function  $\eta$  it follows that

$$\int \int_D \left( \frac{\partial u_2}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial u_2}{\partial y} \frac{\partial \eta}{\partial y} + \lambda_2 u_2 \eta \right) dx dy = 0.$$

Applying an auxiliary function of the special form of

$$\eta = u_1,$$

we obtain

$$\int \int_D \left( \frac{\partial u_2}{\partial x} \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \frac{\partial u_1}{\partial y} + \lambda_2 u_2 u_1 \right) dx dy = 0.$$

The same argument for the first solution results in

$$\int \int_D \left( \frac{\partial u_1}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial u_1}{\partial y} \frac{\partial \eta}{\partial y} + \lambda_1 u_1 \eta \right) dx dy = 0.$$

Applying an auxiliary function of the special form of

$$\eta = u_2,$$

we obtain

$$\int \int_D \left( \frac{\partial u_1}{\partial x} \frac{\partial u_2}{\partial x} + \frac{\partial u_1}{\partial y} \frac{\partial u_2}{\partial y} + \lambda_1 u_1 u_2 \right) dx dy = 0.$$

Subtracting the equations and canceling the identical terms results in

$$(\lambda_2 - \lambda_1) \int \int_D u_1 u_2 dx dy = 0.$$

Since

$$\lambda_1 \neq \lambda_2,$$

it follows that

$$\int \int_D u_2 u_1 dx dy = 0$$

must be true. The two eigensolutions are orthogonal. With similar arguments and specially selected auxiliary functions, it is also easy to show that the second solutions also satisfy

$$\Delta u_2 - \lambda_2 u_2 = 0.$$

The subsequent eigensolutions may be found by the same procedure and the sequence of the eigenpairs attain the extrema of the variational problem under the successive conditions of the orthogonality against the preceding solutions.

### 5.3 Sturm-Liouville problems

The process demonstrated in the last section in connection with Laplace's operator may be applied to arrive at eigenvalues and eigensolutions of other differential equations as well. The differential equations of the form

$$-\frac{d}{dx} \left( p(x) \frac{dy}{dx} \right) + q(x)y(x) = \lambda y(x),$$

are called the Sturm-Liouville differential equations. Here the unknown solution function  $y(x)$  is the eigensolution and  $\lambda$  is the eigenvalue. The known functions  $p(x)$  and  $q(x)$  are continuous and continuously differentiable, respectively. The boundary conditions imposed are

$$y(x_0) = 0$$

and

$$y(x_1) = 0.$$

The corresponding variational problem is posed as

$$I(y) = \int_{x_0}^{x_1} (p(x)y'^2(x) + q(x)y^2(x))dx = \text{extremum},$$

subject to the above boundary conditions and the additional constraint of

$$\int_{x_0}^{x_1} y^2(x) dx = 1.$$

The engineering importance of these problems lies in the fact that, depending on the choice of the coefficient functions  $p(x), q(x)$ , various influential functions may be generated as the eigensolutions. For example, the Bessel functions, Hermite, Chebyshev, and Laguerre polynomials may be derived from Sturm-Liouville equations with various selections of the coefficient functions.

The simplest form of the Sturm-Liouville equations is with

$$p(x) = 1$$

and

$$q(x) = 0,$$

leading to

$$-\frac{d}{dx}\left(\frac{dy}{dx}\right) = \lambda y(x).$$

Straightforward integration indicates the possibility of a solution in the form of trigonometric functions. Surely, a solution of the form

$$y(x) = c_i \sin(ix), i = 1, 2, \dots$$

would satisfy the equation with a judicious choice of the constant  $c_i$  and boundary conditions. We will assume boundary conditions of

$$y(0) = 0$$

and

$$y(\pi) = 0.$$

The selection of this boundary is for the convenience of dealing with this function family and does not restrict the generality of the discussion. Furthermore, it is well established that

$$\int_0^\pi \sin(ix)\sin(jx)dx = 0; i \neq j,$$

hence these solutions are orthogonal. The constraint equation is easy to evaluate, since for  $i = 1$ , for example,

$$\int_0^\pi (c_1 \sin(1x))^2 dx = c_1^2 \int_0^\pi \sin^2(x) dx = c_1^2 \left( \frac{1}{2} x \Big|_0^\pi - \frac{1}{4} \sin(2x) \Big|_0^\pi \right).$$

Substituting the boundary values results in

$$c_1^2 \frac{\pi}{2} = 1,$$

producing the constant necessary to satisfy the constraint:

$$c_1 = \sqrt{\frac{2}{\pi}}.$$

The generic solution may be obtained from the form

$$\int_0^\pi (c_i \sin(ix))^2 dx = c_i^2 \int_0^\pi \sin^2(ix) dx = c_i^2 \left( \frac{1}{2} x \Big|_0^\pi - \frac{1}{4i} \sin(2ix) \Big|_0^\pi \right).$$

Considering the boundary conditions, the second term vanishes and

$$c_i^2 \frac{\pi}{2} = 1,$$

which yields the generic coefficients as

$$c_i = \sqrt{\frac{2}{\pi}}.$$

In order to establish the eigenvalues, we execute the operations posted by the differential equation for the generic solution,

$$-\frac{d}{dx}(c_i \sin'(ix)) = -\frac{d}{dx}(ic_i \cos(ix)) = i^2 c_i \sin(ix),$$

which must be equal to the right-hand side of

$$\lambda c_i \sin(ix).$$

This results in the eigenvalue of

$$\lambda = i^2.$$

A more generic discussion of this case is presented in [10]. It is noteworthy that even this simplest form of Sturm-Liouville problems leads to an engineering application, the vibrating string problem, the subject of Section 11.1.

### 5.3.1 Legendre's equation and polynomials

A very important sub-case of the Sturm-Liouville problems is when

$$p(x) = 1 - x^2,$$

along with

$$q(x) = 0.$$

The form of the equation becomes

$$\frac{d}{dx} \left( (1 - x^2) \frac{dy}{dx} \right) + \lambda y(x) = 0,$$

subject to the boundary conditions

$$y(-1) = 0,$$

and

$$y(1) = 0.$$

The associated variational problem becomes

$$I(y) = \int_{-1}^{+1} (1 - x^2) \left( \frac{dy}{dx} \right)^2 dx = \text{extremum},$$

subject to the constraint

$$\int_{-1}^{+1} y^2 dx = 1.$$

It is easy to obtain the Euler-Lagrange equation of this constrained variational problem as

$$\frac{\partial}{\partial y} ((1 - x^2)y'^2 - \lambda y^2) - \frac{d}{dx} \left( \frac{\partial}{\partial y'} ((1 - x^2)y'^2 - \lambda y^2) \right) = 0.$$

Here  $\lambda$  again is only the Lagrange multiplier connecting the constraint, but its final disposition is pre-ordained. The equation with some algebra simplifies to

$$\frac{d}{dx} ((1 - x^2)y') + \lambda y = 0,$$

which is of course the equation we started from. The eigenvalues of this problem are of the form

$$\lambda = i(i + 1),$$

and the eigensolutions are

$$y(x) = c_i L e_i(x).$$

Here  $c_i$  are constants and  $L e_i$  are the Legendre polynomials. The first few values of the eigenpairs are shown in Table 5.1.

The eigensolution functions are shown graphically in Figure 5.1.

The first eigensolution with

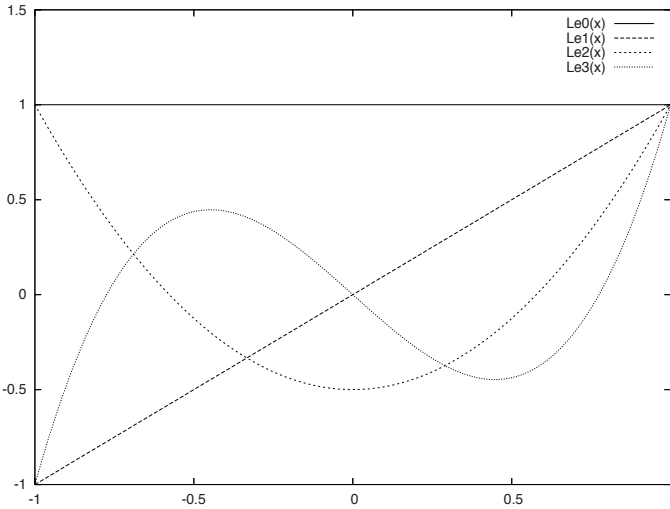
$$\lambda_0 = 0$$

and

$$y_0(x) = 1$$

**TABLE 5.1**  
Eigenpairs of Legendre equation

| $i$ | $\lambda$ | $Le_i(x)$       | $c_i$        |
|-----|-----------|-----------------|--------------|
| 0   | 0         | 1               | $\sqrt{1/2}$ |
| 1   | 2         | $x$             | $\sqrt{3/2}$ |
| 2   | 6         | $(3x^2 - 1)/2$  | $\sqrt{5/2}$ |
| 3   | 12        | $(5x^3 - 3x)/2$ | $\sqrt{7/2}$ |



**FIGURE 5.1** Eigensolutions of Legendre’s equation

is clearly a trivial solution, since both terms of the left-hand side of the equation vanish. The first non-trivial solution may be verified as

$$\frac{d}{dx}((1 - x^2)(x)') + 2x = \frac{d}{dx}(1 - x^2) + 2x = -2x + 2x = 0.$$

Furthermore, the satisfaction of the constraint equations is seen as

$$\int_{-1}^{+1} y^2 dx = c_1^2 \int_{-1}^{+1} x^2 dx = c_1^2 \frac{x^3}{3} \Big|_{-1}^{+1} = c_1^2 \frac{2}{3}.$$

The constant of

$$c_1 = \sqrt{\frac{3}{2}}$$

will enforce the satisfaction of

$$\int_{-1}^{+1} y^2 dx = 1.$$

In fact, the generic form of the constants may be obtained by

$$c_i = \sqrt{\frac{2i+1}{2}}.$$

Finally, Sturm-Liouville problems may also be presented with functions of multiple variables. For example, in three dimensions the equation becomes

$$-\nabla(p(x, y, z)\nabla u(x, y, z)) + q(x, y, z)u(x, y, z) = \lambda u(x, y, z)$$

leading to various elliptic partial differential equations that all have engineering implications.





# 6

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## *Analytic solutions of variational problems*

This chapter presents a handful of analytic methods for solving variational problems. They include the methods of Laplace transformation, separation of variables, complete integrals, and Poisson's integral formula. The method of gradients, with high relevance to engineering optimization, concludes the chapter.

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### 6.1 Laplace transform solution

The first method we discuss in this chapter transforms the original variational problem by applying the Laplace transform and producing an auxiliary differential equation.

Let us consider the variational problem of

$$I(t, x) = \int f(t, x) dt = \text{extremum},$$

and apply the Laplace transform to the function as

$$\int_0^{\infty} e^{-st} f(t, x) dt.$$

During this transform we regard time as the independent variable and  $x$  as a parameter. Note that the transformation of the boundary conditions is also required to obtain the complete auxiliary problem.

Let us illustrate this by the Euler-Lagrange differential equation of one spatial and one temporal independent variable of the form

$$a^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0,$$

with initial condition

$$u(t = 0, x) = 0,$$

and boundary conditions

$$u(t, x = 0) = 0$$

and

$$u(t, x = 1) = B.$$

Executing the Laplace transform on the boundary conditions results in

$$\bar{u}(s, 0) = \int_0^\infty e^{-st} u(t, 0) dt = \int_0^\infty e^{-st} 0 dt = \frac{0}{s} = 0$$

and

$$\bar{u}(s, 1) = \int_0^\infty e^{-st} u(t, 1) dt = \int_0^\infty e^{-st} B dt = \frac{B}{s}.$$

Transforming the yet unknown solution as

$$\bar{u}(s, x) = \int_0^\infty e^{-st} u(t, x) dt,$$

we produce the auxiliary equation in the form of

$$a^2 \frac{d^2 \bar{u}}{dx^2} - s \bar{u} = 0.$$

This equation is now ordinary. By integrating twice and applying Euler's formula the solution of the auxiliary equation becomes

$$\bar{u}(s, x) = \frac{B}{s} \frac{\sinh(x\sqrt{\frac{s}{a}})}{\sinh(\sqrt{\frac{s}{a}})}.$$

Finally, inverse Laplace transformation yields the solution of the original problem in the form of

$$u(t, x) = B \left( x + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} e^{-(k\pi a)^2 t} \sin(k\pi x) \right).$$

This is the analytic solution of the one-dimensional heat conduction problem with constant temperature ( $B$ ) at the boundary. The two- and three-dimensional heat conduction problems will be the subject of further discussion in the next section.

Let us now consider another Euler-Lagrange equation whose temporal derivative is also of second order

$$a^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0.$$

The initial conditions are

$$u(t = 0, x) = 0, \quad \frac{\partial u}{\partial t}(t = 0, x) = 0.$$

The boundary conditions are

$$u(t, x = 0) = 0, \frac{\partial u}{\partial x}(t, x = 1) = B.$$

The boundary conditions are transformed again as

$$\bar{u}(x = 0) = 0$$

and

$$\frac{d\bar{u}}{dx}(x = 1) = \frac{B}{s},$$

where  $s$  is the Laplace variable. The auxiliary equation becomes an ordinary differential equation of

$$\frac{d^2\bar{u}}{dx^2} = \frac{s^2}{a^2}\bar{u},$$

Integrating this equation we obtain the result in the form

$$\bar{u}(s, x) = \frac{aB}{s^2} \frac{\sinh(\frac{sx}{a})}{\cosh(\frac{s}{a})}.$$

Finally the inverse transformation yields the solution of the original problem at any point in the domain at any time as

$$u(t, x) = B(x - \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \sin(\frac{\pi x}{2}(2k+1)) \cos(\frac{\pi at}{2}(2k+1))).$$

This is the analytic solution to the problem of the compression of a unit length beam along its longitudinal axis. The coefficient  $a$  and boundary condition  $B$  represent the physical characteristics of the beam and the problem, respectively. They will be introduced in connection with the solution of a beam bending under its weight, presented in Section 11.3.

## 6.2 Separation of variables

The method discussed here has a resemblance to the Laplace transform solution since it also uses a transformation of the solution. We address the problem of

$$\frac{\partial u}{\partial t} = h^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

We impose uniformly zero boundary conditions as

$$u(x, 0, t) = u(x, b, t) = 0; 0 \leq x \leq a,$$

and

$$u(0, y, t) = u(a, y, t) = 0; 0 \leq y \leq b.$$

The initial solution is given as a non-zero function of the spatial coordinates

$$u(x, y, 0) = f(x, y).$$

The separation of variables method seeks a solution in the form of

$$u(x, y, t) = e^{-\lambda t} u_1(x) u_2(y),$$

where  $\lambda$  is a yet unknown constant. Substitution and differentiation yields

$$-\lambda u_1 u_2 = h^2 (u_1'' u_2 + u_1 u_2'').$$

Conveniently reordering produces

$$\frac{u_1''}{u_1} + \frac{\lambda}{h^2} = -\frac{u_2''}{u_2} = k^2,$$

where  $k$  is a constant since the left-hand side is independent of  $y$  and the right-hand side is independent of  $x$ . Introducing

$$q^2 = \frac{\lambda}{h^2} - k^2,$$

we obtain a system of ordinary differential equations:

$$u_1'' + q^2 u_1 = 0,$$

and

$$u_2'' + k^2 u_2 = 0.$$

Their solutions are easily obtained as

$$u_1(x) = a_1 \sin(qx) + b_1 \cos(qx),$$

and

$$u_2(y) = a_2 \sin(ky) + b_2 \cos(ky).$$

The boundary conditions imply that  $b_1 = b_2 = 0$  as well as

$$\sin(qa) = 0,$$

and

$$\sin(kb) = 0.$$

Here  $a, b$  are the original spatial boundaries. Due to the periodic nature of the trigonometric functions

$$q = \frac{m\pi}{a}, m = 1, 2, \dots$$

and

$$k = \frac{n\pi}{b}, n = 1, 2, \dots$$

Substituting produces the unknown variable as

$$\lambda_{mn} = h^2 \left( \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \right),$$

and the solution function of

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} e^{-\lambda_{mn}t} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}.$$

The final unknown coefficient  $c_{mn}$  is obtained by the satisfaction of the initial condition:

$$f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b},$$

from which the value of

$$c_{mn} = \frac{4}{ab} \int_0^b \int_0^a f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} dx dy$$

emerges. It is easy to generalize this solution to the three-dimensional problem of

$$\frac{\partial u}{\partial t} = h^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right).$$

We impose uniformly zero boundary conditions on three spatial dimensions as

$$u(x, y, 0, t) = u(x, y, c, t) = 0; 0 \leq x \leq a, 0 \leq y \leq b.$$

$$u(x, 0, z, t) = u(x, b, z, t) = 0; 0 \leq x \leq a, 0 \leq z \leq c.$$

and

$$u(0, y, z, t) = u(a, y, z, t) = 0; 0 \leq y \leq b, 0 \leq z \leq c.$$

The initial solution is given as a non-zero function of the three spatial coordinates,

$$u(x, y, z, 0) = f(x, y, z).$$

The solution with

$$\lambda_{mnr} = h^2 \left( \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 + \left( \frac{r\pi}{c} \right)^2 \right),$$

becomes

$$u(x, y, z, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} c_{mnr} e^{-\lambda_{mnr}t} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{r\pi z}{c}.$$

The coefficient of the solution is also a straightforward generalization as

$$c_{mnr} = \frac{8}{abc} \int_0^c \int_0^b \int_0^a f(x, y, z) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sin \frac{r\pi z}{c} dx dy dz.$$

These last two solutions were the analytic solutions to the two- and three-dimensional heat conduction problems. The computational solution of the two-dimensional problem will be further addressed in Chapter 12.

Let us now solve a problem of two spatial variables again, but with a temporal variable whose second derivative is present:

$$h^2 \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial^2 u}{\partial t^2}.$$

We assume constant boundary conditions:

$$u(0, y, t) = u(a, y, t) = 0,$$

and

$$u(x, 0, t) = u(x, b, t) = 0.$$

The  $a, b$  are the dimensions of the domain. We seek the solution in the separated form of

$$u(x, y, t) = e^{i\lambda t} v(x, y).$$

Note the presence of the imaginary unit  $i$  in the exponent for later convenience. We introduce the constant

$$k^2 = \frac{\lambda^2}{h^2}.$$

Substitution yields the new differential equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + k^2 v = 0.$$

The new boundary conditions are

$$v(0, y) = v(a, y) = 0,$$

and

$$v(x, 0) = v(x, b) = 0.$$

Furthermore, we separate the variables of this equation as

$$v(x, y) = u_1(x)u_2(y).$$

This leads to the system of equations

$$\frac{1}{u_1} \frac{d^2 u_1}{dx^2} = -\frac{1}{u_2} \frac{d^2 u_2}{dy^2} - k^2 = -m^2.$$

Here  $m$  is another yet unknown constant. The now familiar system of ordinary differential equations arises again

$$\frac{d^2 u_1}{dx^2} + m^2 u_1 = 0,$$

and

$$\frac{d^2 u_2}{dy^2} + q^2 u_2 = 0,$$

with  $q^2 = k^2 - m^2$ . The new boundary conditions are

$$u_1(0) = u_1(a) = 0,$$

and

$$u_2(0) = u_2(b) = 0.$$

Following the road paved earlier in this section, the first equation yields

$$u_1(x) = A_1 \sin(mx),$$

with  $ma = n\pi, n = 1, 2, 3, \dots$  and the second equation

$$u_2(y) = A_2 \sin(qy),$$

with  $qb = r\pi, r = 1, 2, 3, \dots$ . Exploiting the relation

$$k^2 = m^2 + q^2 = \pi^2 \left( \frac{n^2}{a^2} + \frac{r^2}{b^2} \right)$$

we obtain the original parameter of the transformation

$$\lambda_{nr} = h\pi \sqrt{\frac{n^2}{a^2} + \frac{r^2}{b^2}}.$$

Finally by substituting and using Euler's formula we obtain

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{r=1}^{\infty} c_{nr} \cos(\lambda_{nr} t) \sin \frac{n\pi x}{a} \sin \frac{r\pi y}{b}.$$

The initial displacement represented by  $f(x, y)$  aids in finding the final coefficient as

$$c_{nr} = \frac{4}{ab} \int_0^b \int_0^a f(x, y) \sin \frac{n\pi x}{a} \sin \frac{r\pi y}{b} dx dy.$$

This is the analytic solution of the problem of the vibrating membrane. A more general solution of this problem with variable boundary conditions is presented in Chapter 11.

Finally, let us consider an Euler-Lagrange equation of the first order with many independent variables in the implicit form of

$$F(x_1, x_2, \dots, x_n, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}) = 0,$$



whose generic solution is

$$u(x_1, x_2, \dots, x_n; a_1, a_2, \dots, a_n) = 0.$$

The solution to such problems may be found by a repeated use of the separation of variables and the constant  $k$ . Let us first separate one variable as

$$u(x_1, x_2, \dots, x_n) = u_1(x_1) + u_2(x_2, x_3, \dots, x_n).$$

This corresponds to the following differential equation

$$F_1(x_1, u_1, \frac{\partial u_1}{\partial x_1}) = F_2(x_2, x_3, \dots, x_n, u_2, \frac{\partial u_2}{\partial x_2}, \frac{\partial u_2}{\partial x_3}, \dots, \frac{\partial u_2}{\partial x_n}) = k^2.$$

The equation may be satisfied by solving a pair of equations with an unknown constant:

$$F_1(x_1, u_1, \frac{\partial u_1}{\partial x_1}) = k^2,$$

and

$$F_2(x_2, x_3, \dots, x_n, u_2, \frac{\partial u_2}{\partial x_2}, \frac{\partial u_2}{\partial x_3}, \dots, \frac{\partial u_2}{\partial x_n}) = k^2.$$

The first equation becomes an ordinary differential equation whose solution is easily obtained. The second equation may again be further separated and the same process continued.

### 6.3 Complete integral solutions

For certain types of problems a complete integral solution is available [16]. The complete integral form presents a parametric family of general solutions. The particular solution of a specific problem can then be obtained from the general complete integral solution by selection of the parameters.

We will first demonstrate generating a complete integral solution by exploiting the concept of separation of variables introduced in the last section. For the simplicity of the discussion, and without loss of generality, we will do this with an example of only two independent variables.

$$F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0.$$

We seek the complete integral solution as

$$u(x, y, u, a, b) = 0.$$

where the  $a, b$  are yet unknown coefficients. Let us generate the complete integral solution for the equation of

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1.$$

The separated solution is of the form

$$u(x, y) = u_1(x) + u_2(y).$$

The first differential equation with a constant  $k$  is then

$$F_1(x, u_1, \frac{du_1}{dx}) = \left(\frac{du_1}{dx}\right)^2 = k^2.$$

The solution comes by

$$du_1 = kdx,$$

from which

$$u_1 = kx + k_1$$

emerges. Similarly the second, in this case also ordinary equation is

$$F_2(y, u_2, \frac{du_2}{dy}) = 1 - \left(\frac{du_2}{dy}\right)^2 = k^2.$$

The solution of

$$du_2 = \sqrt{1 - k^2} dy$$

yields

$$u_2 = \sqrt{1 - k^2} y + k_2.$$

Finally the complete integral solution for this problem is

$$u(x, y, a, b) = ax + \sqrt{1 - a^2} y + b,$$

with  $a = k$  and  $b = k_1 + k_2$ . The complete integral solution satisfies the original problem

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1,$$

since

$$\frac{\partial u}{\partial x} = a, \quad \frac{\partial u}{\partial y} = \sqrt{1 - a^2},$$

and

$$a^2 + 1 - a^2 = 1.$$

This is a two-parameter family of solutions from which any particular solution may be obtained. Surely any selection of the parameter  $b$  will satisfy the original equation. As far as the parameter  $a$  is concerned, selecting for

example  $a = 1/2$  will result in

$$\frac{\partial u}{\partial x} = \frac{1}{2}, \frac{\partial u}{\partial y} = \sqrt{1 - \frac{1}{4}},$$

and

$$\frac{1}{4} + 1 - \frac{1}{4} = 1.$$

During generating a complete integral solution, the separation strategy depends on the given differential equation. When second derivatives are also present a product type separation may be used. For example, for the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0$$

the separated solution of the form

$$u(x, y) = u_1(x) \cdot u_2(y)$$

is recommended. The pair of differential equations in this scenario are

$$\frac{1}{u_1} \frac{d^2 u_1}{dx^2} = k^2,$$

and

$$\frac{1}{u_2} \frac{du_2}{dy} = k^2.$$

The solution of this system is of the form

$$u(x, y) = (k_1 e^{kx} + k_2 e^{-kx}) e^{k^2 y}.$$

Note that three parameters are needed because of the presence of the second derivative. Since this is the complete integral solution we have the freedom of choice of the parameters. By setting them all to unity, a particular solution emerges as

$$u(x, y) = (e^x + e^{-x}) e^y = e^{x+y} + e^{y-x}.$$

To validate the solution we compute

$$\frac{\partial^2 u}{\partial x^2} = e^{x+y} + e^{y-x}$$

and

$$\frac{\partial u}{\partial y} = e^{x+y} + e^{y-x},$$

whose difference is the desired zero.

Let us now consider simply using pre-computed complete integrals. Certain complete integral solutions actually contain integrals. Consider the non-homogeneous differential equation type with non-constant coefficients

$$a(x)\left(\frac{\partial u}{\partial x}\right)^2 + b(x)\left(\frac{\partial u}{\partial y}\right)^2 = f(x) + g(y).$$

Such problems have a complete integral solution of

$$u(x, y) = \int^x \sqrt{\frac{f(t) + a_1}{a(t)}} dt + \int^y \sqrt{\frac{g(t) - a_1}{b(t)}} dt + a_2.$$

For example, the equation

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = x + y$$

has a complete integral solution of the form

$$u(x, y) = \int^x \sqrt{t + a_1} dt + \int^y \sqrt{t - a_1} dt + a_2.$$

There are also rather specific, but practical problems where the partial derivatives occur in an exponential expression. The generic form of such problems is

$$\frac{\partial u}{\partial x} = f(x) \frac{\partial u}{\partial y} + g(x) e^{\frac{\partial u}{\partial y}}.$$

The complete integral solution of this problem is in the following form

$$u = a_1 \int_0^x f(t) dt + e^{a_1} \int_0^x g(t) dt + a_1 y + a_2.$$

For an example of this case, the equation

$$\frac{\partial u}{\partial x} = x^2 \frac{\partial u}{\partial y} + x e^{\frac{\partial u}{\partial y}},$$

has a complete integral solution in the following form

$$u = a_1 x^3 / 3 + e^{a_1} x^2 / 2 + a_1 y + a_2.$$

Finding a particular solution from a complete integral solution is not always trivial. From the complete integral solution of the form

$$u = f(x_1, x_2, \dots, x_n, a_1, a_2, \dots, a_n)$$

the introduction of another set of coefficients as

$$\frac{\partial f}{\partial a_i} = b_i$$

results in a new complete integral solution of

$$u = f(a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n).$$

This may provide an easier way toward the particular solution form.

## 6.4 Poisson's integral formula

We consider Laplace's equation in two dimensions that plays a fundamental role in mathematical physics:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

We will assume a circular domain and use the polar coordinate form as

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} = 0,$$

where  $r = \sqrt{x^2 + y^2}$ ,  $\phi = \arctan \frac{y}{x}$ . Here the  $r$  is the radius and the  $\phi$  is the polar angle. Using the separation of variables again, we seek the solution in the form of

$$u(r, \phi) = u_1(r)u_2(\phi),$$

with the notational convention also followed. Substituting into the equation we obtain

$$\frac{1}{u_1(r)} \left( r^2 \frac{d^2 u_1}{dr^2} + r \frac{du_1}{dr} \right) = -\frac{1}{u_2} \frac{d^2 u_2}{d\phi^2} = k^2,$$

where  $k^2$  is the yet unknown coefficient. The resulting pair of ordinary differential equations becomes

$$r^2 \frac{d^2 u_1}{dr^2} + r \frac{du_1}{dr} - k^2 u_1 = 0$$

and

$$\frac{d^2 u_2}{d\phi^2} + k^2 u_2 = 0.$$

The general solutions of these equations were derived in an earlier section. For  $k = 0$  the separate solutions are

$$u_{2,0}(\phi) = a_0 \phi + b_0$$

and

$$u_{1,0}(r) = c_0 \ln(r) + d_0.$$

The complete solution for the  $k = 0$  case is

$$u_0(r, \phi) = (a_0 \phi + b_0)(c_0 \ln(r) + d_0).$$

In the case of  $k \neq 0$  the separated solutions are

$$u_{2,k}(\phi) = a_k \cos(k\phi) + b_k \sin(k\phi),$$

and

$$u_{1,k}(r) = c_k r^k + d_k r^{-k}.$$

The solution of the problem is then

$$u_k(r, \phi) = (a_k \cos(k\phi) + b_k \sin(k\phi))(c_k r^k + d_k r^{-k}).$$

We assume a uniquely defined solution function, therefore

$$u_k(r, \phi) = u_k(r, \phi + 2\pi),$$

which implies that  $k$  can only be an integer. Executing the multiplication and introducing the products

$$\bar{a}_k = a_k c_k,$$

$$\bar{b}_k = b_k c_k,$$

$$\bar{c}_k = a_k d_k$$

and

$$\bar{d}_k = b_k d_k,$$

we obtain

$$u_k(r, \phi) = \sum_{k=1}^{\infty} r^k (\bar{a}_k \cos(k\phi) + \bar{b}_k \sin(k\phi)) + \sum_{k=1}^{\infty} \frac{1}{r^k} (\bar{c}_k \cos(k\phi) + \bar{d}_k \sin(k\phi)).$$

The constants may be found by the boundary conditions. Dictating that the solution should be non-zero and bounded at the origin implies that

$$a_0, c_0, \bar{c}_k, \bar{d}_k = 0.$$

Because the Laplace equation is linear and homogeneous, the solution is the sum of the  $k = 0$  and  $k \neq 0$  solutions:

$$u(r, \phi) = \bar{a}_0 + \sum_{k=1}^{\infty} r^k (\bar{a}_k \cos(k\phi) + \bar{b}_k \sin(k\phi)).$$

Here  $\bar{a}_0 = b_0 d_0$ . Let us impose another, external boundary condition at the outermost radius of our interest as

$$u(r_{max}, \phi) = f(\phi).$$

Substituting into the solution form we get

$$f(\phi) = \bar{a}_0 + \sum_{k=1}^{\infty} r_{max}^k (\bar{a}_k \cos(k\phi) + \bar{b}_k \sin(k\phi)).$$

Hence the coefficients become

$$\bar{a}_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\phi) d\phi,$$

$$\bar{a}_k = \frac{1}{r_{max}^k \pi} \int_0^{2\pi} f(\phi) \cos(k\phi) d\phi,$$

and

$$\bar{b}_k = \frac{1}{r_{max}^k \pi} \int_0^{2\pi} f(\phi) \sin(k\phi) d\phi.$$

Bringing the now resolved coefficients into the generic solution form and introducing a new integral variable  $\psi$  we obtain

$$u(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} f(\psi) d\psi +$$

$$\frac{1}{\pi} \sum_{k=1}^{\infty} \left(\frac{r}{r_{max}}\right)^k (\sin(k\phi) \int_0^{2\pi} f(\psi) \sin(k\psi) d\psi + \cos(k\phi) \int_0^{2\pi} f(\psi) \cos(k\psi) d\psi).$$

Employing the algebraic identity of

$$\cos(\psi - \phi) = \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\psi),$$

we can write

$$u(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} f(\psi) d\psi + \frac{1}{\pi} \sum_{k=1}^{\infty} \int_0^{2\pi} f(\psi) \left(\frac{r}{r_{max}}\right)^k \cos(k(\psi - \phi)) d\psi.$$

Since for  $0 \leq r \leq r_{max}$  the series of

$$\sum_{k=1}^{\infty} \left(\frac{r}{r_{max}}\right)^k \cos(k(\psi - \phi))$$

is monotonically convergent, the order of the integration and summation may be changed. This results in the form:

$$u(r, \phi) = \frac{1}{\pi} \int_0^{2\pi} f(\psi) \left(\frac{1}{2} + \sum_{k=1}^{\infty} \left(\frac{r}{r_{max}}\right)^k \cos(k(\psi - \phi))\right) d\psi.$$

Finally, we use Euler's formula to replace the  $\cos$  term as

$$\cos(k(\psi - \phi)) = \frac{e^{ik(\psi - \phi)} + e^{-ik(\psi - \phi)}}{2}.$$

In the above expression  $i = \sqrt{-1}$  is the imaginary unit. Substituting the above and after some algebraic manipulations we obtain

$$u(r, \phi) = \frac{1}{2\pi} \int_0^{2\pi} f(\psi) \frac{r_{max}^2 - r^2}{r_{max}^2 - 2r_{max}r \cos(\phi - \psi) + r^2} d\psi.$$

This formula is known as Poisson's integral formula. With this the solution value of Laplace's equation on a bounded circular domain may be obtained at any radius  $0 \leq r \leq r_{max}$  and at any angle  $\phi + j2\pi, j = 0, 1, 2, \dots$ , for a given boundary value function  $f(\phi)$ .

Laplace's equation of course also occurs in three-dimensional form frequently as

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Assuming a circular domain as above, the  $z$  dimension's presence leads us to use cylindrical coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Following the method laid out above, we seek the solution in the form of

$$u(r, \phi) = u_1(r)u_2(\phi)u_3(z).$$

Substitution yields

$$\frac{1}{u_3} \frac{d^2 u_3}{dz^2} = -\frac{1}{u_1} \frac{d^2 u_1}{dr^2} - \frac{1}{u_1 r} \frac{du_1}{dr} - \frac{1}{r^2 \phi} \frac{d^2 u_2}{d\phi^2}.$$

Relying on the insight gained in the last sections using an unknown coefficient, for the left-hand side we chose the solution of

$$\frac{1}{u_3} \frac{d^2 u_3}{dz^2} = k^2.$$

This choice yields

$$u_3(z) = c_1 e^{kz} + c_2 e^{-kz}.$$

The right-hand side of the problem may be written as

$$\frac{r^2}{u_1} \frac{d^2 u_1}{dr^2} + \frac{r}{u_1} \frac{du_1}{dr} + k^2 r^2 = -\frac{1}{u_2} \frac{d^2 u_2}{d\phi^2} = m^2,$$

where  $m$  is another yet unknown constant. In order to have a solution as a uniquely defined function in  $\phi$ ,  $m$  is again an integer. The right-hand equation becomes an ordinary differential equation as

$$\frac{d^2 u_2}{d\phi^2} + m^2 u_2 = 0,$$

whose solution is

$$u_2(\phi) = c_3 \cos(m\phi) + c_4 \sin(m\phi).$$



Finally the remaining equation is of the Bessel kind:

$$r^2 \frac{d^2 u_1}{dr^2} + r \frac{du_1}{dr} + (k^2 r^2 - m^2) u_1 = 0.$$

The solution of such a differential equation when  $m$  is not an integer is of the form

$$u_1(r) = c_5 J_m(kr) + c_6 J_{-m}(kr),$$

where  $J$  are the Bessel functions of the first kind, defined by the formula

$$J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} \left(\frac{x}{2}\right)^{m+2n},$$

This is a convergent series for any  $x = kr$  value. The  $J_{-m}$  function in the expression is simply defined by

$$J_{-m}(x) = (-1)^m J_m(x).$$

However, in our case  $m$  is an integer and then the solution is

$$u_1(r) = c_5 J_m(kr) + c_6 Y_m(kr).$$

The Bessel function of the second kind is defined as

$$Y_m(x) = \lim_{p \rightarrow m} \frac{\cos(p\pi) J_p(x) - J_{-p}(x)}{\sin(p\pi)}.$$

The limit is needed since the denominator is zero for any integer multiple of  $\pi$ . Therefore, this function is infinite at the origin, hence to assure that at  $r = 0$  we have a bounded solution, we choose  $c_6 = 0$ . Then the term with  $Y_m$  drops out and the solution of this equation becomes

$$u_1(r) = c_5 J_m(kr).$$

When  $m$  is an integer, Bessel functions of the first kind can also be computed from integral formulae:

$$J_m(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin(t)) \cos(mt) dt,$$

when  $m$  is even, and

$$J_m(x) = \frac{1}{\pi} \int_0^\pi \sin(x \sin(t)) \sin((m+1)t) dt,$$

when  $m$  is odd.

Finally, rejoining the separated solutions we obtain

$$u(r, \phi, z) = \sum_{m=0}^{\infty} (e^{kz} (a_m \cos(m\phi) + b_m \sin(m\phi)) +$$

$$+e^{-kz}(d_m \cos(m\phi) + e_m \sin(m\phi))J_m(kr).$$

Here  $a_m, b_m, d_m, e_m$  are various products of the above  $c_k$  constants to be specified by the boundary conditions. This is the general solution of the three-dimensional Laplace's equation in cylindrical coordinates.

## 6.5 Method of gradients

The final method in this class of solutions is that of the gradients. Let us focus on the very first variational problem we were faced with at the beginning of the book:

$$I(y) = \int_0^1 f(x, y, y') dx = \text{extremum}.$$

The gradient of this functional, based on [10] is

$$I'(y) = \int_0^x \left( \int_t^1 \frac{\partial}{\partial y} f(s, y(s), y'(s)) ds + \frac{\partial}{\partial y'} f(t, y(t), y'(t)) \right) dt.$$

$I(y)$  is at an extremum when  $I'(y) = 0$ . With this gradient, we can construct a procedure as follows. Starting from an initial solution  $y_1(x)$ , we compute successive approximations as

$$y_{i+1} = y_i - \alpha_i I'(y_i), i = 1, 2, \dots$$

where the constant may be determined from

$$\min_{\alpha_i \geq 0} I(y_i - \alpha_i I'(y_i)).$$

The sequence of

$$I(y_1) \geq I(y_2) \geq \dots \geq I(y_i)$$

is continued until the following condition is satisfied:

$$\frac{I(y_i) - I(y_{i+1})}{I(y_i)} < \epsilon, I(y_i) \neq 0.$$

Here  $\epsilon$  is an appropriately chosen small number.

Let us illuminate this procedure with a simple example of seeking the solution of the variational problem of

$$I(y) = \int_0^1 (2xy + y^2 + y'^2) dx = \text{minimum}.$$

The two parts of the gradient of this functional are computed from the above formula as

$$\int_0^x \frac{\partial f}{\partial y'} dt = \int_0^x 2y'(t) dt = 2y(x),$$

and

$$\int_0^x \int_t^1 \frac{\partial f}{\partial y} ds dt = \int_0^x \int_t^1 (2s + 2y(s)) ds dt.$$

The first term of this integral is

$$\int_0^x \int_t^1 2s ds dt = \int_0^x s^2 \Big|_t^1 dt = \int_0^x (1 - t^2) dt = x - \frac{x^3}{3}.$$

Since we do not know  $y(x)$  yet the second term cannot be integrated and kept in integral form. So the gradient of this example is

$$I'(y) = 2y(x) + x - \frac{x^3}{3} + 2 \int_0^x \int_t^1 y(s) ds dt.$$

The successive approximation procedure for this becomes

$$y_{i+1}(x) = y_i(x) - \alpha_i (2y_i(x) + x - \frac{x^3}{3} + 2 \int_0^x \int_t^1 y_i(s) ds dt).$$

From the initial solution  $y_1(x) = 0$  we obtain the starting gradient of

$$I'(y_1) = x - \frac{x^3}{3}.$$

The procedure's first step produces

$$y_2(x) = y_1(x) - \alpha_1 I'(y_1) = -\alpha_1 (x - \frac{x^3}{3}).$$

Substituting into the functional we obtain

$$I(y_2) = \int_0^1 (-2\alpha_1(x^2 - \frac{x^4}{3}) + (x - \frac{x^3}{3})^2 + \alpha_1^2(1 - x^2)^2) dx.$$

Executing the posted operations, grouping, and integrating, will result in the quadratic expression

$$I(y_2) = \frac{4}{15} (\frac{59}{21} \alpha_1^2 - 2\alpha_1)$$

whose minimum is at

$$\alpha_1 = \frac{21}{59}.$$

Hence

$$y_2(x) = -\frac{21}{59} (x - \frac{x^3}{3})$$

and

$$I(y_2) = -\frac{28}{295}.$$

We then compute  $I'(y_2)$  and the solution of

$$\min_{\alpha_2 \geq 0} I(y_2 - \alpha_2 I'(y_2))$$

produces the next iteration in the form of

$$y_3(x) = y_2(x) - \alpha_2 I'(y_2).$$

Then the process continues until convergence is achieved.

The gradient method is fundamental in engineering optimization problems of variational origin. In practical solutions, however, the gradient is computed by evaluating the solution function at discrete locations in the solution domain. This is the approach in structural optimization where the finite element method (to be discussed in detail in Section 12.5) is used.



# 7

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## *Numerical methods of calculus of variations*

In the last chapter we focused on analytical solutions. Application problems in engineering practice, however, may not be easily solved by such techniques, if solvable at all. Hence, before we embark on applications, it seems prudent to discuss solution techniques that are amenable for practical problems. These methods produce approximate solutions and are, as such, called numerical methods.

It was mentioned in the introduction that the solution of the Euler-Lagrange differential equation resulting from a certain variational problem may not be easy. This gave rise to the idea of directly solving the variational problem. The classical method is the Euler method.

The most influential method is that of Ritz. The methods of Galerkin and Kantorovich, both described in [13], could be considered extensions of Ritz's. They are the most well-known by engineers and used in the industry. Finally, the boundary integral method is also useful for certain kind of engineering problems.

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### 7.1 Euler's method

Euler proposed a numerical solution for the variational problem of

$$I(y) = \int_{x_0}^{x_n} f(x, y, y') dx = \text{extremum}$$

with the boundary conditions

$$y(x_0) = y_0; y(x_n) = y_n,$$

by subdividing the interval of the independent variable as

$$x_i = x_0 + i \frac{x_n - x_0}{n}; i = 1, 2, \dots, n.$$

Introducing

$$h = \frac{x_n - x_0}{n},$$

the functional may be approximated as

$$I(y_i) = \int_{x_0}^{x_1} f(x_i, y_i, y'_i) = h \sum_{i=1}^{n-1} f(x_0 + ih, y_i, \frac{y_{i+1} - y_i}{h}) dx = \text{extremum}.$$

Here the approximated solution values  $y_i$  are the unknowns and the extremum may be found by differentiation:

$$\frac{\partial I}{\partial y_i} = 0.$$

The process is rather simple and follows from Euler's other work in the numerical solution of differential equations. For illustration, we consider the following problem:

$$I(y) = \int_0^1 (2xy + y^2 + y'^2) dx = \text{extremum},$$

with the boundary conditions

$$y(0) = y(1) = 0.$$

Let us subdivide the interval into  $n = 5$  equidistant segments with

$$h = 0.2,$$

and

$$x_i = 0.2i.$$

The approximate functional with the appropriate substitutions becomes

$$I(y_i) = 0.2 \sum_{i=1}^4 (0.4iy_i + y_i^2 + (5(y_{i+1} - y_i))^2).$$

The computed partial derivatives are

$$\frac{\partial I}{\partial y_1} = 0.2(0.4 + 2y_1 - \frac{2(y_2 - y_1)}{0.04}) = 0,$$

$$\frac{\partial I}{\partial y_2} = 0.2(0.8 + 2y_2 - \frac{2(y_3 - y_2)}{0.04} + \frac{2(y_2 - y_1)}{0.04}) = 0,$$

$$\frac{\partial I}{\partial y_3} = 0.2(1.2 + 2y_3 - \frac{2(y_4 - y_3)}{0.04} + \frac{2(y_3 - y_2)}{0.04}) = 0,$$

and

$$\frac{\partial I}{\partial y_4} = 0.2(1.6 + 2y_4 + \frac{2y_4}{0.04} + \frac{2(y_4 - y_3)}{0.04}) = 0.$$

**TABLE 7.1**  
Accuracy of Euler's  
method

| i | $x_i$ | $y_i$   | $y(x_i)$ |
|---|-------|---------|----------|
| 1 | 0.2   | -0.0286 | -0.0287  |
| 2 | 0.4   | -0.0503 | -0.0505  |
| 3 | 0.6   | -0.0580 | -0.0583  |
| 4 | 0.8   | -0.0442 | -0.0444  |

This system of four equations yields the values of the approximate solution. The analytic solution of this problem is

$$y(x) = -x + e \frac{e^x - e^{-x}}{e^2 - 1}.$$

The comparison of the Euler solution ( $y_i$ ) and the analytic solution ( $y(x_i)$ ) at the four discrete points is shown in Table 7.1.

The boundary solutions of  $y(0)$  and  $y(1) = 0$  are not shown since they are in full agreement by definition.

## 7.2 Ritz method

Let us consider the variational problem of

$$I(y) = \int_{x_0}^{x_1} f(x, y, y') dx = \text{extremum},$$

under the boundary conditions

$$y(x_0) = y_0; y(x_1) = y_1.$$

The Ritz method is based on an approximation of the unknown solution function with a linear combination of certain basis functions. Finite element or spline-based approximations are the most commonly used and will be the subject of detailed discussion in Chapters 9 and 11. Let the unknown function be approximated with

$$\bar{y}(x) = \alpha_0 b_0(x) + \alpha_1 b_1(x) + \dots + \alpha_n b_n(x),$$

where the basis functions are also required to satisfy the boundary conditions and the coefficients are as yet unknown. Substituting the approximate solution into the variational problem results in



$$I(\bar{y}) = \int_{x_0}^{x_1} f(x, \bar{y}, \bar{y}') dx = \text{extremum}.$$

In order to reach an extremum of the functional, it is necessary that the derivatives with respect to the unknown coefficients vanish:

$$\frac{\partial I(\bar{y})}{\partial \alpha_i} = 0; i = 0, 1, \dots, n.$$

It is not intuitively clear that the approximated function approaches the extremum of the original variational problem, but it has been proven, for example in [13]. Let us just demonstrate the process with a small analytic example. Consider the variational problem of

$$I(y) = \int_0^1 y'^2(x) dx = \text{extremum},$$

with the boundary conditions

$$y(0) = y(1) = 0,$$

and constraint of

$$\int_0^1 y^2(x) dx = 1.$$

Since this is a constrained problem, we apply the Lagrange multiplier technique and rewrite the variational problem as

$$I(y) = \int_0^1 (y'^2(x) - \lambda y^2) dx = \text{extremum}.$$

Let us use, for example, the basis functions of

$$b_0(x) = x(x - 1)$$

and

$$b_1(x) = x^2(x - 1).$$

It is trivial to verify that these also obey the boundary conditions. The approximated solution function is

$$\bar{y} = \alpha_0 x(x - 1) + \alpha_1 x^2(x - 1).$$

The functional of the constrained, approximated variational problem is

$$I(\bar{y}) = \int_0^1 (\bar{y}'^2 - \lambda \bar{y}^2) dx.$$

Evaluating the integral yields

$$I(\bar{y}) = \frac{1}{3}(\alpha_0^2 + \alpha_0\alpha_1 + \frac{2}{5}\alpha_1^2) - \lambda(\frac{1}{30}\alpha_0^2 + \frac{1}{30}\alpha_0\alpha_1 + \frac{1}{105}\alpha_1^2).$$

The extremum requires the satisfaction of

$$\frac{\partial I}{\partial \alpha_0} = \alpha_0 \left( \frac{2}{3} - \frac{\lambda}{15} \right) + \alpha_1 \left( \frac{1}{3} - \frac{\lambda}{30} \right) = 0$$

and

$$\frac{\partial I}{\partial \alpha_1} = \alpha_0 \left( \frac{1}{3} - \frac{\lambda}{30} \right) + \alpha_1 \left( \frac{4}{15} - \frac{2\lambda}{105} \right) = 0.$$

A nontrivial solution of this system of equations is obtained by setting its determinant to zero, resulting in the following quadratic equation

$$\frac{\lambda^2 - 52\lambda + 420}{6300} = 0.$$

Its solutions are

$$\lambda_1 = 10; \lambda_2 = 42.$$

Using the first value and substituting into the second condition yields

$$\alpha_1 = 0$$

with arbitrary  $\alpha_0$ . Hence

$$\bar{y}(x) = \alpha_0 x(x - 1).$$

The condition

$$\int_0^1 \bar{y}^2 dx = \int_0^1 \alpha_0^2 x^2 (x - 1)^2 dx = 1$$

results in

$$\alpha_0 = \pm \sqrt{30}.$$

The approximate solution of the variational problem is

$$\bar{y}(x) = \pm \sqrt{30} x(x - 1).$$

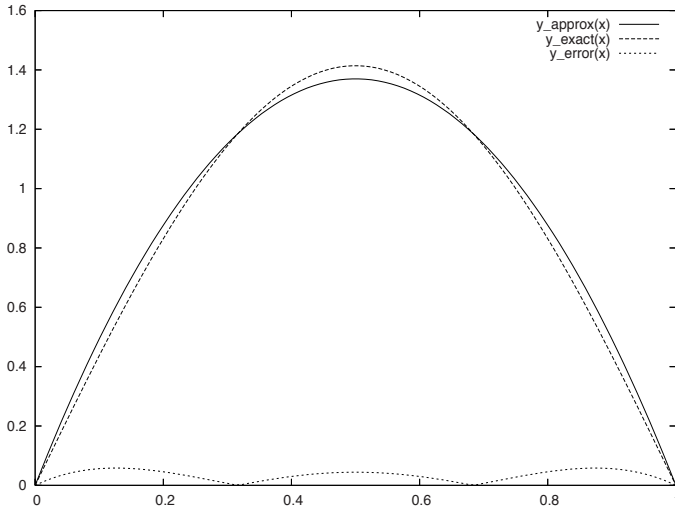
It is very important to point out that the solution obtained as a function of the chosen basis functions is not the analytic solution of the variational problem. For this particular example the corresponding Euler-Lagrange differential equation is

$$y'' + \lambda y = 0$$

whose analytic solution, based on Section 5.3, is

$$y = \pm \sqrt{2} \sin(\pi x).$$

Figure 7.1 compares the analytic and the approximate solutions and plots the error of the latter.



**FIGURE 7.1** Accuracy of the Ritz solution

The figure demonstrates that the Ritz solution satisfies the boundary conditions and shows acceptable differences in the interior of the interval. Finally, the variational problem's extremum is computed for both cases. The analytical solution is based on the derivative

$$y' = \sqrt{2}\pi\cos(\pi x),$$

and obtained as

$$\int_0^1 y'^2(x)dx = 2\pi^2 \int_0^1 \cos^2(\pi x)dx = \pi^2 = 9.87.$$

The Ritz solution's derivative is

$$\bar{y}' = -\sqrt{30}(2x - 1).$$

and the approximate extremum is

$$\int_0^1 \bar{y}'^2(x)dx = 30 \int_0^1 (2x - 1)^2 dx = \frac{30}{3} = 10.$$

The approximate extremum is slightly higher than the analytic extremum, but by only a very acceptable error.

### 7.2.1 Application: solution of Poisson's equation

The second order boundary value problem of Poisson's, introduced earlier, is presented in the variational form of

$$I(y) = \int \int_D \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 + 2f(x, y)u(x, y) \right) dx dy$$

whose Euler-Lagrange equation leads to the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y),$$

as was shown in the last chapter. For the simplicity of the discussion and without the loss of generality, we impose the boundary condition of

$$u = 0$$

on the perimeter of the domain  $D$ . Ritz's method indicates the use of the basis functions

$$b_i(x, y)$$

and demands that they also vanish on the boundary. The approximate solution in this two-dimensional case is

$$\bar{u}(x, y) = \sum_{i=1}^n \alpha_i b_i(x, y).$$

The partial derivatives are

$$\frac{\partial \bar{u}}{\partial x} = \sum_{i=1}^n \alpha_i \frac{\partial b_i(x, y)}{\partial x},$$

and

$$\frac{\partial \bar{u}}{\partial y} = \sum_{i=1}^n \alpha_i \frac{\partial b_i(x, y)}{\partial y}.$$

Substituting the approximate solution into the functional yields

$$I(\bar{u}) = \int \int_D \left( \left( \frac{\partial \bar{u}}{\partial x} \right)^2 + \left( \frac{\partial \bar{u}}{\partial y} \right)^2 + 2f(x, y)\bar{u}(x, y) \right) dx dy.$$

Evaluating the derivatives, this becomes

$$I(\bar{u}) = \int \int_D \left( \left( \sum_{i=1}^n \alpha_i \frac{\partial b_i}{\partial x} \right)^2 + \left( \sum_{i=1}^n \alpha_i \frac{\partial b_i}{\partial y} \right)^2 + 2f(x, y) \sum_{i=1}^n \alpha_i b_i \right) dx dy,$$

which may be reordered into the form

$$I(\bar{u}) = \sum_{i=1}^n \sum_{j=1}^n c_{ij} \alpha_i \alpha_j + \sum_{i=1}^n d_i \alpha_i.$$

The coefficients are

$$c_{ij} = \int \int_D \left( \frac{\partial b_i}{\partial x} \frac{\partial b_j}{\partial x} + \frac{\partial b_i}{\partial y} \frac{\partial b_j}{\partial y} \right) dx dy$$

and

$$d_i = \int \int_D f(x, y) b_i dx dy.$$

As above, the unknown coefficients are solved from the conditions

$$\frac{\partial I(\bar{u})}{\partial \alpha_i} = 0, i = 1, 2, \dots, n,$$

resulting in the linear system of equations

$$\sum_{j=1}^n c_{ij} \alpha_j + d_j = 0, i = 1, 2, \dots, n.$$

It may be shown that the system is nonsingular and always yields a nontrivial solution assuming that the basis functions form a linearly independent set. The computation of the terms of the equations, however, is rather tedious and resulted in the emergence of the next method.

### 7.3 Galerkin's method

The difference between the Ritz method and that of Galerkin's is in the fact that the latter addresses the differential equation form of a variational problem. Galerkin's method minimizes the residual of the differential equation integrated over the domain with a weight function; hence, it is also called the method of weighted residuals.

This difference lends more generality and computational convenience to Galerkin's method. Let us consider a linear differential equation in two variables:

$$L(u(x, y)) = 0$$

and apply Dirichlet boundary conditions. Galerkin's method is also based on the Ritz approximation of the solution as

$$\bar{u} = \sum_{i=1}^n \alpha_i b_i(x, y),$$

in which case, of course there is a residual of the differential equation

$$L(\bar{u}) \neq 0.$$

Galerkin proposed using the basis functions of the approximate solution also as the weights, and requires that the integral vanishes with a proper selection of the coefficients:

$$\int \int_D L(\bar{u})b_j(x, y)dxdy = 0; j = 1, 2, \dots, n.$$

This yields a system for the solution of the coefficients as

$$\int \int_D L\left(\sum_{i=1}^n \alpha_i b_i(x, y)\right)b_j(x, y)dxdy = 0; j = 1, 2, \dots, n.$$

This is also a linear system and produces the unknown coefficients  $\alpha_i$ . We illustrate the computational technique of Galerkin's method also in connection with Poisson's equation:

$$L(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - f(x, y) = 0.$$

For this Galerkin's method is presented as

$$\int \int_D \left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} - f(x, y)\right)b_j dxdy = 0, j = 1, \dots, n.$$

Therefore

$$\int \int_D \left(\sum_{i=1}^n \alpha_i \frac{\partial^2 b_i}{\partial x^2} + \sum_{i=1}^n \alpha_i \frac{\partial^2 b_i}{\partial y^2} - f(x, y)\right)b_j dxdy = 0, j = 1, \dots, n.$$

Reordering yields

$$\sum_{i=1}^n \alpha_i \int \int_D \left(\frac{\partial^2 b_i}{\partial x^2} + \frac{\partial^2 b_i}{\partial y^2}\right)b_j dxdy - \int \int_D f(x, y)b_j dxdy = 0, j = 1, \dots, n.$$

The system of equations becomes

$$B\mathbf{a} = \mathbf{b}$$

with solution vector of

$$\mathbf{a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \dots \\ \alpha_n \end{bmatrix}.$$

The system matrix is of the form

$$B = \begin{bmatrix} B_{1,1} & B_{1,2} & \dots & B_{1,n} \\ B_{2,1} & B_{2,2} & \dots & B_{2,n} \\ \dots & \dots & \dots & \dots \\ B_{n,1} & B_{n,2} & \dots & B_{n,n} \end{bmatrix}$$

whose terms are defined as

$$B_{j,i} = \int \int_D \left( \frac{\partial^2 b_i}{\partial x^2} + \frac{\partial^2 b_i}{\partial y^2} \right) b_j dx dy.$$

Finally, the right-hand side vector is

$$\underline{b} = \begin{bmatrix} \int \int_D f(x,y) b_1 dx dy \\ \int \int_D f(x,y) b_2 dx dy \\ \dots \\ \int \int_D f(x,y) b_n dx dy \end{bmatrix}.$$

Note that if Galerkin's and Ritz's methods are applied to the same problem, the approximate solutions found are identical.

## 7.4 Kantorovich's method

Both the Ritz and Galerkin methods are restricted in their choices of basis functions, because their basis functions are required to satisfy the boundary conditions. The clever method of Kantorovich, described in [13], relaxes this restriction and enables the use of simpler basis functions.

Consider the variational problem of two variables

$$I(u) = \text{extremum}, (x, y) \in D,$$

with boundary conditions

$$u(x, y) = 0, (x, y) \in \partial D.$$

Here  $\partial D$  again denotes the boundary of the domain.

The method proposes the construction of a function, such that

$$\omega(x, y) \geq 0, (x, y) \in D,$$

and

$$\omega(x, y) = 0, (x, y) \in \partial D.$$

This function assumes the role of enforcing the boundary condition and the following set of simpler, low order basis functions are adequate to present the solution:

$$\begin{aligned} b_1(x, y) &= \omega(x, y), \\ b_2(x, y) &= \omega(x, y)x, \\ b_3(x, y) &= \omega(x, y)y, \\ b_4(x, y) &= \omega(x, y)x^2, \\ b_5(x, y) &= \omega(x, y)xy, \\ b_6(x, y) &= \omega(x, y)y^2, \end{aligned}$$

and so on, following the same pattern. It is clear that all these basis functions vanish on the boundary by the virtue of  $\omega(x, y)$ , even though the power function components do not.

The question is how to generate  $\omega(x, y)$  for various shapes of domains. For a centrally located circle with radius  $r$ , the equation

$$x^2 + y^2 = r^2$$

implies very intuitively the form of

$$\omega(x, y) = r^2 - x^2 - y^2.$$

Obviously the function is zero everywhere on the circle and non-zero in the interior of the domain. It is also non-zero on the outside of the domain, but that is irrelevant in connection with our problem.

One can also consider a domain consisting of overlapping circular regions, some of which represent voids in the domain. Figure 7.2 shows a domain of two circles with equations

$$x^2 + y^2 = r^2$$

and

$$(x - r/2)^2 + y^2 = (r/2)^2.$$

Reordering the latter yields

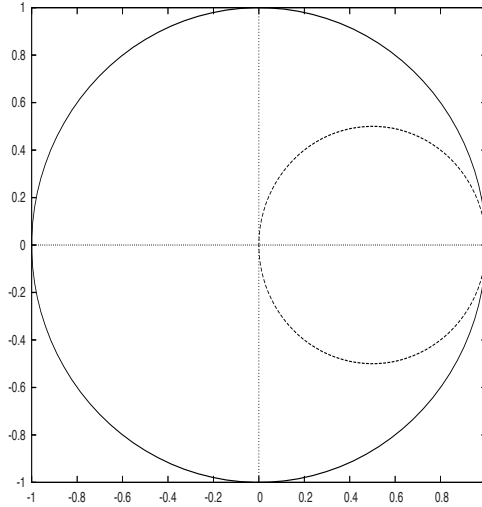
$$x^2 - xr + y^2 = 0,$$

and in turn results in

$$\omega(x, y) = (r^2 - x^2 - y^2)(x^2 - rx + y^2).$$

Clearly on the boundary of the larger circle the left term is zero and on the boundary of the smaller circle the right term is zero. Hence the product





**FIGURE 7.2** Domain with overlapping circular regions

function vanishes on the perimeter of both circles, which constitutes the now nontrivial boundary.

Let us now consider the boundary of a rectangle of width  $2w$  and height  $2h$ , also centrally located around the origin. The equations of the sides

$$x = \pm w,$$

and

$$y = \pm h,$$

imply the very simple form of

$$\omega(x, y) = (w^2 - x^2)(h^2 - y^2).$$

The verification is very simple,

$$\omega(x, y) = 0, (x, y) = (\pm w, \pm h).$$

The construction technique clearly shows signs of difficulties to come with very generic, and especially three-dimensional domains. In fact such difficulties limited the practical usefulness of this otherwise innovative method until more recent work enabled the automatic creation of the  $\omega$  functions for generic

two- or three-dimensional domains with the help of spline functions, a topic that will be discussed in Chapter 11 at length.

We shall now demonstrate the correctness of such a solution. For this we consider the solution of a specific Poisson's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -2,$$

with

$$u(x, y) = 0, (x, y) \in \partial D,$$

where we designate the domain to be the rectangle whose  $\omega$  function was specified above. We will search for Kantorovich's solution in the form of

$$u(x, y) = (w^2 - x^2)(h^2 - y^2)(\alpha_1 + \alpha_2 x + \alpha_3 y + \dots).$$

The method, as all direct methods, is approximate, so we may truncate the sequence of power function terms at a certain power. It is sufficient for the demonstration to use only the first term.

We will apply the method in connection with Galerkin's method of the last section. Therefore the extremum is sought from

$$\int_{-w}^{+w} \int_{-h}^{+h} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2 \right) \omega(x, y) dy dx = 0.$$

Executing the posted differentiations and substituting results in

$$\int_{-w}^{+w} \int_{-h}^{+h} -2\alpha_1(w^2 - x^2)(h^2 - y^2)^2 - 2\alpha_1(w^2 - x^2)^2(h^2 - y^2) + 2(w^2 - x^2)(h^2 - y^2) dy dx = 0.$$

Since we only have a single coefficient, the system of equations developed earlier boils down to a single scalar equation of

$$b\alpha_1 = f,$$

with

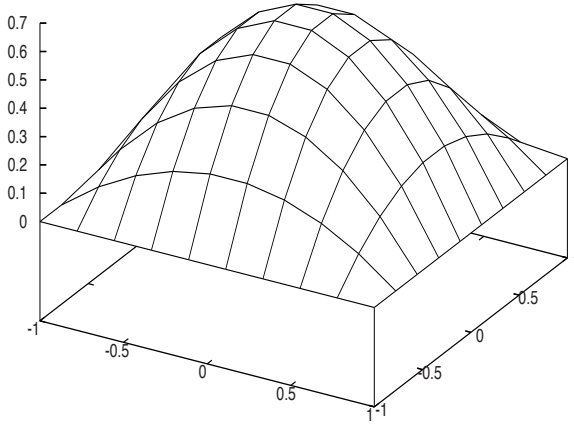
$$b = \int_{-w}^{+w} \int_{-h}^{+h} ((w^2 - x^2)(h^2 - y^2)^2 + (w^2 - x^2)^2(h^2 - y^2)) dy dx,$$

and

$$f = \int_{-w}^{+w} \int_{-h}^{+h} (w^2 - x^2)(h^2 - y^2) dy dx.$$

After the (tedious) evaluation of the integrals, the value of

$$\alpha_1 = \frac{5}{4(w^2 + h^2)}$$



**FIGURE 7.3** Solution of Poisson's equation

emerges. In turn, the approximate Kantorovich-Galerkin solution is

$$u(x, y) = \frac{5}{4} \frac{(w^2 - x^2)(h^2 - y^2)}{w^2 + h^2}.$$

The solution is depicted graphically in Figure 7.3 using

$$w = h = 1.$$

The figure demonstrates that the solution function satisfies the zero boundary condition on the circumference of the square. To increase accuracy, additional terms of the power series may be used. The method also enables the exploitation of the symmetry of the domain. For example, if the above domain would exhibit the same height as width,

$$s = w = h,$$

the solution may be sought in the form of

$$u(x, y) = (s^2 - x^2)(s^2 - y^2)(\alpha_1 + \alpha_{23}(x + y)),$$

where  $\alpha_{23}$  denotes the single constant accompanying both the second and third terms.

A generalization of this approach is necessary to eliminate the difficulties of producing an analytic  $\omega$  function for practical domains with convoluted boundaries. The idea is to use an approximate solution to generate the  $\omega$  function as well.

Let us consider the two-dimensional domain case and generate a surface approximation over the domain in the form of

$$\omega(x, y) = \sum_{i=0}^n \sum_{j=0}^m C_{i,j} B_i(x) B_j(y),$$

where the two sets of  $B$  basis functions are of common form, but dependent on either of the independent variables. The coefficients  $C_{i,j}$  are either sampling points of the domain, or control points used to generate the surface. The latter case applies mainly to the interior points, and the earlier to the boundary.

This requires a simple Cartesian discretization of the domain along topological (possibly even equidistant) lines. The B-spline fitting technique introduced in Chapter 9 will provide the means for accomplishing this.

## 7.5 Boundary integral method

The boundary integral method is related to Kantorovich's method in the sense that both make use of the boundary-interior distinction of a variational problem. We will discuss this method in connection with a two-dimensional variational problem; however, the technique and conclusions apply to three dimensions as well. Let us consider the problem of

$$L(x, y)u(x, y) = f(x, y),$$

where  $L(x, y)$  is a two-dimensional linear operator and the problem is defined on the domain  $(x, y) \in \Omega$ . The domain's boundary is  $\Gamma$  and the outward normal of the boundary,  $n$ , is defined.

The boundary integral method finds the solution in the form of

$$u(x, y) = \int_{\Omega} G(P, Q) f(x, y) d\Omega.$$

Here  $G(P, Q)$  is Green's function corresponding to the particular linear operator. It is defined in terms of two points,  $P = (x_p, y_p)$ ,  $Q = (x_q, y_q)$  as

$$L(G(P, Q)) = \delta(P - Q),$$

where  $\delta$  is the Dirac function. Let us work with the two-dimensional Poisson's equation of the form

$$\Delta u(x, y) = f(x, y).$$

Here  $L(x, y) = \Delta = \nabla^2$  and its Green's function is

$$G(P, Q) = \frac{1}{2\pi} \ln(r),$$

where

$$r = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}.$$

Green's theorem's generic form (a consequence of Gauss' divergence theorem) may be written as

$$\int_{\Omega} (u \nabla^2 v - v \nabla^2 u) d\Omega = \int_{\Gamma} (u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}) d\Gamma.$$

Using Green's function in place of  $v$  we obtain

$$\int_{\Omega} (u \nabla^2 G - G \nabla^2 u) d\Omega = \int_{\Gamma} (u \frac{\partial G}{\partial n} - G \frac{\partial u}{\partial n}) d\Gamma.$$

By definition

$$L(G(P, Q)) = \nabla^2 G(P, Q) = \delta(P - Q),$$

and due to the characteristics of the Dirac function the first term on the left-hand side reduces to  $u(x, y)$ . Substituting the original equation into the second term the resulting boundary integral solution becomes

$$u(x, y) = \int_{\Omega} G f(x, y) d\Omega + \int_{\Gamma} u \frac{\partial G}{\partial n} d\Gamma - \int_{\Gamma} G \frac{\partial u}{\partial n} d\Gamma.$$

The first term on the right-hand side is the applied load in the domain and it is zero when the homogeneous Laplace problem is solved. The second term contains the Dirichlet boundary conditions via given boundary values of the function. The third term represents the Neumann boundary conditions by given derivatives with respect to the normal. It is possible that both types are given at the same time.

Assuming that the set of discretized points on the boundary are  $q_j, j = 1, \dots, m$  and boundary conditions are given, the solution at any point in the interior may be computed as

$$u(x, y) = \int_{\Omega} G f(x, y) d\Omega + \sum_{j=1}^m u(x_{q_j}, y_{q_j}) \int_{\Gamma_j} \frac{\partial G(p, q_j)}{\partial n} d\Gamma_j -$$

$$-\sum_{j=1}^m \frac{\partial u(x_{q_j}, y_{q_j})}{\partial n} \int_{\Gamma_j} G(p, q_j) d\Gamma_j.$$

Here the boundary segments are assigned to the given boundary points as

$$\Gamma = \sum_{j=1}^{m-1} \Gamma_j.$$

It is also possible to produce a discretized solution at a set of given interior points  $p_i, i = 1, \dots, n$ . In this case a matrix formulation is possible (using the homogeneous case for simplicity of the presentation) as

$$u(x_{p_i}, y_{p_i}) = \sum_{j=1}^m A_{i,j} u(x_{q_j}, y_{q_j}) - \sum_{j=1}^m B_{i,j} \frac{\partial u(x_{q_j}, y_{q_j})}{\partial n},$$

where the matrix coefficients contain the pre-computed integrals

$$A_{i,j} = \int_{\Gamma_j} \frac{\partial G(p_i, q_j)}{\partial n} d\Gamma_j,$$

and

$$B_{i,j} = \int_{\Gamma_j} G(p_i, q_j) d\Gamma_j.$$

Let us now gather the solution points into the array

$$\underline{u} = \begin{bmatrix} u(x_{p_1}, y_{p_1}) \\ u(x_{p_2}, y_{p_2}) \\ \dots \\ u(x_{p_n}, y_{p_n}) \end{bmatrix}.$$

Then the solution may be written as a simple matrix equation:

$$\underline{u} = A\underline{v} - B\underline{t},$$

where the vector containing the boundary condition displacement values is

$$\underline{v} = \begin{bmatrix} u(x_{q_1}, y_{q_1}) \\ u(x_{q_2}, y_{q_2}) \\ \dots \\ u(x_{q_m}, y_{q_m}) \end{bmatrix}$$

and the vector holding the tangents is

$$\underline{t} = \begin{bmatrix} \frac{\partial u(x_{q_1}, y_{q_1})}{\partial n} \\ \frac{\partial u(x_{q_2}, y_{q_2})}{\partial n} \\ \dots \\ \frac{\partial u(x_{q_m}, y_{q_m})}{\partial n} \end{bmatrix}.$$

This is the approach of software tools using the boundary element method. The method is of engineering importance when the solution of a problem in the interior is largely homogeneous and the important solution variation is at

or close to the boundary.

Let us now consider the case when only boundary tangents (Neumann boundary conditions) are given. Then the unknowns may be both in the interior and on the boundary as

$$u(x_{p_i}, y_{p_i}) = u(x_{q_j}, y_{q_j}),$$

when  $i = j$ . By the definition, the Green's function for the Laplacian operator is singular when the solution point coincides with a boundary condition point and the solution integrals become improper. Hence the evaluation of the matrix coefficients must deal with that issue.

Nevertheless, the problem can be reformulated as

$$\sum_{j=1}^m (A_{i,j} + \frac{1}{2}\delta_{ij})u(x_{q_j}, y_{q_j}) = \sum_{j=1}^m B_{i,j} \frac{\partial u(x_{q_j}, y_{q_j})}{\partial n},$$

where  $\delta_{ij}$  is the Kronecker delta. The problem is then of the form

$$\overline{A}\underline{u} = B\underline{t}.$$

Since the matrix on the left-hand side is now square, the system of equations may be formally solved as

$$\underline{u} = \overline{A}^{-1} B\underline{t}.$$

The singularity of the integrals carry into the system matrix by making it numerically ill-conditioned and requiring specialized solution techniques that avoid computing an explicit inverse.

This formulation is very similar to the finite element solution, a subject of detailed discussion in Chapter 12. This similarity is exploited in industrial applications where part of the physical problem is solved by the finite element method and part by the boundary integral method.

The method is easily generalized to the three-dimensional Laplace operator whose Green's function is of the form

$$G(P, Q) = \frac{-1}{4\pi r}.$$

Finding the Green's function for other operators is also possible. For example, the Green's function for the operator

$$L(x, y, t) = \partial_t^2 - \nabla^2$$

is defined also in terms of the Dirac function and the radius  $r$  as

$$G(P, Q) = \frac{\delta(t-r)}{4\pi r}.$$

This is the so-called d'Alembert operator of the wave equation that will be the subject of a mechanical problem (the elastic string) in Section 11.1.





## Part II

# Engineering applications



# 8

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## *Differential geometry*

Differential geometry is a classical mathematical area that has become very important for engineering applications in the recent decades. This importance is based on the rise of computer-aided visualization and geometry generation technologies.

The fundamental problem of differential geometry, the finding of geodesic curves, has practical implications in manufacturing. Development of non-mathematical surfaces used in ships and airplanes has serious financial impact in reducing material waste and improving the quality of the surfaces.

While the discussion in this chapter will focus on analytically solvable problems, the methods and concepts we introduce will provide a foundation applicable in various engineering areas.

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### 8.1 The geodesic problem

Finding a geodesic curve on a surface is a classical problem of differential geometry. Variational calculus seems uniquely applicable to this problem. Let us consider a parametrically given surface

$$\underline{r} = x(u, v)\underline{i} + y(u, v)\underline{j} + z(u, v)\underline{k}.$$

Let two points on the surface be

$$\underline{r}_0 = x(u_0, v_0)\underline{i} + y(u_0, v_0)\underline{j} + z(u_0, v_0)\underline{k},$$

and

$$\underline{r}_1 = x(u_1, v_1)\underline{i} + y(u_1, v_1)\underline{j} + z(u_1, v_1)\underline{k}.$$

The shortest path on the surface between these two points is the **geodesic curve**. Consider the square of the arc length

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2,$$

and compute the differentials related to the parameters.

$$ds^2 = E(u, v)(du)^2 + 2F(u, v)dudv + G(u, v)(dv)^2.$$

Here the so-called first fundamental quantities are defined as

$$E(u, v) = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 = (\underline{r}'_u)^2,$$

$$F(u, v) = \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} + \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} + \frac{\partial z}{\partial u} \frac{\partial z}{\partial v} = \underline{r}'_u \underline{r}'_v,$$

and

$$G(u, v) = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 = (\underline{r}'_v)^2.$$

Assume that the equation of the geodesic curve in the parametric space is described by

$$v = v(u).$$

Then the geodesic curve is the solution of the variational problem

$$I(v) = \int_{u_0}^{u_1} \sqrt{E(u, v) + 2F(u, v) \frac{dv}{du} + G(u, v) \left(\frac{dv}{du}\right)^2} du = \text{extremum}$$

with boundary conditions

$$v(u_0) = v_0,$$

and

$$v(u_1) = v_1.$$

The corresponding Euler-Lagrange differential equation is

$$\frac{E_v + 2v'F_v + v'^2G_v}{2\sqrt{E(u, v) + 2F(u, v)v' + G(u, v)v'^2}} - \frac{d}{du} \frac{F + Gv'}{\sqrt{E(u, v) + 2F(u, v)v' + G(u, v)v'^2}} = 0,$$

with the notation of

$$E_v = \frac{\partial E}{\partial v}, F_v = \frac{\partial F}{\partial v}, G_v = \frac{\partial G}{\partial v},$$

and

$$v' = \frac{dv}{du}.$$

The equation is rather difficult in general, and exploitation of special cases arising from the particular surface definitions is advised.

When the first fundamental quantities are only functions of the  $u$  parameter, the equation simplifies to

$$\frac{F + Gv'}{\sqrt{E(u, v) + 2F(u, v)v' + G(u, v)v'^2}} = c_1.$$

A further simplification is based on the practical case when the  $u$  and  $v$  parametric lines defining the surface are orthogonal. In this case

$$F = 0,$$

and the equation may easily be integrated as

$$v = c_1 \int \frac{\sqrt{E}}{\sqrt{G^2 - c_1^2 G}} du + c_2.$$

The integration constants may be resolved from the boundary conditions.

When the function in the integral only contains the  $v$  function explicitly, and the  $F = 0$  assumption still holds, then the equation becomes

$$\frac{Gv'^2}{\sqrt{E + Gv'^2}} - \sqrt{E + Gv'^2} = c_1.$$

Reordering and another integration brings

$$u = c_1 \int \frac{\sqrt{G}}{\sqrt{E^2 - c_1^2 E}} dv + c_2.$$

### 8.1.1 Geodesics of a sphere

For an enlightening example, we consider a sphere, given by

$$x(u, v) = R \sin(v) \cos(u),$$

$$y(u, v) = R \sin(v) \sin(u),$$

and

$$z(u, v) = R \cos(v).$$

The first fundamental quantities encapsulating the surface information are

$$E = R^2 \sin^2(v),$$

$$F = 0,$$

and

$$G = R^2.$$

Since this is the special case consisting of only  $v$ , the equation of the geodesic curve becomes

$$u = c_1 \int \frac{R}{\sqrt{R^4 \sin^4(v) - c_1^2 R^2 \sin^2(v)}} dv + c_2.$$

After the integration by substitution and some algebraic manipulations, we get

$$u = -a \sin \frac{\cot(v)}{\sqrt{\left(\frac{R}{c_1}\right)^2 - 1}} + c_2.$$

It follows that

$$\sin(c_2)(R \sin(v) \cos(u)) - \cos(c_2)(R \sin(v) \sin(u)) - \frac{R \cos(v)}{\sqrt{\left(\frac{R}{c_1}\right)^2 - 1}} = 0.$$

Substituting the surface definition of the sphere yields

$$x \sin(c_2) + y \cos(c_2) - \frac{z}{\sqrt{\left(\frac{R}{c_1}\right)^2 - 1}} = 0$$

and that represents an intersection of the sphere with a plane. By substituting boundary conditions, it would be easy to see that the actual plane contains the origin and defines the great circle going through the two given points. This fact is manifested in everyday practice by the transoceanic airplane routes' well-known northern swing in the Northern Hemisphere.

## 8.2 A system of differential equations for geodesic curves

Let us now seek the geodesic curve in the parametric form of

$$u = u(t),$$

and

$$v = v(t).$$

The curve goes through two points

$$P_0 = (u(t_0), v(t_0)),$$

and

$$P_1 = (u(t_1), v(t_1)).$$

Then the following variational problem provides the solution:

$$I(u, v) = \int_{t_0}^{t_1} \sqrt{Eu'^2 + 2Fu'v' + Gv'^2} dt = \text{extremum.}$$

Here

$$u' = \frac{du}{dt}, v' = \frac{dv}{dt}.$$

The corresponding Euler-Lagrange system of differential equations is

$$\frac{E_u u'^2 + 2F_u u'v' + G_u v'^2}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}} - \frac{d}{dt} \frac{2(Eu' + Fv')}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}} = 0,$$

and

$$\frac{E_v u'^2 + 2F_v u'v' + G_v v'^2}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}} - \frac{d}{dt} \frac{2(Fu' + Gv')}{\sqrt{Eu'^2 + 2Fu'v' + Gv'^2}} = 0.$$

In the equations the notation

$$E_u = \frac{\partial E}{\partial u}, F_u = \frac{\partial F}{\partial u}, G_u = \frac{\partial G}{\partial u}$$

was used.

A more convenient and practically useful formulation, without the explicit derivatives, based on [8] is

$$u'' + \Gamma_{11}^1 u'^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 v'^2 = 0,$$

and

$$v'' + \Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2 = 0.$$

Here

$$u'' = \frac{d^2 u}{dt^2}, v'' = \frac{d^2 v}{dt^2}.$$

The  $\Gamma$  are the Christoffel symbols that are defined as

$$\Gamma_{11}^1 = \frac{GE_u - 2FF_u + FE_v}{2(EG - F^2)},$$

$$\Gamma_{12}^1 = \frac{GE_v - FG_u}{2(EG - F^2)},$$

$$\Gamma_{22}^1 = \frac{2GF_v - GG_u - FG_v}{2(EG - F^2)},$$

$$\Gamma_{11}^2 = \frac{2EF_u - EE_v - FE_u}{2(EG - F^2)},$$



$$\Gamma_{12}^2 = \frac{EG_u - FE_v}{2(EG - F^2)},$$

and

$$\Gamma_{22}^2 = \frac{EG_v - 2FF_v + FG_u}{2(EG - F^2)}.$$

These formulae all require that

$$EG - F^2 \neq 0$$

which is true when a parameterization is regular.

### 8.2.1 Geodesics of surfaces of revolution

Another practically important special case is represented by surfaces of revolution. Their generic description may be of the form

$$x = u \cos(v),$$

$$y = u \sin(v),$$

and

$$z = f(u).$$

Here the last equation describes the meridian curve generating the surface. The first order fundamental terms are

$$E = 1 + f'^2(u),$$

$$F = 0,$$

and

$$G = u^2.$$

The solution following the discussion in Section 8.1 becomes

$$v = c_1 \int \frac{\sqrt{1 + f'^2(u)}}{u\sqrt{u^2 - c_1^2}} du + c_2.$$

A simple sub-case of this class is a unit cylinder, described as

$$x = \cos(v),$$

$$y = \sin(v),$$

and

$$z = u.$$

The geometric meaning of the  $v$  parameter is the rotation angle generating the cylinder's circumference and the  $u$  parameter is the axial direction of the

surface. The fundamental terms are

$$E = 1,$$

$$F = 0,$$

and

$$G = 1.$$

The equation of the geodesic curve on the cylinder following above is

$$v = c_1 \int \frac{1}{1\sqrt{1-c_1^2}} du + c_2,$$

or

$$v = c_1 \frac{1}{\sqrt{1-c_1^2}} \int du + c_2.$$

With

$$c_3 = c_1 \frac{1}{\sqrt{1-c_1^2}}$$

and integration we obtain

$$v = c_3 u + c_2.$$

In the general case, this is a helix on the surface of the cylinder going through the two points. This is also a line in the parametric space. This fact is geometrically easy to explain because the cylinder is a developable surface. Such a surface may be rectified onto a plane. In such a case the geodesic curve is a straight line on the rectifying plane. The only curvature of the helix will be that of the cylinder.

The constants of integration may be determined from the boundary conditions. For example, assume the case shown in Figure 8.1, where the starting point is at

$$P_0 = (x_0, y_0, z_0) = (1, 0, 0)$$

corresponding to parametric coordinates

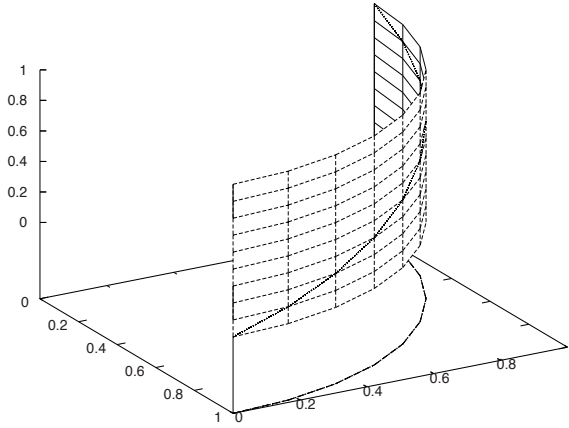
$$u(t_0) = 0, v(t_0) = 0.$$

The endpoint is located at

$$P_1 = (x_1, y_1, z_1) = (0, 1, 1)$$

corresponding to parametric coordinates

$$u(t_1) = 1, v(t_1) = \pi/2.$$



**FIGURE 8.1** Geodesic curve of a cylinder

Substituting the starting point yields

$$0 = c_3 \cdot 0 + c_2,$$

which results in

$$c_2 = 0.$$

The endpoint substitution produces

$$\pi/2 = c_3 \cdot 1 + c_2,$$

and in turn

$$c_3 = \frac{\pi}{2}.$$

The specific solution for this case in the parametric space is

$$v = \frac{\pi}{2}u.$$

The Cartesian solution is obtained in the form of

$$x = \cos(v),$$

$$y = \sin(v),$$

and

$$z = \frac{v}{\pi/2}.$$

It is easy to see that the latter equation makes the elevation change from zero to 1, in accordance with the turning of the helix.

Since the parametric space of the cylinder is simply the rectangle of the developed surface, it is easy to see some special sub-cases. If the two points are located at the same rotational position ( $v = \text{constant}$ ), but at different heights, the geodesic curve is a straight line. If the two points are on the same height ( $u = \text{constant}$ ), but at different rotational angles, the geodesic curve is a circular arc.

The last two sections demonstrated the difficulties of finding the geodesic curves even on regular surfaces like the sphere or the cylinder. On a general three-dimensional surface these difficulties increase significantly and may render using the differential equation of the geodesic curve unfeasible.

### 8.3 Geodesic curvature

Let us consider the parametric curve

$$\underline{r}(t) = x(t)\underline{i} + y(t)\underline{j} + z(t)\underline{k}$$

on the surface

$$S(u, v) = x(u, v)\underline{i} + y(u, v)\underline{j} + z(u, v)\underline{k}.$$

Let  $\underline{n}$  denote the unit normal of the surface. The curvature vector of a three-dimensional curve is defined as

$$\underline{k} = \frac{d\underline{t}}{dt} = \underline{t}',$$

where  $\underline{t}$  is the tangent vector computed as

$$\underline{t} = \frac{d\underline{r}}{dt},$$

and also assumed to be a unit vector (a unit speed curve) for the simplicity of the derivation. Then the unit bi-normal vector is

$$\underline{b} = \underline{n} \times \underline{t}.$$

We represent the curvature vector with components along the bi-normal vector and the normal vector  $\underline{n}$  at any point as

$$\underline{k} = \kappa_n \underline{n} + \kappa_g \underline{b}.$$

The coefficients are the normal curvature and the geodesic curvature, respectively. Taking the inner product of the last equation with the  $\underline{b}$  vector and exploiting the perpendicularity conditions present, we obtain

$$\underline{b} \cdot \underline{k} = \kappa_g.$$

Substituting the definition of the bi-normal and the curvature vector results in

$$\kappa_g = (\underline{n} \times \underline{t}) \cdot \underline{t}'.$$

For the more generic case when the tangent and normal vectors are not of unit length, the geodesic curvature of a curve is defined as

$$\kappa_g = \frac{\underline{r}''(t) \cdot (\underline{n} \times \underline{r}'(t))}{\|\underline{r}'(t)\|^3}.$$

A curve on a surface is called geodesic if at each point of the curve its principal normal and the surface normal are collinear. Therefore:

**A curve  $\underline{r}(t)$  on the surface  $S(u, v)$  is geodesic if the geodesic curvature of the curve is zero.**

In order to prove that, the terms are computed from the surface information, such as

$$\underline{r}' = \underline{t} = \frac{\partial f}{\partial u} u' + \frac{\partial f}{\partial v} v' = f_u u' + f_v v'.$$

The application of the chain rule results in

$$\underline{r}'' = \underline{t}' = f_{uu}(u')^2 + 2f_{uv}u'v' + f_{vv}(v')^2 + f_u u'' + f_v v''.$$

After substitution into the equation of the geodesic curvature and some algebraic work, while employing again the Christoffel symbols, [8] produces the form

$$\begin{aligned} \kappa_g = \sqrt{EG - F^2} & (\Gamma_{11}^2 (u')^3 + (2\Gamma_{12}^1 - \Gamma_{11}^1)(u')^2 v' + \\ & (\Gamma_{22}^2 - 2\Gamma_{12}^1)u'(v')^2 - \Gamma_{22}^1(v')^3 + u'v'' - u''v'). \end{aligned}$$

Since

$$EG - F^2 \neq 0,$$

the term in the parenthesis must be zero for zero geodesic curvature. That happens when the following terms vanish

$$u'' + \Gamma_{11}^1 u'^2 + 2\Gamma_{12}^1 u'v' + \Gamma_{22}^1 v'^2 = 0,$$

and

$$v'' + \Gamma_{11}^2 u'^2 + 2\Gamma_{12}^2 u'v' + \Gamma_{22}^2 v'^2 = 0.$$

This result is the decoupled system of equations of the geodesic, introduced in Section 8.1, hence the vanishing of the geodesic curvature is indeed a characteristic of a geodesic curve.

Finally, since the recent discussions were mainly on parametric forms, the equation of the geodesic for an explicitly given surface

$$z = z(x, y(x))$$

is quoted from [8] for completeness' sake without derivation:

$$\begin{aligned} (1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2) \frac{d^2 y}{dx^2} &= \frac{\partial z}{\partial x} \frac{\partial^2 z}{\partial y^2} (\frac{dy}{dx})^3 + \\ & (2 \frac{\partial z}{\partial x} \frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial y} \frac{\partial^2 z}{\partial y^2}) (\frac{dy}{dx})^2 + \\ & (\frac{\partial z}{\partial x} \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial z}{\partial y} \frac{\partial^2 z}{\partial x \partial y}) \frac{dy}{dx} - \frac{\partial z}{\partial y} \frac{\partial^2 z}{\partial x^2}. \end{aligned}$$

The formula is rather overwhelming and useful only in connection with the simplest surfaces.

### 8.3.1 Geodesic curvature of helix

Let us enlighten this further by reconsidering the case of the geodesic curve of the cylinder discussed in the last section. The geodesic curve we obtained was the helix:

$$r(t) = \cos(t)\underline{i} + \sin(t)\underline{j} + \frac{t}{\pi/2}\underline{k}.$$

The appropriate derivatives are

$$r'(t) = -\sin(t)\underline{i} + \cos(t)\underline{j} + \frac{1}{\pi/2}\underline{k}$$

and

$$r''(t) = -\cos(t)\underline{i} - \sin(t)\underline{j} + 0\underline{k}.$$

The surface normal is computed as

$$\underline{n} = \frac{\partial S}{\partial u} \times \frac{\partial S}{\partial v}.$$

In the specific case of the cylinder

$$S(u, v) = \cos(v)\underline{i} + \sin(v)\underline{j} + u\underline{k},$$

it becomes

$$\underline{n} = \cos(t)\underline{i} + \sin(t)\underline{j} + 0\underline{k}.$$

The cross-product and substitution of  $v = t$  yields

$$\underline{n}(t) \times r'(t) = \frac{2}{\pi}\sin(t)\underline{i} - \frac{2}{\pi}\cos(t)\underline{j} + (\sin^2(t) + \cos^2(t))\underline{k}.$$

The numerator of the curvature becomes zero, as

$$(-\cos(t)\underline{i} - \sin(t)\underline{j} + 0\underline{k}) \cdot \left(\frac{2}{\pi}\sin(t)\underline{i} - \frac{2}{\pi}\cos(t)\underline{j} + (\sin^2(t) + \cos^2(t))\underline{k}\right) = 0.$$

Since the denominator

$$\|r'(t)\|^3 = \left(\sqrt{1 + \left(\frac{1}{\pi/2}\right)^2}\right)^3$$

is not zero, the geodesic curvature becomes zero. Hence the helix is truly the geodesic curve of the cylinder.

The concept of geodesic curves may be generalized to higher-dimensional spaces. The geodesic curve notation, however, while justified on a three-dimensional surface, needs to be generalized as well. In such cases one talks about finding a geodesic object, or just a geodesic, as opposed to a curve on a surface.

## 8.4 Generalization of the geodesic concept

Heretofore we confined the geodesic problem to finding a curve on a three-dimensional surface, but the concept may be generalized to higher dimensions. Physicists use the space-time continuum as a four-dimensional (Minkowski) space and find geodesic paths in that space. The arc length definition of this space is

$$ds^2 = dx^2 + dy^2 + dz^2 - cdt^2,$$

where  $t$  is the time dimension and  $c$  is the speed of light. The variational problem of minimal arc length may be posed similarly as in Section 8.1 and may be solved with similar techniques. Einstein used this generalization to explain planetary motion as a geodesic phenomenon in the four-dimensional space.

Another important generalization is in the area of dynamical analysis of mechanical systems, a topic of Chapter 12. The motion of the system will be described in terms of  $n$  generalized coordinates  $q_i$  and their time derivatives  $\dot{q}_i$ , the generalized velocities. These generalized coordinates, along with their constraint relationships, define an  $n$ -dimensional space, sometimes called the mechanical space.

The system will strive to reach an energy minimal equilibrium described by a variational problem in the form of

$$I = \int_{t_0}^{t_1} f(t, q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n) dt = \textit{extremum}.$$

The solution of this is mathematically equivalent to finding the geodesic path in the  $n$ -dimensional mechanical space.

To demonstrate the meaning of a mechanical space, let us consider a particle that is constrained to move on a two-dimensional surface, say a cylinder, enforced by some constraint. Then the particle's move in this two-dimensional mechanical space (the surface) is along the geodesic curve of the surface. This abstraction to higher dimensions is sometimes called the geometrization of mechanics.

Hamilton spearheaded this generalization and this will lead to Lagrange's equations of motion. The approach is of utmost importance in computational mechanics, the subject of extensive discussions in Chapter 12.





# 9

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## *Computational geometry*

The geodesic concept, introduced in the last chapter purely on variational principles, has interesting engineering aspects. On the other hand, the analytic solution of a geodesic curve by finding the extremum of a variational problem may not be easy in practical cases.

It is reasonable to assume, however, that the quality of a curve in a geodesic sense is related to its curvature. This observation proposes a strategy for creating good quality (albeit not necessarily geodesic) curves by minimizing the curvature.

Since the curvature is difficult to compute, one can use the second derivative of the curve in lieu of the curvature. This results in the following variational problem statement for a smooth curve: Find the curve  $s(t)$  that results in

$$I(s) = \int_{t_0}^{t_n} k(s'')^2 dt = \textit{extremum}.$$

The constant  $k$  represents the fact that this is an approximation of the curvature, but will be left out from our work below. This variational problem leads to the well-known spline functions.

---

### 9.1 Natural splines

Let us consider the following variational problem. Find the curve between two points  $P_0, P_3$  such that

$$I(s) = \int_{t_0}^{t_3} \left(\frac{d^2s}{dt^2}\right)^2 dt = \textit{extremum},$$

with boundary conditions of

$$P_0 = s(t_0), P_3 = s(t_3),$$

and additional discrete internal constraints of

$$P_1 = s(t_1), P_2 = s(t_2).$$

In essence, we are constraining two interior points of the curve, along with the fixed beginning and endpoints. We will, for the sake of simplicity, assume unit equidistant parameter values as

$$t_i = i, i = 0, \dots, 3.$$

While the functional does not contain the independent variable  $t$  and the dependent variable  $s(t)$ , it is of higher order. Hence the Euler-Poisson equation of second order applies:

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} = 0,$$

and in this case it simplifies to

$$\frac{d^4}{dt^4} s(t) = 0.$$

Straightforward integration yields the solution of the form

$$s(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3,$$

where  $c_i$  are constants of integration to be satisfied by the boundary conditions. Imposing the boundary conditions and constraints yields the system of equations

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}.$$

The inversion of the system matrix results in the generating matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -11/6 & 3 & -3/2 & 1/3 \\ 1 & -5/2 & 2 & -1/2 \\ -1/6 & 1/2 & -1/2 & 1/6 \end{bmatrix}$$

for the natural spline. For any given set of four points

$$P = \begin{bmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{bmatrix}$$

the coefficients of the solution may be obtained by

$$C = MP,$$

with distinct coefficients for all coordinate directions as

$$C = \begin{bmatrix} c_0^x & c_0^y & c_0^z \\ c_1^x & c_1^y & c_1^z \\ c_2^x & c_2^y & c_2^z \\ c_3^x & c_3^y & c_3^z \end{bmatrix}.$$

For example, the points

$$P = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 2 \\ 4 & 3 \end{bmatrix}$$

result in coefficients

$$C = \begin{bmatrix} 1 & 1 \\ 1 & 13/6 \\ 0 & -3/2 \\ 0 & 1/3 \end{bmatrix}.$$

The parametric solution curve is of the form

$$\begin{aligned} x(t) &= 1 + t, \\ y(t) &= 1 + 13/6t - 3/2t^2 + 1/3t^3. \end{aligned}$$

The example solution curve is shown in Figure 9.1, where the input points are connected by the straight lines that represent the chords of the spline. The spline demonstrates a good smoothness while satisfying the constraints.

Several extensions of this problem are noteworthy. It is possible to pose the variational problem in two-parameter form as

$$I(s) = \int \int_D \left( \left( \frac{\partial}{\partial u} s(u, v) \right)^2 + \left( \frac{\partial}{\partial v} s(u, v) \right)^2 \right) dudv = \text{extremum}.$$

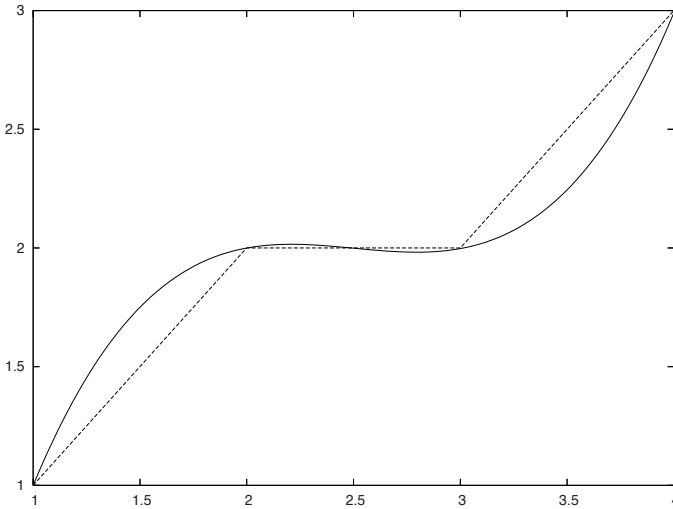
The Euler-Lagrange equation corresponding to this problem arrives at Laplace's equation:

$$\frac{\partial^2}{\partial u^2} s(u, v) + \frac{\partial^2}{\partial v^2} s(u, v) = 0.$$

This is sometimes called the harmonic equation, hence the splines so obtained are also called harmonic splines.

It is also often the case in practice that many more than four points are given:

$$P_i = (x_i, y_i, z_i), i = 0, \dots, n.$$



**FIGURE 9.1** Natural spline approximation

This enables the generation of a multitude of natural spline segments, and a curvature continuity condition between the segments may also be enforced. Finally, the direct (for example Ritz) solution of the above variational problem leads to the widely used B-splines, a topic of the next chapter.

---

## 9.2 B-spline approximation

As shown in Chapter 7, when using numerical methods an approximate solution is sought in terms of suitable basis functions:

$$\bar{s}(t) = \sum_{i=0}^n B_{i,k}(t) Q_i,$$

where  $Q_i$  are the yet unknown control points ( $i=0, \dots, n$ ) and  $B_{i,k}$  are the basis functions of degree  $k$  in the parameter  $t$ . For industrial computational work, the class of basis functions resulting in the so-called B-splines are defined in [1] as

$$B_{i,0}(t) = \begin{cases} 1, & t_i \leq t < t_{i+1} \\ 0, & t < t_i, t \geq t_{i+1} \end{cases}$$

The higher order terms are recursively computed:

$$B_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i} B_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(t).$$

The basis functions are computed from specific parameter values, called knot values. If their distribution is not equidistant, the splines are called non-uniform B-splines. If they are uniformly distributed, they are generating uniform B-splines.

The knot values are a subset of the parameter space and their selection enables a unique control of the behavior of the spline. For example, the use of duplicate knot values inside the parameters span of the spline results in a local change. The use of multiple knot values at the boundaries enforces various end conditions, such as the frequently used clamped-end condition. This control mechanism contributes to the popularity of the method in computer-aided design (CAD), but will not be further explored here.

Smoothing a B-spline is defined by the variational problem

$$I_s(Q) = \int_{t_0}^{t_n} \left( \sum_{i=0}^n B''_{i,k}(t) Q_i \right)^2 dt = \text{extremum}.$$

The derivative of the basis functions may be recursively computed. For  $k = 1$ , since  $B_{i,0}$  are constant

$$\frac{d}{dt} B_{i,1}(t) = B'_{i,1}(t) = \frac{1}{t_{i+1} - t_i} B_{i,0}(t) - \frac{1}{t_{i+2} - t_{i+1}} B_{i+1,0}(t).$$

For  $k = 2$

$$\begin{aligned} \frac{d}{dt} B_{i,2}(t) = B'_{i,2}(t) &= \frac{1}{t_{i+2} - t_i} B_{i,1}(t) + \frac{t - t_i}{t_{i+2} - t_i} B'_{i,1}(t) - \\ &\frac{1}{t_{i+3} - t_i} B_{i+1,1}(t) + \frac{t_{i+3} - t}{t_{i+3} - t_{i+1}} B'_{i+1,1}(t). \end{aligned}$$

Substituting the  $k = 1$  derivative into the second term results in

$$\begin{aligned} \frac{t - t_i}{t_{i+2} - t_i} B'_{i,1}(t) &= \frac{t - t_i}{t_{i+2} - t_i} \left( \frac{1}{t_{i+1} - t_i} B_{i,0}(t) - \frac{1}{t_{i+2} - t_{i+1}} B_{i+1,0}(t) \right) = \\ &\frac{1}{t_{i+2} - t_i} \left( \frac{t - t_i}{t_{i+1} - t_i} B_{i,0}(t) + \frac{t_i - t}{t_{i+2} - t_{i+1}} B_{i+1,0}(t) \right). \end{aligned}$$

The content of the parenthesis is easily recognizable as  $B_{i,1}(t)$ , hence this term is identical to the first. Similar arithmetic on the second two terms results in

the derivative for  $k = 2$  as

$$\frac{d}{dt}B_{i,2}(t) = B'_{i,2}(t) = \frac{2}{t_{i+2} - t_i}B_{i,1} - \frac{2}{t_{i+3} - t_{i+1}}B_{i+1,1}.$$

By induction, for any  $k$  value the first derivative is as follows:

$$\frac{d}{dt}B_{i,k}(t) = B'_{i,k}(t) = \frac{k}{t_{i+k} - t_i}B_{i,k-1}(t) - \frac{k}{t_{i+k+1} - t_{i+1}}B_{i+1,k-1}(t).$$

A repeated application of the same step will produce the needed second derivative  $B''$  as

$$\frac{d}{dt}B'_{i,k}(t) = B''_{i,k}(t) = \frac{k}{t_{i+k} - t_i}B'_{i,k-1}(t) - \frac{k}{t_{i+k+1} - t_{i+1}}B'_{i+1,k-1}(t).$$

The spline, besides being smooth (minimal in curvature), is expected to approximate a given set of points  $P_j; j = 0, \dots, m$ , with associated prescribed parameter values (not necessarily identical to the knot values) of  $t_j; j = 0, \dots, m$ . If such parameter values are not given, the parameterization may be via the simple method of uniform spacing defined as  $t_j = j; 0 \leq j \leq m$ . Assuming that the point set defined is geometrically semi-equidistant this is proven in industry to be a good method for the problem at hand. If that is not the case, a parameterization based on the chord length may also be used.

Approximating the given points with the spline is another variational problem that requires finding a minimum of the squares of the distances between the spline and the points.

$$I_a(\bar{s}) = \sum_{j=0}^m (\bar{s}(t_j) - P_j)^2 = \text{extremum}.$$

Substituting the B-spline formulation and the basis functions results in

$$I_a(Q) = \sum_{j=0}^m \left( \sum_{i=0}^n B_{i,k}(t_j) Q_i - P_j \right)^2.$$

Similarly, in the smoothing variational problem we also replace the integral with a sum over the given points in the parameter span, resulting in

$$I_s(Q) = \sum_{j=0}^m \left( \sum_{i=0}^n B''_{i,k}(t_j) Q_i \right)^2.$$

Finally, the functional to produce a smooth spline approximation is the sum of the two functionals

$$I(Q) = I_a(Q) + I_s(Q).$$

The notation is to demonstrate the dependence upon the yet unknown control points of the spline.

According to the numerical method developed in Chapter 7, the problem has an extremum when

$$\frac{\partial I}{\partial Q_i} = 0,$$

for each  $i = 0, \dots, n$ . The control points will be, of course, described by Cartesian coordinates; hence, each of the above equations represents three scalar equations.

The derivative of the approximating component with respect to an unknown control point  $Q_p$  yields

$$2 \sum_{j=0}^m B_{p,k}(t_j) \left( \sum_{i=0}^n B_{i,k}(t_j) Q_i - P_j \right),$$

where  $p = 0, 1, \dots, n$ . This is expressed in matrix form as

$$B^T B Q - B^T P$$

with the matrices

$$B = \begin{bmatrix} B_{0,k}(t_0) & B_{1,k}(t_0) & B_{2,k}(t_0) & \dots & B_{n,k}(t_0) \\ B_{0,k}(t_1) & B_{1,k}(t_1) & B_{2,k}(t_1) & \dots & B_{n,k}(t_1) \\ \dots & \dots & \dots & \dots & \dots \\ B_{0,k}(t_m) & B_{1,k}(t_m) & B_{2,k}(t_m) & \dots & B_{n,k}(t_m) \end{bmatrix},$$

$$P = \begin{bmatrix} P_0 \\ P_1 \\ \dots \\ P_m \end{bmatrix},$$

and

$$Q = \begin{bmatrix} Q_0 \\ Q_1 \\ \dots \\ Q_n \end{bmatrix}.$$

For degree  $k = 3$  the basis functions may be analytically computed as:

$$B_{0,3} = \frac{1}{6}(1-t)^3,$$

$$B_{1,3} = \frac{1}{6}(3t^3 - 6t^2 + 4),$$

$$B_{2,3} = \frac{1}{6}(-3t^3 + 3t^2 + 3t + 1),$$



and

$$B_{3,3} = \frac{1}{6}t^3.$$

Furthermore, for uniform parameterization the  $B$  matrix is easily computed by hand as

$$B = \frac{1}{6} \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ -1 & 4 & -5 & 8 \\ -8 & 31 & -44 & 27 \\ -27 & 100 & -131 & 64 \end{bmatrix}.$$

The derivative of the smoothing component of the functional, with respect to an unknown control point  $Q_p$  yields

$$2 \sum_{j=1}^m B''_{p,k}(t_j) \sum_{i=0}^n B''_{i,k}(t_j) Q_i,$$

where  $p \in [0, \dots, n]$ . This results in a smoothing matrix

$$D = \begin{bmatrix} B''_{0,k}(t_0) & B''_{1,k}(t_0) & \dots & B''_{n,k}(t_0) \\ B''_{0,k}(t_1) & B''_{1,k}(t_1) & \dots & B''_{n,k}(t_1) \\ \dots & \dots & \dots & \dots \\ B''_{0,k}(t_m) & B''_{1,k}(t_m) & \dots & B''_{n,k}(t_m) \end{bmatrix}.$$

These second derivatives for the cubic case are

$$B''_{0,3} = 1 - t,$$

$$B''_{1,3} = 3t - 2,$$

$$B''_{2,3} = -3t + 1,$$

and

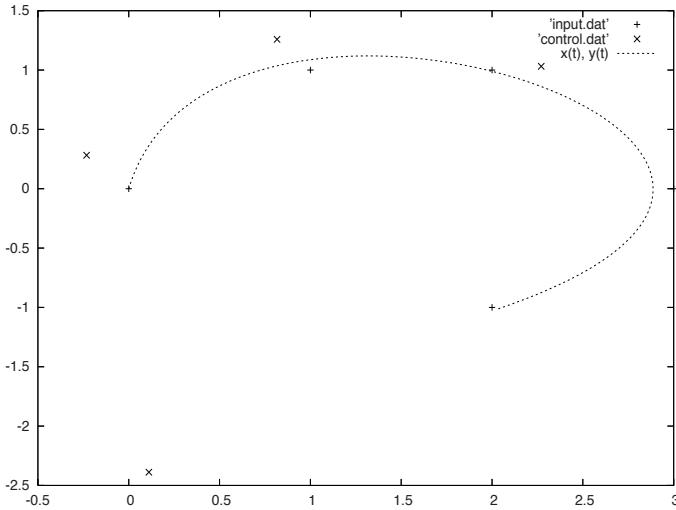
$$B''_{3,3} = t.$$

For uniform parameterization, the smoothing matrix is computed as

$$D = \frac{1}{6} \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ -1 & 4 & -5 & 2 \\ -2 & 7 & -8 & 3 \\ -3 & 10 & -11 & 4 \end{bmatrix}.$$

The simultaneous solution of both the smoothing and approximating problem is now represented by the matrix equation

$$AQ = B^T P$$



**FIGURE 9.2** Smooth B-spline approximation

where the system matrix is

$$A = B^T B + D^T D.$$

The solution of this system produces the control points for a smooth approximation.

Figure 9.2 shows the smooth spline approximation for a set of given points. The input points as well as the computed control points are also shown. Note that, as opposed to the natural spline, the curve does not go through the points exactly, but it is very smooth.

### 9.3 B-splines with point constraints

It is possible to require some of the points to be exactly satisfied. For the generic case of multiple enforced points, a constrained variational problem is formed.

$$I(Q_i) = \text{extremum},$$

subject to

$$\bar{s}(t_l) = R_l; l = 0, 1, \dots, o.$$

Here the enforced points  $R_l$  represent a subset of the given points ( $P_j$ ), while the remainder are to be approximated. The subset is specified as

$$R_l = MP_j; l = 0, 1, \dots, o; j = 0, 1, \dots, m, o < m,$$

where the mapping matrix  $M$  has  $o$  rows and  $m$  columns and contains a single non-zero term in each row, in the column corresponding to a selected interpolation point. For example, the matrix

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

would specify the two internal points of four  $P_j$  points, i.e.,

$$R_0 = P_1$$

and

$$R_1 = P_2.$$

This approach could be used to specify any pattern, such as every second or third term, or some specific points at intermittent locations.

Introducing the specifics of the splines and Lagrange multipliers, the constrained variational problem is presented as

$$I(Q_i, \lambda_l) = I(Q_i) + \sum_{l=0}^o \lambda_l \left( \sum_{i=0}^n B_{i,k}(t_l) Q_i - R_l \right).$$

The derivatives of  $I(Q_i)$  with respect to the  $Q_p$  control point were computed earlier, but need to be extended with the derivative of the term containing the Lagrange multiplier:

$$\sum_{l=0}^o B_{p,k}(t_l) \lambda_l \sum_{i=0}^n B_{i,k}(t_l).$$

Utilizing the earlier introduced matrices, this term is

$$B^T M^T \Lambda,$$

where  $\Lambda$  is a column vector of  $o + 1$  Lagrange multipliers,

$$\begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \dots \\ \lambda_o \end{bmatrix}.$$

The derivatives with respect to the Lagrange multipliers  $\lambda_l$  produce

$$\left(\sum_{i=0}^n B_{i,k}(t_l)Q_i - R_l\right) = 0; l = 0, 1, \dots, o.$$

This results in  $o + 1$  new equations of the form

$$\sum_{i=0}^n B_{i,k}(t_l)Q_i = R_l,$$

or in matrix form, using the earlier matrices:

$$MBQ = R,$$

where  $R$  is a vector of the interpolated points. The two sets of equations may be assembled into a single matrix equation with  $n + 1 + o + 1$  rows and columns of the form

$$\begin{bmatrix} A & B^T M^T \\ MB & 0 \end{bmatrix} \begin{bmatrix} Q \\ \Lambda \end{bmatrix} = \begin{bmatrix} B^T P \\ MP \end{bmatrix}.$$

The first matrix row represents the constrained functional and the second row represents the constraints. The simultaneous solution of this (symmetric, indefinite, but still linear) system produces the optimized (approximated and smoothed) and selectively interpolated solution.

The solution of this problem is accomplished in the following steps. First the unknown control points are expressed from the first row of

$$AQ + B^T M^T \Lambda = B^T P$$

as functions of the unknown Lagrange multipliers. Substituting into the second equation is the way to compute the multipliers:

$$\Lambda = (MBA^{-1}B^T M^T)^{-1}(MBA^{-1}B^T P - MP).$$

Finally, the set of control points, which are solutions of the constrained variational problem, are obtained explicitly from the first equation as

$$Q = A^{-1}(B^T P - B^T M^T \Lambda).$$

The effect of point constraints is shown in Figure 9.3 in connection with the earlier example, constraining the spline to go through the second and the fourth points. The dashed curve is the original curve while the dotted curve is the new curve and it demonstrates the adherence to the constraint, at the same time maintaining the quality of the approximation and the smoothness.

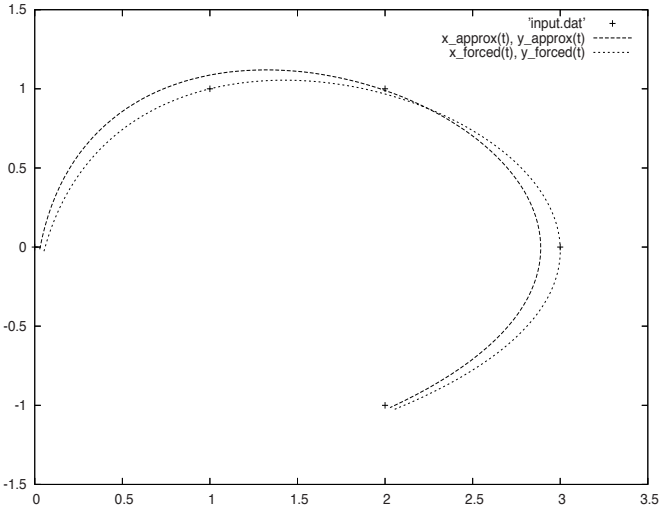


FIGURE 9.3 Point constrained B-spline

### 9.4 B-splines with tangent constraints

It may be desirable for the engineer to be able to enforce constraints posed by specifying tangents at certain points. These are of the form

$$\vec{s}'(t_l) = T_l; l = 0, 1, \dots, o,$$

assuming for now that the tangents are given at the same points where interpolation constraints were also given. The constrained problem shown in the prior section will be extended with the additional constraints and Lagrange multipliers:

$$\sum_{l=0}^o \lambda_l \left( \sum_{i=0}^n B'_{i,k}(t_l) Q_i - T_l \right); l = 0, 1, \dots, o.$$

The derivative with respect to the new Lagrange multipliers is

$$\sum_{i=0}^n B'_{i,k}(t_l) Q_i - T_l = 0; l = 0, 1, \dots, o.$$

This results in  $o + 1$  new equations of the form

$$\sum_{i=0}^n B'_{i,k}(t_l) Q_i = T_l,$$

or in matrix form, using some of the earlier matrices:

$$MCQ = T,$$

where  $T$  is a vector of the given tangents and the matrix of first derivatives is

$$C = \begin{bmatrix} B'_{0,k}(t_0) & B'_{1,k}(t_0) & \dots & B'_{n,k}(t_0) \\ B'_{0,k}(t_1) & B'_{1,k}(t_1) & \dots & B'_{n,k}(t_1) \\ \dots & \dots & \dots & \dots \\ B'_{0,k}(t_m) & B'_{1,k}(t_m) & \dots & B'_{n,k}(t_m) \end{bmatrix}.$$

The first derivatives of the basis functions for the cubic case are

$$B'_{0,3} = -\frac{1}{2}(1-t)^2,$$

$$B'_{1,3} = \frac{3}{2}t^2 - 2t,$$

$$B'_{2,3} = -\frac{3}{2}t^2 + t + \frac{1}{2},$$

and

$$B'_{3,3} = \frac{1}{2}t^2.$$

For the uniform case the  $C$  matrix containing the first derivatives is

$$C = \frac{1}{2} \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & 4 & -7 & 4 \\ -4 & 15 & -20 & 9 \\ -9 & 32 & -39 & 16 \end{bmatrix}.$$

The three sets of equations may be assembled into one matrix equation with  $n + 1 + 2(o + 1)$  rows and columns of the form

$$\begin{bmatrix} A & B^T M^T & C^T M^T \\ MB & 0 & 0 \\ MC & 0 & 0 \end{bmatrix} \begin{bmatrix} Q \\ \Lambda_p \\ \Lambda_t \end{bmatrix} = \begin{bmatrix} B^T P \\ MP \\ MT \end{bmatrix}.$$

The index of the Lagrange multipliers refers to points ( $p$ ) or tangents ( $t$ ).

The restriction of giving tangents at all the same points where interpolation constraints are also given may be relaxed and the points with tangential constraints may only be a subset of the points where interpolation constraints

are placed. In this case the final equation is

$$\begin{bmatrix} A & B^T M^T & C^T N^T \\ MB & 0 & 0 \\ NC & 0 & 0 \end{bmatrix} \begin{bmatrix} Q \\ \Lambda_p \\ \Lambda_t \end{bmatrix} = \begin{bmatrix} B^T P \\ MP \\ NT \end{bmatrix}.$$

Here the  $N$  mapping matrix is a subset of the  $M$  mapping matrix. The solution of this problem is similar to the solution of the simply constrained case, albeit a bit more tedious, due to the fact that the constraints are now of two different kinds. First the solution in terms of the multipliers is expressed

$$Q = A^{-1} B^T P - A^{-1} [B^T M^T \ C^T N^T] \begin{bmatrix} \Lambda_p \\ \Lambda_t \end{bmatrix}.$$

Then there is a matrix equation to compute the multipliers from

$$\begin{bmatrix} MB \\ NC \end{bmatrix} Q = \begin{bmatrix} MP \\ NT \end{bmatrix}.$$

The sets of multipliers are obtained as

$$\begin{bmatrix} \Lambda_p \\ \Lambda_t \end{bmatrix} = \left( \begin{bmatrix} MB \\ NC \end{bmatrix} A^{-1} [B^T M^T \ C^T N^T] \right)^{-1} \left( \begin{bmatrix} MB \\ NC \end{bmatrix} A^{-1} (B^T P) - \begin{bmatrix} MP \\ NT \end{bmatrix} \right).$$

Finally, the desired set of control points satisfying the constrained variational problem are

$$Q = A^{-1} (B^T P - B^T M^T \Lambda_p - C^T N^T \Lambda_t).$$

Let us use again the same set of points, but enforce the curve going through the second point with a tangent of 45 degrees.

The dotted curve in Figure 9.4 demonstrates the satisfaction of both constraints, going through the second point and having a 45-degree tangent. It is also very clear that such a strong constraint imposed upon one point has a significant, and in this case detrimental, effect on the shape of the curve at least as far as the approximation is concerned. The smoothness of the curve is still impeccable.

In practical applications, some heuristics, like setting the tangent at a certain point parallel to the chord between the two neighboring points, can be used successfully. Then

$$T_i = \frac{\overline{P_{i+1} - P_i}}{\| \overline{P_{i+1} - P_i} \|}.$$

This would result in different control points and a much smaller deformation of the overall curve may be obtained. Systematic and possibly interactive application of this concept should result in good shape preservation and general smoothness.

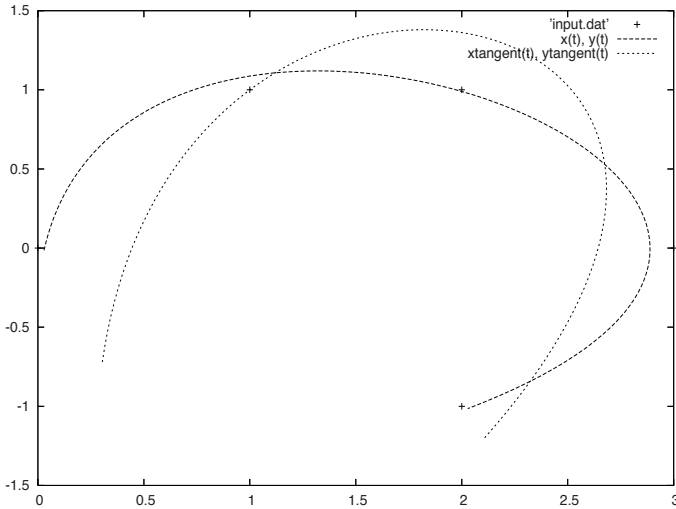


FIGURE 9.4 Tangent constrained B-spline

### 9.5 Generalization to higher dimensions

The spline technology discussed above is easily generalized to higher-dimensional spaces. Let us consider surfaces given in the form of

$$z(x, y) = f(x, y)$$

first. A B-spline surface is defined by a set of control points as

$$\bar{s}(u, v) = \sum_{i=0}^n \sum_{j=0}^m B_{i,k}(u) B_{j,k}(v) Q_{ij},$$

where now we have two distinct knot value sequences of

$$u_i; i = 0, 1, \dots, n,$$

and

$$v_j; j = 0, 1, \dots, m.$$

The topological rectangularity of the control points is not necessary and may be overcome by coalescing certain points or adjustments of the knot points.



The control points to provide a smooth approximation of the given geometric surface are selected from the variational problem of

$$I(\bar{s}) \int \int (f(x, y) - \bar{s}(x, y))^2 dx dy = \text{extremum.}$$

Substituting the surface spline definition and sampling of the given surface results in another, albeit more difficult, system of linear equations from which the control point locations may be resolved in a similar fashion as in the case of spline curves before.

Finally, it is also possible to describe some geometrical volumes with the B-spline technology. Consider the form

$$\bar{s}(u, v, t) = \sum_{i=0}^n \sum_{j=0}^m \sum_{l=0}^p B_{i,k}(u) B_{j,k}(v) B_{l,k}(t) Q_{ijl},$$

where now the third set of knot values

$$t_l; l = 0, 1, \dots, p$$

defines the direction through the volume starting from a surface. Topological rectangularity considerations again apply, but may be overcome with some inconvenience.

Both of these generalizations are important in computer-aided design (CAD) tools in the industry. They represent an efficient way to describe general (non-mathematical) surfaces and volumes.

Another, very important higher than three-dimensional extension occurs in computer-aided manufacturing (CAM). The calculation of smooth tool-paths of five-axis machines includes the three Cartesian coordinates and two additional quantities related to the tool position. This is important in machining of surface parts comprised of valleys and walls.

The positioning of the cutting tool is customarily described by two angles. The tool's "leaning" in the normal plane is one that may be construed as a rotation with respect to the bi-normal of the path curve. The tool's "swaying" from the normal plane, which constitutes a rotation around the path tangent as an axis, may be another angle.

Abrupt changes in the tool axes are detrimental to the machined surface quality as well as to the operational efficiency. Hence it is a natural desire to smooth these quantities as well. The variational formulation for the geometric smoothing of the spline, shown above, accommodates any number of such additional considerations.

Finally, the recent emergence of the additive manufacturing technology brought further importance to the spline based surfaces and volumes. In the additive manufacturing process, also known as 3D printing, the desired shape of a product is described by the two- and three-dimensional splines developed above. The process is systematically depositing material until the final shape is achieved. The spline formulations enable easy computation of the intermediate shapes of the printing process.



# 10

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## *Variational equations of motion*

We encountered variational forms of equations of motion in prior chapters, for example, when solving the brachistocrone problem in Section 1.4.2. Several dynamic equations of motion will be derived from variational principles in this chapter.

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### 10.1 Legendre's dual transformation

This transformation invented by Legendre is of fundamental importance. Let us consider the function of  $n$  variables

$$f = f(u_1, u_2, \dots, u_n).$$

Legendre proposed to introduce a new set of variables by the transformation of

$$v_i = \frac{\partial f}{\partial u_i}, i = 1, 2, \dots, n.$$

The Hessian matrix of this transformation is

$$H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial u_1^2} & \frac{\partial^2 f}{\partial u_1 \partial u_2} & \cdots & \frac{\partial^2 f}{\partial u_1 \partial u_n} \\ \frac{\partial^2 f}{\partial u_2 \partial u_1} & \frac{\partial^2 f}{\partial u_2^2} & \cdots & \frac{\partial^2 f}{\partial u_2 \partial u_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 f}{\partial u_n \partial u_1} & \frac{\partial^2 f}{\partial u_n \partial u_2} & \cdots & \frac{\partial^2 f}{\partial u_n^2} \end{bmatrix}.$$

If the determinant of this matrix, sometimes called the Hessian, is not zero, then the variables of the new set are independent. This means that they could also be obtained as functions of the original variables.

We define a new function in terms of the new variables

$$g = g(v_1, v_2, \dots, v_n).$$

The two functions are related as

$$g = \sum_{i=1}^n u_i v_i - f.$$

The notable consequence is the spectacular duality between the two sets. The original variables can now be expressed as

$$u_i = \frac{\partial g}{\partial v_i}, i = 1, 2, \dots, n.$$

and the original function regained as

$$f = \sum_{i=1}^n u_i v_i - g.$$

Legendre's transformation is completely symmetric.

Let us now look at a function of two sets of variables:

$$f = f(u_1, u_2, \dots, u_n, w_1, w_2, \dots, w_n).$$

If the variables of the second set are independent of the first, they are considered to be parameters and the transformation will retain them as such:

$$g = g(v_1, v_2, \dots, v_n, w_1, w_2, \dots, w_n).$$

The relationship between the two functions regarding the parameters is

$$\frac{\partial f}{\partial w_i} = -\frac{\partial g}{\partial w_i}, i = 1, 2, \dots, n.$$

This transformation will be instrumental when applied to the functions introduced in the next sections.

## 10.2 Hamilton's principle for mechanical systems

Hamilton's principle was briefly mentioned earlier in Section 1.4.2 in connection with the problem of a particle moving under the influence of a gravity field. The principle, however, is much more general and it is applicable to complex mechanical systems. For conservative (energy preserving) systems, Hamilton's principle states that the motion between two points is defined by the variational problem of

$$\int_{t_0}^{t_1} (E_k - E_p) dt = \text{extremum},$$

where  $E_k$  and  $E_p$  are the kinetic and potential energy, respectively. Introducing the Lagrange function

$$L = E_k - E_p,$$

the principle may also be stated as

$$\int_{t_0}^{t_1} L dt = \text{extremum},$$

where the extremum is not always zero. The advantageous feature of Hamilton's principle is that it is stated in terms of energies, which are independent of the selection of coordinate systems. Hamilton's principle is of fundamental importance because many of the general physical laws may be derived from it as we will see in the next sections.

### 10.2.1 Newton's law of motion

We consider the simplest mechanical system of a mass particle, but since any complex mechanical system may be considered a collection of many mass particles, the following is valid for those as well. Let the mass of the particle be  $m$  and its position defined at a certain time  $t$  by the coordinates:

$$q_i(t), i = 1, 2, 3,$$

where  $q_1(t) = x(t), q_2(t) = y(t), q_3(t) = z(t)$ .

The kinetic energy of the particle is then

$$E_k = \sum_{i=1}^3 \frac{1}{2} m \dot{q}_i^2.$$

The mass particle is moving from its position at time  $t_0$  to a position at time  $t_1$ . We assume that there is a force  $F$  acting on the particle to result in this motion and the mechanical system is conservative, hence there exists a force potential such that

$$F_i = -\frac{\partial E_p}{\partial q_i}, i = 1, 2, 3.$$

Here  $F_i$  are the components of the force in the coordinate directions. Hamilton's principle dictates that

$$\int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} (E_k - E_p) dt = \text{extremum}.$$

Substituting the kinetic energy results in

$$\int_{t_0}^{t_1} (\sum_{i=1}^3 \frac{1}{2} m \dot{q}_i^2 - E_p) dt = \text{extremum}.$$

Applying the Euler-Lagrange equation

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = 0$$

for this variational problem yields

$$\frac{\partial E_p}{\partial q_i} + \frac{d}{dt}(m\dot{q}_i), i = 1, 2, 3.$$

The first term yields the acting force components, the differentiation produces the acceleration, and the formula becomes

$$m\ddot{q}_i = F_i, i = 1, 2, 3.$$

This is Newton's second law of motion, better known in the form of

$$F = ma.$$

### 10.3 Lagrange's equations of motion

We now consider a mechanical system of  $n$  mass points with distinct masses  $m^j, j = 1, \dots, n$  and generalize the coordinates as

$$q_1 = x_1, q_2 = y_1, q_3 = z_1; q_4 = x_2, q_5 = y_2, q_6 = z_2; \dots$$

and

$$q_{3n-2} = x_n, q_{3n-1} = y_n, q_{3n} = z_n.$$

They are gathered into a vector  $q$ . The mass particles are

$$m_1 = m_2 = m_3 = m^1; m_4 = m_5 = m_6 = m^2; \dots$$

and

$$m_{3n-2} = m^n, m^{3n-1} = m^n, m_{3n} = m^n.$$

Note the distinction between the subscripts and superscripts. To represent the system with Newton's law we simply extend the form derived in the last section to this case as

$$m_i \ddot{q}_i = f_i, i = 1, 2, 3, \dots, 3n.$$

Now our variational problem contains multiple functions

$$I(\underline{q}, \underline{\dot{q}}, \underline{\ddot{q}}) = \text{extremum}.$$

The system of Euler-Lagrange equations of this problem is called Lagrange's equations of motion. Note the emphasis on the plural. With the Lagrangian expression

$$L = E_k - E_p,$$

Lagrange's equations of motion are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0; i = 1, 2, \dots, 3n.$$

These equations provide the computational solution for practical mechanical systems. In most cases the kinetic energy is only a function of the locations and the potential energy is only a function of the velocities. Hence

$$\frac{\partial E_k}{\partial q_i} = 0,$$

and

$$\frac{\partial E_p}{\partial \dot{q}_i} = 0.$$

This produces a simpler version of Lagrange's equations as

$$\frac{d}{dt} \frac{\partial E_k}{\partial \dot{q}_i} + \frac{\partial E_p}{\partial q_i} = 0; i = 1, 2, \dots, 3n.$$

This form will be used in several later sections.

## 10.4 Hamilton's canonical equations

Hamilton's canonical equations are the result of the application of Legendre's transformation to the Lagrangian function.

Since the Lagrangian is a function of generalized coordinates and time as

$$L = L(q_i, \dot{q}_i, t),$$

for  $i = 1, 2, \dots, 3n$ . One can select either the displacements or the velocities as the set of variables to transform, following Legendre's dual transformation steps. Hamilton's insightful idea was to transform the velocity components:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

The result is a significant simplification of the equation of motion as we will see. The total differential of the Lagrangian is

$$dL = \frac{\partial L}{\partial t} dt + \sum_{i=1}^n \left( \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right).$$



Differentiating and applying Lagrange's equation we obtain

$$\dot{p}_i = \frac{d}{dt}p_i = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}.$$

Hence the total differential of the Lagrangian becomes

$$dL = \frac{\partial L}{\partial t}dt + \sum_{i=1}^n (\dot{p}_i dq_i + p_i d\dot{q}_i).$$

Exploiting that

$$d(p_i \dot{q}_i) = dp_i \dot{q}_i + p_i d\dot{q}_i,$$

and reordering yields

$$d\left(\sum_{i=1}^n p_i \dot{q}_i - L\right) = -\frac{\partial L}{\partial t}dt - \sum_{i=1}^n (\dot{p}_i dq_i - \dot{q}_i dp_i).$$

The left-hand side term is called the Hamiltonian

$$H = \sum_{i=1}^n (p_i \dot{q}_i - L) = f(p_i, q_i, t),$$

which is now only a function of the new and old generalized displacement variables and time. Its total differential is

$$dH = \frac{\partial H}{\partial t}dt + \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i\right).$$

Matching terms between the  $dH$  and  $dL$  differentials produces the relationship

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

Hamilton's canonical equations are then

$$\dot{q}_i = \frac{\partial H}{\partial p_i},$$

and

$$\dot{p}_i = -\frac{\partial H}{\partial q_i},$$

for  $i = 1, 2, \dots, 3n$ . The  $p_i, q_i$  are called canonical variables. These bring twice as many first order equations as second order equations of Lagrange; otherwise they are equivalent. In many cases these equations are simpler to solve.

Legendre's duality is clearly present. The time variable is the parameter unchanged by the Legendre transformation and it satisfies the same equation derived in Section 10.1 as

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

The dual relationship of the Lagrangian and Hamiltonian functionals is also clearly satisfied with

$$H = \sum_{i=1}^n p_i \dot{q}_i - L,$$

and

$$L = \sum_{i=1}^n p_i \dot{q}_i - H.$$

The double dimensional space of the canonical variables is called the phase-space, sometimes also called the  $q - p$  space. When the time variable  $t$  is added, the space is called the state-space, an instrumental platform in structural mechanics.

### 10.4.1 Conservation of energy

The relationship between the two functionals is not always easy to establish. Let us consider conservative systems in which the potential energy is only a function of the displacement generalized variables as

$$E_p = E_p(q),$$

while the kinetic energy is a quadratic function of the derivative generalized variables (or generalized velocities):

$$E_k = E_k(\dot{q}_i^2).$$

Hence

$$2E_k = \sum_{i=1}^n \frac{\partial E_k}{\partial \dot{q}_i} \dot{q}_i.$$

Substituting the canonical variables from the last section we obtain

$$2E_k = \sum_{i=1}^n \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i = \sum_{i=1}^n p_i \dot{q}_i.$$

Therefore the Hamiltonian becomes

$$H = \sum_{i=1}^n p_i \dot{q}_i - L = 2E_k - (E_k - E_p) = E_k + E_p.$$

This relationship states that the Hamiltonian is the sum of kinetic and potential energy. Let us now further investigate the Hamiltonian. Since it is of the form

$$H = H(q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n),$$

its derivative with respect to time is

$$\frac{dH}{dt} = \sum_{i=1}^n \left( \frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i \right).$$

By virtue of the canonical equations

$$\frac{dH}{dt} = 0,$$

from which it follows that

$$H = \text{constant} = E_{total}.$$

This is the law of conservation of energy, stating that for a conservative system the total energy (which is the Hamiltonian) is constant.

## 10.5 Orbital motion

We now address the orbital motion of two celestial bodies moving under each other's gravitational influence. It is known from Newtonian mechanics that such a motion is planar and we can confine our discussion to the  $x - y$  plane.

We will assume that the central body is located at the origin and the moving body has unit mass. The location of the moving body at time  $t$  is at coordinates  $x, y$ , and it is moving on a path

$$u(x, y, t).$$

The distance between these bodies is

$$\sqrt{x^2(t) + y^2(t)}.$$

Newton's law of gravitation states that the gravitational potential acting on the moving body is

$$E_p = -\frac{\gamma}{\sqrt{x^2 + y^2}}.$$

Here the constant  $\gamma$  is the universal gravitational constant generated by the mass of the central body. The velocity of the orbiting body is

$$\sqrt{\dot{x}^2 + \dot{y}^2},$$

hence its kinetic energy is

$$E_k = \frac{1}{2}(\dot{x}^2 + \dot{y}^2).$$

Let us first observe this in the Lagrangian framework. The Lagrangian becomes

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{-\gamma}{\sqrt{x^2 + y^2}}.$$

We have two generalized displacement variables and their velocities as

$$q_1 = x, q_2 = y, \dot{q}_1 = \dot{x}, \dot{q}_2 = \dot{y}.$$

Lagrange's equations of motion are obtained by evaluating

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}, i = 1, 2.$$

The results are  $\ddot{x}$  and  $\ddot{y}$ . The other component comes from computing

$$\frac{\partial L}{\partial q_i}, i = 1, 2,$$

and produces the equations

$$-\gamma x(x^2 + y^2)^{-3/2}, -\gamma y(x^2 + y^2)^{-3/2}.$$

The two Lagrange equations of motion are

$$\ddot{x} + \gamma x(x^2 + y^2)^{-3/2} = 0$$

and

$$\ddot{y} + \gamma y(x^2 + y^2)^{-3/2} = 0.$$

Now turning to Hamilton's formulation, we introduce the canonical variables

$$p_i = \frac{dq_i}{dt}.$$

Specifically in our case they are

$$p_1 = \dot{x}, p_2 = \dot{y}.$$

The other generalized variables remain as  $q_1 = x, q_2 = y$ . The Hamiltonian is, as shown in the last section, of the form

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{-\gamma}{\sqrt{q_1^2 + q_2^2}}.$$

The canonical equations are

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} = -\gamma q_i(q_1^2 + q_2^2)^{-3/2}, i = 1, 2,$$

and

$$\dot{q}_i = \frac{\partial H}{\partial p_i} = p_i, i = 1, 2.$$

The resulting canonical system of four equations becomes

$$\begin{aligned} p_1 &= \frac{dx}{dt}, \\ p_2 &= \frac{dy}{dt}, \\ \dot{p}_1 &= -\gamma x(x^2 + y^2)^{-3/2}, \end{aligned}$$

and

$$\dot{p}_2 = -\gamma y(x^2 + y^2)^{-3/2}.$$

It is easy to see by differentiation and substitution that the solution of this system is identical to the Lagrange solution.

Polar coordinate transformation enables us to further simplify this problem. The kinetic energy is then

$$E_k = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2)$$

and the potential energy becomes

$$E_p = -\gamma\frac{1}{r}.$$

The solution of this problem by either Lagrange's or Hamilton's method is left to the reader but the final result is shown as

$$r = \frac{b^2}{\gamma} \frac{1}{1 + e\cos(\phi)}.$$

This equation describes the trajectory of the orbiting body in relationship to the one in the center. The  $e$  term, called the eccentricity, is

$$e = \sqrt{1 + \frac{2ab^2}{\gamma^2}},$$

and  $a, b$  are constants of integration that occurred during the solution. They are resolved from the physical conditions. The  $b$  constant is the  $\theta$  angular momentum of the orbiting body and the  $a$  is the energy of the orbit.

Specifically the orbital trajectory is an ellipse when  $e < 1$ , a parabola when  $e = 1$ , and a hyperbola when  $e > 1$ . The minimum value of the elliptical trajectory radius is

$$r_{min} = \frac{\theta^2}{\gamma} \frac{1}{1 + e},$$

while its maximum is at

$$r_{max} = \frac{\theta^2}{\gamma} \frac{1}{1 - e}.$$

These are the lengths of the minor and major axis of the elliptical path of the orbiting body.

### 10.6 Variational foundation of fluid motion

Until now we have focused on particles of mechanical systems. To provide a foundation for a later topic we now consider a fluid "particle" in the form of an infinitesimally small volume  $\nu$ . We will assume that this elementary volume of fluid will not change but, true to the behavior of fluid, could move by the displacement vector  $u = (u_x, u_y, u_z)$ . The condition of the unchanged volume may be expressed as

$$\int_{\nu} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) d\nu = \int_{\nu} \nabla \cdot u d\nu = 0.$$

Let us now follow the Hamiltonian avenue again. The mass of the infinitesimal volume of fluid is

$$dm = \rho d\nu,$$

where  $\rho$  is the density of the fluid. Then the kinetic energy of the infinitesimal mass of fluid is

$$e_k = \frac{1}{2} \rho v^2 d\nu.$$

The potential energy of the fluid element is in the form

$$e_p = \rho \phi d\nu,$$

where  $\phi$  is the gravitational potential. The lower case  $e$  letters indicate the energy of the small fluid volume as opposed to the total fluid. The variational form of our problem then is

$$\int_{t_0}^{t_1} (e_k - e_p) dt = \text{extremum}.$$

Combining all the elementary fluid volumes and substituting yields

$$\int_{t_0}^{t_1} \int_{\nu} \rho \left( \frac{1}{2} v^2 - \phi \right) d\nu dt = 0.$$

This is subject to an auxiliary condition we usually add by a scalar Lagrange multiplier  $\lambda$ . Exploiting the identity

$$\nabla \cdot (\lambda u) = \lambda (\nabla \cdot u) + \nabla(\lambda) \cdot u,$$

we obtain

$$\lambda(\nabla \cdot u) = \nabla \cdot (\lambda u) - \nabla(\lambda) \cdot u.$$

Substituting into the volume condition

$$\int_{\nu} \lambda(\nabla \cdot u) d\nu = \int_{\nu} (\nabla \cdot (\lambda u) - \nabla(\lambda) \cdot u) d\nu = 0.$$

The first term may be transformed into a surface integral of the volume and as such vanishes, hence the second term represents the condition in the variational problem as

$$\int_{t_0}^{t_1} \int_{\nu} (\rho(\frac{1}{2}v^2 - \phi) - \nabla(\lambda) \cdot u) d\nu dt.$$

The Euler-Lagrange differential equation corresponding to this variational problem now becomes

$$-\rho \frac{dv}{dt} - \rho \nabla(\phi) - \nabla(\lambda) = 0.$$

By reordering we obtain

$$\frac{dv}{dt} = -\nabla(\phi) - \frac{1}{\rho} \nabla(\lambda).$$

What remains to be found is the physical meaning of the Lagrange multiplier. Let us assume that the fluid is in equilibrium, then  $v = 0$ . The equation then simplifies to

$$\nabla(\phi) + \frac{1}{\rho} \nabla(\lambda) = 0.$$

In the case of incompressible fluid  $\rho = \text{constant} = \rho_0$  and may be moved into the differential operator. Hence the equation may be simplified to

$$\phi + \frac{\lambda}{\rho_0} = \text{constant}.$$

The gravitational potential  $\phi$  at a height (or depth as we will see) is

$$\phi = -gz,$$

hence we obtain

$$\lambda = \rho_0 g(z - z_0).$$

The integration constant above is captured in the reference height  $z_0$ . This is really Archimedes' law of hydrostatics, known in the form of

$$p = \rho_0 g(z - z_0).$$

Hence the physical meaning of the Lagrange multiplier is the pressure  $p$ .

If we relinquish the incompressibility condition but assume that the density is a function of the pressure, then

$$\frac{\nabla p}{\rho} = \frac{\nabla p}{f(p)} = \nabla P,$$

where

$$P = \int \frac{dp}{f(p)}.$$

The hydrostatic equilibrium is then

$$\phi + P = \text{constant}.$$

For isothermic (constant temperature) fluids the form

$$P = \frac{p_0}{\rho_0} \log \frac{\rho}{\rho_0}$$

applies, resulting in

$$p = p_0 e^{-\alpha z}.$$

This fluid pressure solution is based on the gravitational potential and  $\alpha$  is a constant specific to the fluid medium. For air it is  $0.1184 \text{ km}^{-1}$  resulting in Laplace's atmospheric formula of

$$p = p_0 e^{-0.1184z},$$

where  $z$  is measured in kilometers and  $p_0$  is the pressure at sea level. The negative exponent indicates the decrease of atmospheric pressure at higher elevations.

In fluid dynamics applications at the same height the gravity potential is constant and its derivative vanishes. Using this, and introducing the pressure instead of the multiplier in our above solution and further differentiation yields

$$\rho \ddot{u} = -\nabla p,$$

which is the well known Euler equation of fluid dynamics. This will be the starting equation of the computational formulation discussion in Section 12.4.

Finally, fluid dynamics problems with viscous flows around structural components or confined by structures are governed by the general Navier-Stokes equations. These equations are beyond our present focus, but may also be derived by applying Newton's second law to fluid motion.





## *Analytic mechanics*

Analytic mechanics is a mathematical science, but it is of high importance for engineers as it provides analytic solutions to fundamental problems of engineering mechanics. At the same time it establishes generally applicable procedures. Mathematical physics texts, such as [5] and [6], laid the foundation for these analytic approaches addressing physical problems.

In the following sections we find analytic solutions for classical mechanical problems of elasticity utilizing Hamilton's principle. The most fitting application is the excitation of an elastic system by displacing it from its equilibrium position. In this case the system will vibrate with a frequency characteristic to its geometry and material, while constantly exchanging kinetic and potential energy.

The case of non-conservative systems, where energy loss may occur due to dissipation of the energy, will not be discussed. Hamilton's principle may be extended to non-conservative systems, but the added difficulties do not enhance the discussion of the variational aspects, which is our main focus.

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### 11.1 Elastic string vibrations

We now consider the vibrations of an elastic string. Let us assume that the equilibrium position of the string is along the  $x$  axis, and the endpoints are located at  $x = 0$  and  $x = L$ . We will stretch the string (since it is elastic) by displacing it from its equilibrium with some

$$\Delta L$$

value, resulting in a certain force  $F$  exerted on both endpoints to hold it in place. We assume there is no damping and the string will vibrate indefinitely if displaced, i.e., the system is conservative.

A particle of the string located at the coordinate value  $x$  at the time  $t$  has a yet unknown displacement value of  $y(x, t)$ . The boundary conditions are:

$$y(0, t) = y(L, t) = 0,$$

in other words the string is clamped at the ends. In order to use Hamilton's principle, we need to compute the kinetic and potential energies.

With unit length mass of  $\rho$ , the kinetic energy is of the form

$$E_k = \frac{1}{2} \int_0^L \rho \left( \frac{\partial y}{\partial t} \right)^2 dx.$$

The potential energy is related to the elongation (stretching) of the string. The arc length of the elastic string is

$$\int_0^L \sqrt{1 + \left( \frac{\partial y}{\partial x} \right)^2} dx,$$

and the elongation due to the transversal motion is

$$\Delta L = \int_0^L \sqrt{1 + \left( \frac{\partial y}{\partial x} \right)^2} dx - L.$$

Assuming that the elongation is small, i.e.,

$$\left| \frac{\partial y}{\partial x} \right| < 1,$$

it is reasonable to approximate

$$\sqrt{1 + \left( \frac{\partial y}{\partial x} \right)^2} \approx 1 + \frac{1}{2} \left( \frac{\partial y}{\partial x} \right)^2.$$

The elongation by substitution becomes

$$\Delta L \approx \frac{1}{2} \int_0^L \left( \frac{\partial y}{\partial x} \right)^2 dx.$$

Hence the potential energy contained in the elongated string is

$$E_p = \frac{1}{2} F \Delta L = \frac{F}{2} \int_0^L \left( \frac{\partial y}{\partial x} \right)^2 dx.$$

We are now in a position to apply Hamilton's principle. The variational problem becomes

$$I(y) = \int_{t_0}^{t_1} (E_k - E_p) dt = \frac{1}{2} \int_{t_0}^{t_1} \int_0^L \left( \rho \left( \frac{\partial y}{\partial t} \right)^2 - F \left( \frac{\partial y}{\partial x} \right)^2 \right) dx dt = \text{extremum}.$$

The Euler-Lagrange differential equation for a function of two variables, derived in Section 3.2, is applicable and results in

$$F \frac{\partial^2 y}{\partial x^2} = \rho \frac{\partial^2 y}{\partial t^2}. \quad (11.1)$$

This is the well-known differential equation of the elastic string, also known as the **wave equation**.

The solution of the problem may be solved by separation. We seek a solution in the form of

$$y(x, t) = a(t)b(x),$$

separating it into time and geometry dependent components. Then

$$\frac{\partial^2 y}{\partial x^2} = b''(x)a(t)$$

and

$$\frac{\partial^2 y}{\partial t^2} = a''(t)b(x),$$

where

$$b''(x) = \frac{\partial^2 b}{\partial x^2},$$

$$a''(t) = \frac{\partial^2 a}{\partial t^2}.$$

Substituting into Equation (11.1) yields

$$\frac{b''(x)}{b(x)} = \frac{1}{f^2} \frac{a''(t)}{a(t)},$$

where for future convenience we introduced

$$f^2 = \frac{F}{\rho}.$$

The two sides of this differential equation are dependent on  $x$  and  $t$ , respectively. Their equality is required at any  $x$  and  $t$  values, which implies that the two sides are constant. Let's denote the constant by  $-\lambda$  and separate the (partial) differential equation into two ordinary differential equations:

$$\frac{\partial^2 b}{\partial x^2} + \lambda b(x) = 0,$$

and

$$\frac{\partial^2 a}{\partial t^2} + f^2 \lambda a(t) = 0.$$

The solution of these equations may be obtained by the techniques learned in Chapter 5 for the eigenvalue problems. The first equation has the eigensolutions of the form

$$b_k(x) = \sin\left(\frac{k\pi}{L}x\right); k = 1, 2, \dots,$$

corresponding to the eigenvalues

$$\lambda_k = \frac{k^2 \pi^2}{L^2}.$$

Applying these values we obtain the time-dependent solution from the second equation by means of classical calculus in the form of

$$a_k(t) = c_k \cos\left(\frac{k\pi f}{L}t\right) + d_k \sin\left(\frac{k\pi f}{L}t\right),$$

with  $c_k, d_k$  arbitrary coefficients. Considering that at  $t = 0$  the string is in a static equilibrium position

$$a'(t = 0) = 0$$

we obtain  $d_k = 0$  and the solution of

$$a_k(t) = c_k \cos\left(\frac{k\pi f}{L}t\right).$$

The fundamental solutions of the problem become

$$y_k(x, t) = c_k \cos\left(\frac{k\pi f}{L}t\right) \sin\left(\frac{k\pi}{L}x\right); k = 1, 2, \dots$$

For any specific value of  $k$  the solution is a periodic function with period

$$\frac{2L}{kf}$$

and frequency

$$\frac{k\pi f}{L}.$$

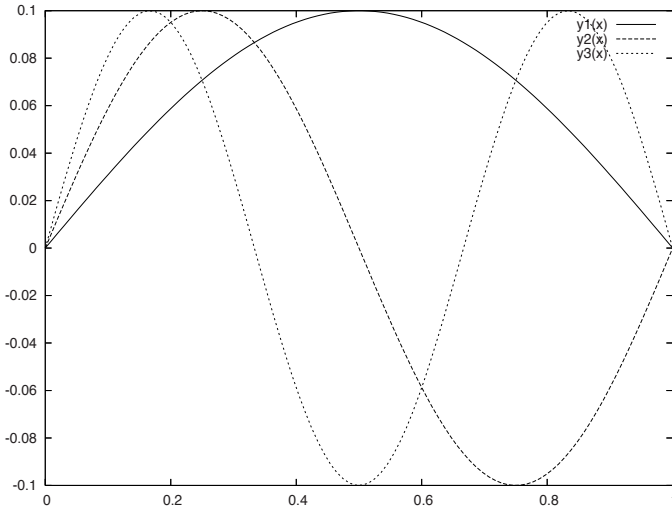
The quantities

$$\lambda_k = \frac{k\pi}{L}$$

for a specific  $k$  value are the natural frequencies of the string. The corresponding fundamental solutions are natural vibration modes shapes, or the normal modes. The first three normal modes are shown in Figure 11.1 for an elastic spring of unit tension force, mass density, and span. The figure demonstrates that the half-period decreases along with increasing mode number.

The motion is initiated by displacing the string and releasing it. Let us define this initial enforced amplitude as

$$y(x_m, 0) = y_m,$$



**FIGURE 11.1** Normal modes of elastic string

where the  $x_m$  describes the location of the initial stationary displacement of the string as an internal value of the span

$$x_m \in (0, L).$$

Then the initial shape of the string is a triangle over the span, described by the function

$$f(x) = \begin{cases} \frac{y_m}{x_m}x, & 0 \leq x \leq x_m, \\ y_m + \frac{y_m}{x_m-L}(x - x_m), & x_m \leq x \leq L. \end{cases}$$

The unknown coefficient may be solved from the initial condition as

$$y(x_m, 0) = f(x_m) = c_k \cos\left(\frac{k\pi}{L}0\right) \sin\left(\frac{k\pi}{L}x_m\right) = y_m,$$

from which

$$c_k = \frac{y_m}{\sin\left(\frac{k\pi}{L}x_m\right)}.$$

Note that if the interior point is the middle point of the span,

$$x_m = \frac{L}{2},$$

then the first coefficient will be simply the  $y_m$  amplitude:

$$c_1 = y_m,$$

since

$$\sin\left(\frac{\pi}{L} \frac{L}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$

Similar, but not identical, considerations may be applied for the coefficients of the higher normal modes.

The natural frequencies depend on the physical conditions, such as the pre-applied tension force distribution and the material characteristics embodied in the unit weight  $\rho$ . Specifically, the higher the tension force  $F$  in the string, the higher the frequency becomes. A very tight string vibrates very quickly (with high frequency), while a very loose string vibrates slowly.

## 11.2 The elastic membrane

We now turn our attention to the case of an elastic membrane. We assume that the membrane is fixed on its perimeter  $L$ , which surrounds the domain  $D$  of the membrane. We further assume that the initial, equilibrium position of the membrane is coplanar with the  $x - y$  plane.

$$z(x, y, t) = 0.$$

The membrane is displaced by a certain amount and released. The ensuing vibrations are the subject of our interest. The vibrations are a function of the location of the membrane and the time as

$$z = z(x, y, t).$$

We will again use Hamilton's principle after the kinetic and potential energy of the membrane are found. Let us assume that the unit area mass of the membrane does not change with time, and is not a function of the location:

$$\rho(x, y) = \rho = \text{constant}.$$

The velocity of the membrane point at  $(x, y)$  is

$$v = \frac{\partial z}{\partial t},$$

resulting in kinetic energy of

$$E_k = \frac{1}{2} \int \int_D \rho v^2 dx dy$$

or

$$E_k = \frac{1}{2} \int \int_D \rho \left( \frac{\partial z}{\partial t} \right)^2 dx dy.$$

We consider the source of the potential energy to be the stretching of the surface of the membrane. The initial surface is

$$\int \int_D dx dy,$$

and the extended surface is

$$\int \int_D \sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} dx dy.$$

Assuming small vibrations, we approximate as earlier in the case of the string

$$\sqrt{1 + \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2} \approx 1 + \frac{1}{2} \left( \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 \right).$$

Hence the surface change is

$$\frac{1}{2} \int \int_D \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 dx dy.$$

The stretching of the surface results in a surface tension  $\sigma$  per unit surface area. The potential energy is equal to  $\sigma$  multiplied by the surface change.

$$E_p = \sigma \frac{1}{2} \int \int_D \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 dx dy.$$

We are now in the position to apply Hamilton's principle. Since

$$I(z) = \int_{t_0}^{t_1} (E_k - E_p) dt = \text{extremum},$$

substitution yields the variational problem of the elastic membrane:

$$I(z) = \frac{1}{2} \int_{t_0}^{t_1} \int \int_D \rho \left( \frac{\partial z}{\partial t} \right)^2 - \sigma \left( \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 \right) dx dy dt = \text{extremum}.$$

The Euler-Lagrange differential equation for this class of problems following Section 3.4 becomes

$$\sigma \left( \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right) = \rho \frac{\partial^2 z}{\partial t^2},$$



or using Laplace's symbol

$$\sigma \Delta z = \rho \frac{\partial^2 z}{\partial t^2}.$$

The solution will follow the insight gained at the discussion of the elastic string and we seek a solution in the form of

$$z(x, y, t) = a(t)b(x, y).$$

The derivatives of this solution are

$$\Delta z(x, y, t) = a(t)\Delta b(x, y),$$

and

$$\frac{\partial^2 z(x, y, t)}{\partial t^2} = b(x, y) \frac{\partial^2 a(t)}{\partial t^2}.$$

Substitution yields

$$\frac{\sigma \Delta b}{\rho b} = \frac{\partial^2 a(t)}{\partial t^2} / a(t).$$

Again, since the left-hand side is only a function of the spatial coordinates and the right-hand side is of time, they must be equal and constant, assumed to be  $-\lambda$ . This separates the partial differential equation into two ordinary ones:

$$\frac{\partial^2 a(t)}{\partial t^2} + \lambda a(t) = 0,$$

and

$$\sigma \Delta b(x, y) + \lambda \rho b(x, y) = 0.$$

The solution of the first differential equation is

$$a(t) = c_1 \cos(\sqrt{\lambda}t) + c_2 \sin(\sqrt{\lambda}t).$$

Since initially the membrane is in equilibrium,

$$\left. \frac{da}{dt} \right|_{t=0} = 0$$

this indicates that

$$c_2 = 0.$$

Hence

$$a(t) = c_1 \cos(\sqrt{\lambda}t).$$

In order to demonstrate the solution for the second equation, let us omit the tension and material density for ease of discussion. The differential equation

of the form

$$\Delta b(x, y) + \lambda b(x, y) = 0,$$

is the same we solved analytically in the case of the elastic string; however, it is now with a solution function of two variables. The solution strategy will consider the variational form of this eigenvalue problem introduced in Section 5.2:

$$I(b) = \int \int_D \left( \left( \frac{\partial b}{\partial x} \right)^2 + \left( \frac{\partial b}{\partial y} \right)^2 - \lambda b^2(x, y) \right) dx dy = \textit{extremum}.$$

### 11.2.1 Circular membrane vibrations

Let us restrict ourselves to the domain of the unit circle for simplicity. The domain  $D$  is defined by

$$D : (1 - x^2 - y^2 \geq 0).$$

We use Kantorovich's method and seek an approximate solution in the form of

$$b(x, y) = \alpha \omega(x, y) = \alpha(x^2 + y^2 - 1),$$

where  $\alpha$  is a yet unknown constant. It follows that on the boundary  $\partial D$

$$\omega(x, y) = x^2 + y^2 - 1 = 0,$$

hence the approximate solution satisfies the zero boundary condition. With this choice

$$I(\alpha) = \alpha^2 \int \int_D (4x^2 + 4y^2 - \lambda(x^2 + y^2 - 1)^2) dx dy = \textit{extremum}.$$

Introducing polar coordinates for ease of integration yields

$$I(\alpha) = \alpha^2 \int_0^{2\pi} \int_0^1 (4r^3 - \lambda r(r^2 - 1)^2) dr d\phi = \textit{extremum}.$$

The evaluation of the integral results in the form

$$I(\alpha) = (2\pi - \lambda \frac{\pi}{3}) \alpha^2 = \textit{extremum}.$$

The necessary condition of the extremum is

$$\frac{\partial I(\alpha)}{\partial \alpha} = 0,$$

which yields an equation for  $\lambda$

$$2\alpha(2\pi - \lambda \frac{\pi}{3}) = 0.$$

The eigenvalue as the solution of this equation is

$$\lambda = 6.$$

The unknown solution function coefficient may be solved by normalizing the eigensolution as

$$\int \int_D b^2(x, y) dx = 1.$$

Substituting yields

$$\alpha^2 \int_0^{2\pi} \int_0^1 r(r^2 - 1)^2 dr d\phi = 1.$$

Integrating results in

$$\alpha^2 \frac{\pi}{3} = 1.$$

Hence

$$\alpha = \sqrt{\frac{3}{\pi}}.$$

The solution of the second equation is

$$b(x, y) = \sqrt{\frac{3}{\pi}}(x^2 + y^2 - 1).$$

The complete solution of the differential equation of the elastic membrane of the unit circle is finally

$$z(x, y, t) = c_1 \cos(\sqrt{6}t) \sqrt{\frac{3}{\pi}}(x^2 + y^2 - 1).$$

The remaining coefficient may be established by the initial condition.

Assuming the center of the membrane is displaced by an amplitude  $A$ ,

$$z(0, 0, 0) = A = c_1 \sqrt{\frac{3}{\pi}}(-1).$$

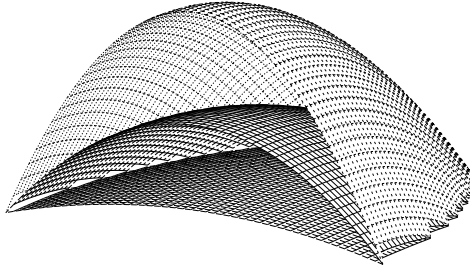
from which follows

$$c_1 = -A \sqrt{\frac{\pi}{3}}.$$

The final solution is

$$z(x, y, t) = -A \cos(\sqrt{6}t)(x^2 + y^2 - 1).$$

The shape of the solution is shown in Figure 11.2. The figure shows the solution of the half-membrane at three distinct time steps. The jagged edges are artifacts of the discretization; the shape of membrane was the unit circle.



**FIGURE 11.2** Vibration of elastic membrane

### 11.2.2 Non-zero boundary conditions

So far we restricted ourselves to trivial boundary conditions for the sake of clarity. In engineering practice, however, non-zero boundary conditions are very often imposed. These, also called enforced motion, boundary conditions are the subject of our focus here.

Let us consider the membrane with a flexible boundary allowing some or all of the boundary points to attain non-zero displacement from the plane. Let the arc length differential of a section of the boundary in equilibrium be denoted by  $ds$ . The reactive force on the section due to displacement  $z$  is

$$-p(s)z(x, y, t)ds,$$

where the negative sign indicates the force's effort to pull the boundary back toward the equilibrium position and opposite from the displacement. The potential energy of the boundary section may be computed by

$$p(s)ds \int z dz = \frac{1}{2}p(s)z^2 ds.$$

The total potential energy due to the reactive force on the boundary  $L$  is

$$E_p^L = \frac{1}{2} \int_L p(s)z^2 ds.$$

Applying Hamilton's principle for this scenario now yields

$$I(z) = \frac{1}{2} \int_{t_0}^{t_1} \left( \int \int_D \rho \left( \frac{\partial z}{\partial t} \right)^2 - \sigma \left( \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 \right) dx dy - \int_L p(s)z^2 ds \right) dt.$$

The newly introduced boundary integral's inconvenience may be avoided as follows. First, it may also be written as

$$\int_L p(s)z^2 ds = \frac{1}{2} \int_L (p(s)z^2 \frac{ds}{dy} dy + p(s)z^2 \frac{ds}{dx} dx).$$

Introducing the twice differentiable

$$P = \frac{1}{2} p z^2 \frac{ds}{dy}$$

and

$$Q = -\frac{1}{2} p z^2 \frac{ds}{dx}$$

functions that are defined on the boundary curve  $L$  the integral further changes to

$$\int_L p z^2 ds = \int_L (P dy - Q dx).$$

Finally, with the help of Green's theorem we obtain

$$\int_L p z^2 ds = \int \int_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx dy.$$

Hence the variational form of this problem becomes

$$I(z) = \frac{1}{2} \int_{t_0}^{t_1} \int \int_D \left( \rho \left( \frac{\partial z}{\partial t} \right)^2 - \sigma \left( \left( \frac{\partial z}{\partial x} \right)^2 + \left( \frac{\partial z}{\partial y} \right)^2 \right) - \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \right) dx dy dt.$$

This problem is identical to the one in Section 3.5, the case of a functional with three independent variables. The two spatial independent variables are augmented in this case with time as the third independent variable. The corresponding Euler-Lagrange differential equation becomes the same as in the case of the fixed boundary

$$\sigma \Delta z = \rho \left( \frac{\partial^2 z}{\partial t^2} \right),$$

with the addition of the constraint on the boundary as

$$\sigma \frac{\partial z}{\partial n} + p z = 0,$$

where  $n$  is the normal of the boundary. The solution may again be sought in the form of

$$z(x, y, t) = a(t)b(x, y),$$

and as before, based on the same reasoning

$$a(t) = c_1 \cos(\sqrt{\lambda}t) + c_2 \sin(\sqrt{\lambda}t).$$

The  $b(x, y)$  now must satisfy the following two equations.

$$\sigma \Delta b + \lambda \rho b = 0; (x, y) \in D,$$

and

$$\sigma \frac{\partial b}{\partial n} + \rho b = 0; (x, y) \in L.$$

The solution of these two equations follows the procedure established in the last section.

### 11.3 Bending of a beam under its own weight

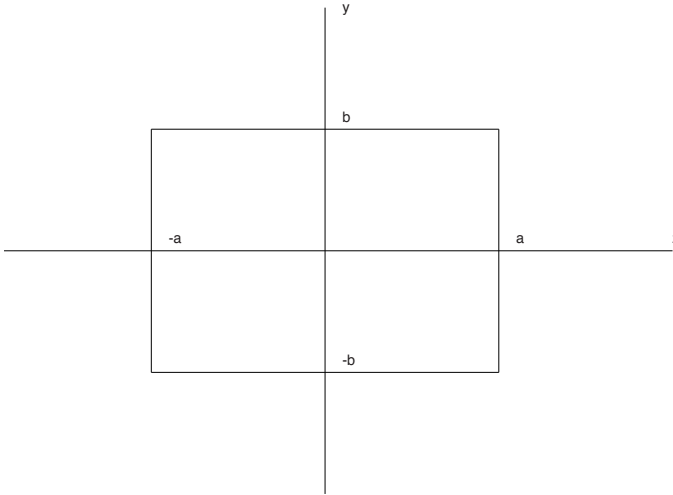
The two analytic elasticity examples presented so far were one- and two-dimensional, respectively. The additional dimensions (the string's cross-section or the thickness of the membrane) were negligible and ignored in the presentation. In this section we address the phenomenon of the bending of a beam with a non-negligible cross-section and consider all three dimensions.

In order to deal with the problem of the beam, we introduce some basic concepts of elasticity for this specific case only. A fuller exposition of the topic will be in the next chapter. Let us consider an elastic beam with length  $L$  and cross-section area  $A$ . We consider the beam fully constrained at one end and free on the other, known as a cantilever beam, with a rectangular cross-section of width  $a$  along the  $z$  axis and height  $b$  along the  $y$  axis as shown in Figure 11.3. The axis of the beam is aligned along the  $x$  axis.

The relationship between the stress resulting from an axial force exerted on the free end of the beam and its subsequent deformation is expressed by the well-known Hooke's law

$$\sigma = E\epsilon,$$

where the constant  $E$ , called Young's modulus, expresses the inherent elasticity of the material with regards to elongation. The relationship between the

**FIGURE 11.3** Beam cross-section

stress ( $\sigma$ ) and the force ( $F$ ) is

$$\sigma = \frac{F}{A}.$$

The strain ( $\epsilon$ ) describes the relative deformation of the beam, and in the axial case this is

$$\epsilon = \frac{dl}{l},$$

where  $dl$  is the elongation along the beam's longitudinal direction. In different deformation scenarios, like the ensuing bending, the formulation for the strain may vary and will be discussed in more detail later.

The energy equilibrium of this problem is again based on Hamilton's principle; however, since in this particular example we consider a static equilibrium, there is no kinetic energy. In this case Hamilton's principle becomes the principle of minimal potential energy. The two components of the potential energy are the internal strain energy and the work of forces acting on the body.

The internal energy related to the strain is

$$E_s = \frac{1}{2} \int_V \sigma \epsilon dV.$$

Substitution of Hooke's law yields

$$E_s = \frac{1}{2} E \int_V \epsilon \epsilon dAdx.$$

The strain energy of a particular cross-section is obtained by integrating as

$$E_s(x) = \frac{1}{2} E \int_{-b}^b \int_{-a}^a \epsilon^2 dz dy.$$

The bending will result in a curved shape with a radius of curvature  $r$  and a strain in the beam. Note that the radius of curvature is a function of the lengthwise location since the shape of the beam (the subject of our interest) is not a circle.

The relationship between the radius of curvature and the strain is established as follows. Above the neutral plane of the bending, which is the  $x - z$  plane in our case, the beam is elongated and it is compressed below the plane. Based on that at a certain distance  $y$  above or below the plane the strain is

$$\epsilon = \frac{y}{r}.$$

Note that since  $y$  is a signed quantity, above yields zero strain in the neutral plane, positive (tension) above the plane and negative (compression) below. Using this in the strain energy of a particular cross-section yields

$$E_s(x) = \frac{E}{2} \int_{-b}^b \int_{-a}^a \frac{y^2}{r^2} dz dy = \frac{E}{2} \frac{4ab^3}{3} \frac{1}{r^2} = \frac{EI}{2} \frac{1}{r^2},$$

where

$$I = \frac{4ab^3}{3}$$

is the moment of inertia of the rectangular cross-section. The total strain energy in the volume is

$$E_s = \frac{1}{2} EI \int_0^L \frac{1}{r^2} dx.$$

We assume that the only load on the beam is its weight. We denote the weight of the unit length with  $w$ . The bending moment generated by the weight of a cross-section with respect to the neutral plane is

$$dM = y dG,$$



where  $y$  is the distance from the neutral plane and  $dG$  is the weight of the cross-section. Using the unit length weight of the beam, we obtain the moment as

$$dM = ywdx = M(x)dx.$$

The total work of bending will be obtained by integrating along the length of the beam:

$$W = \int dM = \int_0^L M(x)dx = w \int_0^L ydx,$$

since the unit weight is constant. We are now ready to state the equilibrium of the beam

$$E_s = W$$

as a variational problem of the form

$$I(y) = \int_0^L (E_s(x) - M(x))dx = \textit{extremum},$$

or

$$I(y) = \int_0^L \left( \frac{1}{2}EI\frac{1}{r^2} - wy \right) dx = \textit{extremum}.$$

Since the radius of curvature is reciprocal of the second derivative of the bent curve of the beam,

$$r = \frac{1}{y''(x)}$$

it follows that

$$I(y) = \int_0^L \left( \frac{1}{2}EIy''^2(x) - wy \right) dx = \textit{extremum}.$$

The problem above is a special case of the form

$$I(y) = \int_L f(y, y'') dx = \textit{extremum},$$

where neither the  $y$  nor the  $y'$  term exists explicitly. It is also a case of higher order derivatives discussed in Section 4.2 and results in the Euler-Poisson equation of order 2:

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial f}{\partial y''} = 0.$$

Since in this case

$$f(y, y'') = \frac{1}{2}EIy''^2 - wy,$$

the first term is simply

$$\frac{\partial f}{\partial y} = -w.$$

The second term vanishes as the first derivative of the unknown function is not explicitly present. With

$$\frac{\partial f}{\partial y''} = 2\frac{1}{2}EIy'',$$

the third term becomes

$$\frac{d^2}{dx^2} \frac{\partial f}{\partial y''} = EI \frac{d^4}{dx^4} y.$$

Hence the solution obtained from the Euler-Poisson equation tailored for this case is

$$\frac{d^4 y}{dx^4} = \frac{w}{EI}.$$

Direct integration yields the final solution of

$$y(x) = \frac{w}{24EI}(x^4 + 4c_1x^3 + 12c_2x^2 + 24c_3x + c_4),$$

where the  $c_i$  are constants of integrations. The solution curve yields the shape of the bent beam shown in Figure 11.4.

In the figure unit physical coefficients were used for the sake of simplicity and the coefficients of integration are resolved from the boundary conditions as follows. At the fixed end, the beam is not deflected, hence

$$y(x=0) = 0,$$

which implies

$$c_4 = 0.$$

Furthermore, at the fixed end, the tangent of the curve is horizontal as

$$y'(x=0) = 0,$$

implying

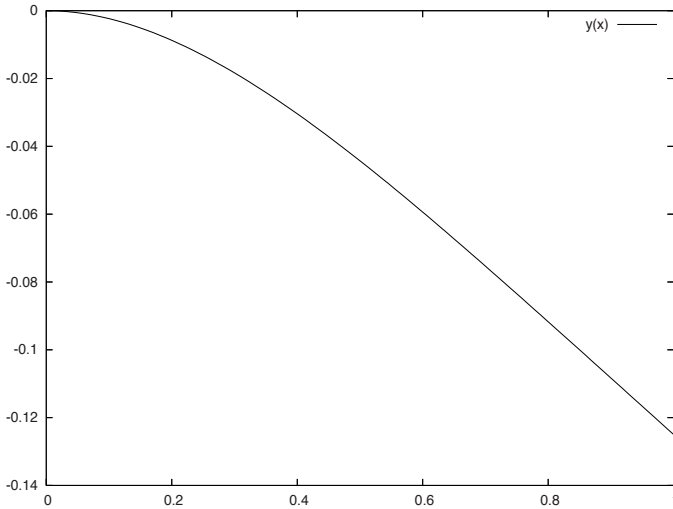
$$c_3 = 0.$$

Finally at the free end the beam has no curvature, hence both second and third derivatives vanish. Therefore

$$y'''(L) = 0$$

yields

$$c_1 = -L,$$



**FIGURE 11.4** Beam profile under its weight

and

$$y''(L) = 0$$

results in

$$c_2 = \frac{L^2}{2}.$$

With these, the final solution becomes

$$y(x) = \frac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2),$$

which is the source of maximum deflection often quoted in engineering handbooks:

$$y(L) = \frac{wL^4}{8EI}.$$

Finally, it is worthwhile to point out the intriguing similarities between this problem and the natural spline solution of Chapter 9.

The scenario is also often presented as a problem of optimization. In engineering practice it is a natural desire to minimize the deflection of the beam under its own weight, since very likely there is an additional load applied to

it as well.

In our case it is easy to see that to minimize the deflection, either of the quantities in the denominator, the Young's modulus or the moment of inertia, should be increased. Assuming that the material type is dictated, hence  $E$  is fixed, we can still address the shape. Since the chosen cross section is rectangular, it follows that the higher the  $b$  dimension, the smaller the deflection is.

This, however, cannot be carried to the extreme: Some minimal and maximal ratios of dimensions of rectangular cross sections are usually given as constraints. The mathematical problem becomes

$$I = \frac{4ab^3}{3} = \textit{maximum},$$

subject to

$$r_{min} \leq \frac{b}{a} \leq r_{max}.$$

This is a constrained optimization problem whose solution is intuitively at the maximum ratio

$$I_{max} = \frac{4r_{max}^3 a^4}{3}.$$

This is a simplest problem of shape optimization, a topic of high importance in structural engineering. The method of gradients in Section 6.5 provides the foundation for the variational solution of such problems.



# 12

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## *Computational mechanics*

The algebraic difficulties of analytic mechanical solutions are rather overwhelming and become insurmountable when solving real-world problems. Computational mechanics is based on the discretization of the geometric continuum and describing its physical behavior in terms of generalized coordinates. Its focus is on computing numerical solutions to practical problems of engineering mechanics.

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### 12.1 Three-dimensional elasticity

One of the fundamental concepts necessary to understanding continuum mechanical systems is a generic treatment of elasticity described in detail in the classical reference of the subject [17]. When an elastic continuum undergoes a one-dimensional deformation, like in the case of the bar discussed in Section 11.3, Young's modulus was adequate to describe the changes.

For a general three-dimensional elastic continuum we need another coefficient, introduced by Poisson, to capture the three-dimensional elastic behavior. Poisson's ratio measures the contraction of the cross-section while an object such as a beam is stretched. The ratio  $\nu$  is defined as the ratio of the relative contraction and the relative elongation:

$$\nu = -\frac{dr}{r} / \frac{dl}{l}.$$

Here a beam with circular cross-section and radius  $r$  is assumed. Poisson's ratio is in the range of zero to 1/2 and expresses the compressibility of the material. The two constants are also often related as

$$\mu = \frac{E}{2(1 + \nu)},$$

and

$$\lambda = \frac{E\nu}{(1 + \nu)(1 - 2\nu)}.$$

Here  $\mu$  and  $\lambda$  are the so-called Lamé constants.

In a three-dimensional elastic body, the elasticity relations could vary significantly. Let us consider isotropic materials, whose elastic behavior is independent of the material orientation. In this case Young's modulus is replaced by an elasticity matrix whose terms are only dependent on the Lamé constants as follows

$$D = \begin{bmatrix} \lambda + 2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda + 2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda + 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}.$$

Viewing an infinitesimal cube of the three-dimensional body, there are six stress components on the element,

$$\underline{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{bmatrix}.$$

The first three are normal and the second three are shear stresses. There are also six strain components

$$\underline{\epsilon} = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix}.$$

The first three are extensional strains and the last three are rotational strains. The stress-strain relationship is described by the generalized Hooke's law as

$$\underline{\sigma} = D\underline{\epsilon}.$$

This will be the fundamental component of the computational techniques for elastic bodies. Let us further designate the location of an interior point of the elastic body with

$$\underline{r}(x, y, z) = x\underline{i} + y\underline{j} + z\underline{k} = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and the displacements of the point with

$$\underline{u}(x, y, z) = u\underline{i} + v\underline{j} + w\underline{k} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}.$$

Then the following strain relations hold:

$$\epsilon_x = \frac{\partial u}{\partial x},$$

$$\epsilon_y = \frac{\partial v}{\partial y},$$

and

$$\epsilon_z = \frac{\partial w}{\partial z}.$$

These extensional strains manifest the change of rate of the displacement of an interior point of the elastic continuum with respect to the coordinate directions.

The rotational strains are computed as

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y},$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x},$$

and

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}.$$

These terms define the rate of change of the angle between two lines crossing at the interior point that were perpendicular in the un-deformed body and get distorted during the elastic deformation.

The strain energy contained in the three-dimensional elastic continuum is

$$E_s = \frac{1}{2} \int_V \underline{\sigma}^T \underline{\epsilon} dV = \frac{1}{2} \int_V \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z & \tau_{yz} & \tau_{xz} & \tau_{xy} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{bmatrix} dV.$$

We will also consider distributed forces acting at every point of the volume (like the weight of the beam in Section 11.3), described by



$$\underline{f} = f_x \underline{i} + f_y \underline{j} + f_z \underline{k} = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}.$$

The work of these forces is based on the displacements they caused at the certain points and computed as

$$W = \int_V \underline{u}^T \underline{f} dV. = \int_V [u \ v \ w] \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} dV.$$

The above two energy components constitute the total potential energy of the volume as

$$E_p = E_s - W.$$

In order to evaluate the dynamic behavior of the three-dimensional body, the kinetic energy also needs to be computed. Let the velocities at every point of the volume be described by

$$\underline{\dot{u}}(x, y, z) = \dot{u} \underline{i} + \dot{v} \underline{j} + \dot{w} \underline{k} = \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix}.$$

With a mass density of  $\rho$ , assumed to be constant throughout the volume, the kinetic energy of the body is

$$E_k = \frac{1}{2} \rho \int_V \underline{\dot{u}}^T \underline{\dot{u}} dV.$$

We are now in the position to write the variational statement describing the equilibrium of the three-dimensional elastic body:

$$I(\underline{u}(x, y, z)) = \int_V (E_k - E_p) dV = \textit{extremum},$$

which is of course Hamilton's principle.

The unknown displacement solution of the body at every  $(x, y, z)$  point is the subject of the computational solution discussed in the next sections.

## 12.2 Lagrangian formulation

The equations will be obtained by finding an approximate solution of the variational problem based on the total energy of the system as follows. For

$$I(\underline{u}) = \int_v f(\underline{u}(x, y, z)) dv = \textit{extremum},$$

we seek the (approximate) solution in the form

$$\underline{\bar{u}}(x, y, z) = \sum_{i=1}^n \underline{q}_i N_i(x, y, z).$$

The yet unknown coefficients, the  $\underline{q}_i$  values, are displacements at  $i = 1, 2, \dots$  discrete locations inside the volume. These are also known as generalized displacements and discussed in an earlier section [15].

The variational problem, based on the total energy of the system, in this case is

$$I(\underline{u}) = \int_v \left( \frac{1}{2} \rho \underline{\dot{u}}^T \underline{\dot{u}} - \left( \frac{1}{2} \underline{\sigma}^T \underline{\epsilon} - \underline{u}^T \underline{f} \right) \right) dv = \textit{extremum}.$$

Let us organize the generalized displacements as

$$\underline{q} = \begin{bmatrix} \underline{q}_1 \\ \dots \\ \underline{q}_n \end{bmatrix},$$

where, in adherence to our three-dimensional focus

$$\underline{q}_i = \begin{bmatrix} q_{i,x} \\ q_{i,y} \\ q_{i,z} \end{bmatrix}.$$

Using this, the approximate solution becomes

$$\underline{\bar{u}}(x, y, z) = N \underline{q}$$

with the matrix of basis functions

$$N(x, y, z) = N_x \underline{i} + N_y \underline{j} + N_z \underline{k} = \begin{bmatrix} N_1 & 0 & 0 & \dots & N_n & 0 & 0 \\ 0 & N_1 & 0 & \dots & 0 & N_n & 0 \\ 0 & 0 & N_1 & \dots & 0 & 0 & N_n \end{bmatrix}.$$

The basis functions are usually low order polynomials of  $x, y, z$ , and will be discussed in detail in Section 12.5.3.

Let us apply this to the terms of our variational problem, starting with the kinetic energy. Assuming that the velocity is also a function of the generalized velocities,

$$\dot{\underline{u}}(x, y, z) = N \dot{\underline{q}},$$

where

$$\underline{\dot{q}} = \begin{bmatrix} \dot{q}_1 \\ \dots \\ \dot{q}_n \end{bmatrix},$$

we obtain

$$E_k = \int_V \frac{1}{2} \rho \underline{\dot{u}}^T \underline{\dot{u}} dV = \frac{1}{2} \underline{\dot{q}}^t \int_V N^T \rho N dV \underline{\dot{q}}.$$

Introducing the mass matrix

$$M = \int_V N^T \rho N dV,$$

the final form of the kinetic energy becomes

$$E_k = \frac{1}{2} \underline{\dot{q}}^T M \underline{\dot{q}}.$$

Now let's focus on the strain energy. Note that the strain is now also expressed in terms of the basis functions. Hence

$$\underline{\epsilon}(N) = \begin{bmatrix} \sum_{i=1}^n q_i^t \frac{\partial N}{\partial x} \\ \sum_{i=1}^n q_i^t \frac{\partial N}{\partial y} \\ \sum_{i=1}^n q_i^t \frac{\partial N}{\partial z} \\ \sum_{i=1}^n q_i^t \left( \frac{\partial N}{\partial z} + \frac{\partial N}{\partial y} \right) \\ \sum_{i=1}^n q_i^t \left( \frac{\partial N}{\partial z} + \frac{\partial N}{\partial x} \right) \\ \sum_{i=1}^n q_i^t \left( \frac{\partial N}{\partial y} + \frac{\partial N}{\partial x} \right) \end{bmatrix},$$

or in matrix form

$$\underline{\epsilon}(N) = B \underline{q},$$

where the columns of  $B$  are

$$B_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial z} + \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} + \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} + \frac{\partial N_i}{\partial x} \end{bmatrix}.$$

With this, the integral becomes

$$\int_V \underline{\epsilon}^T(N) D \underline{\epsilon}(N) dV = \int_V \underline{q}^T B^T D B \underline{q} dV.$$

The total strain energy in the system is

$$E_s = \frac{1}{2} \underline{q}^T \int_V B^T DB dV \underline{q}.$$

Introducing the stiffness matrix of the system as

$$K = \int_V B^T DB dV,$$

the strain energy term is of final form

$$E_s = \frac{1}{2} \underline{q}^T K \underline{q}.$$

A similar approach on the second term of the potential energy yields

$$\int_V \underline{q}^T N^T \underline{f} dV.$$

Introducing the total active force vector on the system as

$$F = \int_V N^T \underline{f} dV,$$

this term becomes

$$\underline{q}^T F.$$

The total potential energy is their difference

$$E_p = \frac{1}{2} \underline{q}^T K \underline{q} - \underline{q}^T F.$$

We are ready to find the value of the unknown solution components and will use the simpler form of Lagrange's equations of motion from the earlier section as

$$\frac{d}{dt} \frac{\partial E_k}{\partial \dot{q}_i} + \frac{\partial E_p}{\partial q_i} = 0; i = 1, 2, \dots, n.$$

The first term is evaluated as

$$\frac{\partial E_k}{\partial \dot{q}} = M \dot{\underline{q}}.$$

Then

$$\frac{d}{dt} (M \dot{\underline{q}}) = M \ddot{\underline{q}}.$$

Here the generalized accelerations are

$$\ddot{\underline{q}} = \begin{bmatrix} \ddot{q}_1 \\ \dots \\ \ddot{q}_n \end{bmatrix}.$$

The second part results in the two terms

$$\frac{\partial E_p}{\partial \underline{q}} = K\underline{q} - F = 0.$$

The final result is

$$M\ddot{\underline{q}} + K\underline{q} = f.$$

This is the well-known equation of the forced undamped vibration of a three-dimensional elastic body.

### 12.3 Heat conduction

While staying on the mechanics territory, we now explore the area of heat conduction. This phenomenon occurs when the temperature between two areas of a body differs. In this application every point in space is associated with a scalar quantity, the temperature, hence these type of problems are called scalar field problems.

For our discussion, we will assume that the body does not deform under the temperature load. This assumption, of course, may be violated in real life. Serious warping of objects left in the sun is a strong example of that scenario.

Two more restrictions we impose. We'll consider two-dimensional problems for simplification of the discussion. We will also only consider the steady state solution case, when the temperature at a certain point is independent of the time.

The differential equation of the heat conduction per [4] for this case is

$$\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + Q = 0,$$

where the temperature field

$$T = T(x, y)$$

is a function of the location and  $k$  is the thermal conductivity of the material of the object. In general the conductivity may be a function of the location as well. The  $Q$  is a heat source in the model, called such when it generates heat, but called a sink when it absorbs heat.

Since the differential equation is given, we'll use the techniques developed in Chapter 5 when dealing with the inverse variational problem. Accordingly, the solution of a differential equation of the form

$$Au + f = 0,$$

corresponds to the extremum value of the functional

$$I(u) = (Au, u) - 2(u, f).$$

For the heat conduction differential equation, this means

$$I(T) = \frac{1}{2} \int \int_D (k((\frac{\partial T}{\partial x})^2 + (\frac{\partial T}{\partial y})^2) - 2QT) dx dy = \text{extremum}.$$

Following the avenue charted in the last section for the elasticity problem, we'll approximate the temperature field in terms of basis functions by

$$T(x, y) = \sum_{i=1}^n T_i N_i.$$

Then

$$\frac{\partial T(x, y)}{\partial x} = \sum_{i=1}^n T_i \frac{\partial N_i}{\partial x},$$

and

$$\frac{\partial T(x, y)}{\partial y} = \sum_{i=1}^n T_i \frac{\partial N_i}{\partial y}.$$

We introduce a vector of the generalized temperatures

$$\underline{T} = \begin{bmatrix} T_1 \\ \dots \\ T_n \end{bmatrix}$$

and  $B_i, i = 1, \dots, n$  columns of a  $B$  matrix as

$$B_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}.$$

We also concatenate the  $N_i$  into the matrix  $N$

$$N = [N_1 \dots N_n].$$

This architecture of the  $N$  matrix is simpler than in the case of the elasticity, reflecting the fact that this is a scalar field problem. The elasticity was a vector field problem as the solution quantity at each point was the displacement vector of three dimensions.

Then in matrix form we obtain the approximate temperature field solution as

$$\bar{T}(x, y) = N\underline{T},$$

and

$$\left(\frac{\partial T}{\partial x}\right)^2 + \left(\frac{\partial T}{\partial y}\right)^2 = (B\underline{T})^T.$$

We apply these to the variational problem. The first part becomes

$$\frac{1}{2} \int \int_D k \left( \left( \frac{\partial T}{\partial x} \right)^2 + \left( \frac{\partial T}{\partial y} \right)^2 \right) dx dy = \frac{1}{2} \int \int_D k \underline{T}^T B^T B \underline{T} dx dy.$$

The second part changes as

$$- \int \int_D Q T dx dy = - \int \int_D Q N \underline{T} dx dy.$$

With the introduction of a “temperature stiffness matrix” of

$$K = \int \int_D k B^T B dx dy,$$

and the load vector of

$$\underline{Q} = \int \int_D Q N dx dy,$$

finally, we obtain the variational problem of

$$I(\bar{T}) = \frac{1}{2} \underline{T}^T K \underline{T} - \underline{Q} \underline{T} = \text{extremum},$$

from which the now familiar result of

$$K \underline{T} = \underline{Q}$$

emerges.

This simpler two-dimensional case will be used to illuminate the computational details of the generalized coordinate selection and processing in Section 12.5, after reviewing another mechanical application involving fluids.

## 12.4 Fluid mechanics

As a final application we discuss a problem of fluid mechanics discussed in detail by [19], where the fluid is partially or fully surrounded by an external

structure and the dissipation of energy into the surrounding space is negligible.

Assuming small motions, the equilibrium of a compressible fluid inside a cavity is governed by the Euler equation derived in Section 10.5

$$\rho \ddot{u} = -\nabla p,$$

where  $\ddot{u}$  is the acceleration of the particles and  $p$  is the pressure in the fluid. Furthermore,  $\rho$  is the density and  $\nabla$  is the differential operator.

We also assume locally linear pressure-velocity behavior of the fluid as

$$p = -b\nabla u,$$

where  $b$  is the so-called bulk modulus related to the density of the fluid and the speed of sound. Differentiating twice with respect to time and substituting the Euler equation we get Helmholtz's equation describing the behavior of the fluid:

$$\frac{1}{b}\ddot{p} = \nabla\left(\frac{1}{\rho}\nabla p\right).$$

The following boundary conditions are also applied. At a structure-fluid interface

$$\frac{\partial p}{\partial \underline{n}} = -\rho \ddot{u}_n, \quad (12.1)$$

where  $\underline{n}$  is the direction of the outward normal. At free surfaces:

$$u = p = 0.$$

Since the equilibrium differential equation of the physical phenomenon is given at this time, the inverse problem approach introduced in Chapter 5 will be used again. Accordingly, for the differential equation of

$$Au = 0,$$

the variational problem of

$$I(u) = (Au, u)$$

applies. Here the inner product is defined over the continuum. For our case this results in

$$\int \int \int_V \left[ \frac{1}{b}\ddot{p} - \frac{1}{\rho}\nabla \cdot \nabla p \right] p dV = 0. \quad (12.2)$$



Following the earlier sections, we will also assume that the pressure field is approximated by basis functions as:

$$p(x, y, z) = \sum_{i=1}^n N_i p_i = N \underline{p}.$$

The same holds for the derivatives:

$$\dot{p}(x, y, z) = N \dot{\underline{p}}.$$

Separating the two parts of the Equation (12.2), the first yields

$$\int_V \frac{1}{b} \ddot{p} p dV = \int_V \frac{1}{b} p \ddot{p} dV = \underline{p}^T \int_V \frac{1}{b} N^T N dV \underline{\ddot{p}}.$$

Introducing the mass matrix

$$M = \int_V \frac{1}{b} N^T N dV,$$

this term simplifies to

$$\int_V \frac{1}{b} \ddot{p} p dV = \underline{p}^T M \underline{\ddot{p}}.$$

Let us now turn our attention to the second part of Equation (12.2). Integrating by parts yields

$$- \int_V \left( \frac{1}{\rho} \nabla \cdot \nabla p p \right) dV = \int_V \frac{1}{\rho} \nabla p \cdot \nabla p dV - \int_S \frac{1}{\rho} \nabla p p dS.$$

From the above assumptions, it follows that

$$\nabla p = \nabla N \underline{p},$$

and we obtain

$$\underline{p}^T \int_V \left( \frac{1}{\rho} \nabla N^T \right) \nabla N dV \underline{p} + \underline{p}^T \int_S N^T \ddot{u}_n dS.$$

Here the boundary condition stated in Equation (12.1) was used. Introducing

$$K = \int_V \frac{1}{\rho} \nabla N^T \nabla N dV,$$

the first part simplifies to

$$\underline{p}^T K \underline{p}.$$

The force exerted on the boundary by the surrounding structure is

$$F = \int_S N^T \ddot{u}_n dS.$$

Substituting and reordering yields

$$\underline{p}^T M \ddot{\underline{p}} + \underline{p}^T K \underline{p} + \underline{p}^T F = 0.$$

Finally, the equilibrium equation is

$$M \ddot{\underline{p}} + K \underline{p} + F = 0.$$

This, as all the similar problems of this chapter, may be solved by efficient numerical linear algebra computations and will not be discussed further here.

## 12.5 The finite element method

In order to conclude the mechanical applications presented in the past three sections, we need to elucidate their common, and thus far omitted, computational details. The most practical and wide spread general purpose computational method is that of the finite elements.

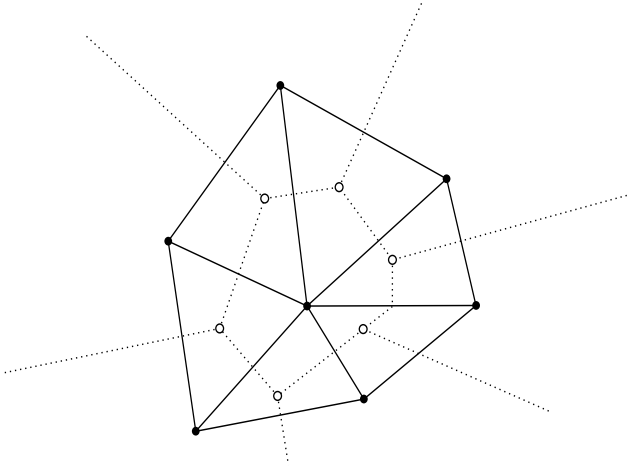
Computational solutions via the finite element method achieved an unparalleled industrial success. The topic's details cover an extensive territory [14] but from this point on we'll discuss only the main components.

These components are: the discretization of the geometric domain by finite elements, the computation of the basis functions used in the approximate solutions, and the assembly of the system matrices. These components are described here once to further emphasize the point of their commonality, or even application independence.

### 12.5.1 Finite element meshing

The discretization of generic three- or two-dimensional domains is usually by finite elements of simple shapes, such as tetrahedra or triangles. The foundation of many general methods of discretization (commonly called meshing) is the classical Delaunay triangulation method. The Delaunay triangulation technique in turn is based on Voronoi polygons. The Voronoi polygon, assigned to a certain point of a set of points in the plane, contains all the points that are closer to the selected point than to any other point of the set.

In Figure 12.1 the dots represent such a set of points. The irregular (dotted line) hexagon containing one point in the middle is the Voronoi polygon of the point in the center. It is easy to see that the points inside the polygon



**FIGURE 12.1** Delaunay triangularization

are closer to the center point than to any other points of the set. It is also quite intuitive that the edges of the Voronoi polygon are the perpendicular bisectors of the line segments connecting the points of the set.

The union of the Voronoi polygons of all the points in the set completely covers the plane. It follows that the Voronoi polygon of two points of the set do not have common interior points; at most they share points on their common boundary.

The definition and process generalizes to three dimensions very easily. If the set of points are in space, the points closest to a certain point define a Voronoi polyhedron.

The Delaunay triangulation process is based on the Voronoi polygons by constructing Delaunay edges connecting those points whose Voronoi polygons have a common edge. Constructing all such possible edges will result in the covering of the planar region of our interest with triangular regions, the Delaunay triangles.

The process starts with placing vertices on the boundary of the domain in an equally spaced fashion. The Voronoi polygons of all boundary points

are created and interior points are generated gradually proceeding inward by creating Delaunay triangles. This is called the advancing front technique.

The process extends quite naturally and covers the plane as shown in Figure 12.1 with six Delaunay triangles where the dotted lines are the edges of the Voronoi polygons and the solid lines depict the Delaunay edges. It is known that under the given definitions no two Delaunay edges cross each other.

Finally, in three dimensions the Delaunay edges are defined as lines connecting points that share a common Voronoi facet (a face of a Voronoi polyhedron). Furthermore, the Delaunay facets are defined by points that share a common Voronoi edge (an edge of a Voronoi polyhedron). In general each edge is shared by exactly three Voronoi polyhedra; hence, the Delaunay regions' facets are going to be triangles. The Delaunay regions connect points of Voronoi polyhedra that share a common vertex. Since in general the number of such polyhedra is four, the generated Delaunay regions will be tetrahedra. The triangulation method generalized into three dimensions is called tessellation.

### 12.5.2 Shape functions

We will demonstrate the finite element method by assuming that the meshed domain in the prior section represents an irregularly shaped membrane problem and only out-of-plane deformations of the membrane are considered. This will simplify the presentation of the method while still capturing its intricacies. The three-dimensional elasticity formulation is simply a generalization of the process presented below.

In the prior sections we used the  $N_i$  basis functions to describe the approximate solutions. In order to approximate the solution inside the domain, the finite element method uses a collection of low order polynomial basis functions. For a triangular element discretization of a two-dimensional domain, as shown in Figure 12.1, bilinear interpolation functions are commonly used in the form:

$$u(x, y) = a + bx + cy.$$

Here  $u$  represents any of the  $q$ ,  $T$ , or  $p$  physical solution quantities introduced in the past three sections.

In order to find the coefficients, let us consider a triangular element in the  $x - y$  plane with corner nodes  $(x_1, y_1)$ ,  $(x_2, y_2)$ , and  $(x_3, y_3)$ . For this particular triangle we seek three specific basis functions, called shape functions in the finite element field, satisfying

$$N_1 + N_2 + N_3 = 1.$$

We also require that these functions at a certain node point reduce to zero at the other two nodes. This is called the Kronecker property and presented as

$$N_i = \begin{cases} 1 & \text{at node } i, \\ 0 & \text{at node } \neq i. \end{cases}$$

Furthermore, a shape function is zero along the edge opposite to the particular node at which the shape function is non-zero.

The solution for the nodes of a particular triangular element  $e$  can be expressed in matrix form as

$$u_e = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

This system of equations is solved for the unknown coefficients that produce the shape functions

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} N_{1,1} & N_{1,2} & N_{1,3} \\ N_{2,1} & N_{2,2} & N_{2,3} \\ N_{3,1} & N_{3,2} & N_{3,3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

By substituting into the matrix form of the bilinear interpolation function

$$u(x, y) = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} N_{1,1} & N_{1,2} & N_{1,3} \\ N_{2,1} & N_{2,2} & N_{2,3} \\ N_{3,1} & N_{3,2} & N_{3,3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix},$$

we get

$$u(x, y) = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Here the  $N_1, N_2, N_3$  shape functions are

$$N_1(x, y) = N_{1,1} + N_{2,1}x + N_{3,1}y,$$

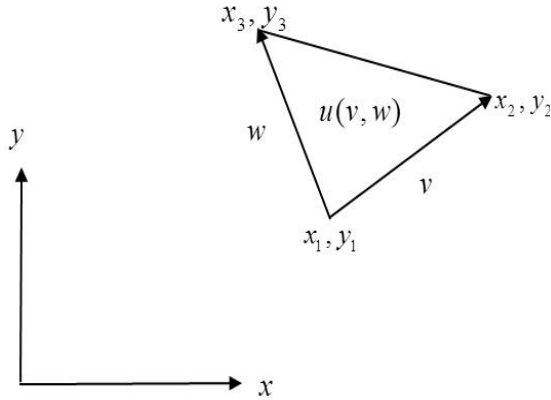
$$N_2(x, y) = N_{1,2} + N_{2,2}x + N_{3,2}y,$$

and

$$N_3(x, y) = N_{1,3} + N_{2,3}x + N_{3,3}y.$$

The shape functions, as their name indicates, solely depend on the coordinates of the corner nodes and the shape of the particular triangular element of the domain. With these we are able to approximate the solution value inside an element in terms of the solutions at the corner node points as

$$u(x, y) = N_1(x, y)u_1 + N_2(x, y)u_2 + N_3(x, y)u_3.$$



**FIGURE 12.2** Parametric coordinates of triangular element

The shortcoming of this direct approach is that the coefficients of the shape functions are different for each element and they would have to be computed for all elements in the domain.

It is practical therefore to generate the shape functions for a standard, parametrically defined element. In that case the shape functions and their derivatives may be pre-computed and appropriately transformed as was originally proposed in [20]. Let us define a parametric coordinate system for the triangular element as shown in Figure 12.2.

The relationship between the geometric and parametric coordinates is defined by the bidirectional mapping

$$x = x(v, w), y = y(v, w)$$

and

$$v = v(x, y), w = w(x, y).$$

The  $v$  axis is directed from node 1 with coordinates  $x_1, y_1$  to node 2 with coordinates  $x_2, y_2$ . The  $w$  axis is directed from node 1 with coordinates  $x_1, y_1$  to node 3 with coordinates  $x_3, y_3$ . The pairing between the geometric and parametric coordinates of the nodes of the triangle is shown in Table 12.1.

Let us now compute the shape functions in terms of these parametric coordinates:

**TABLE 12.1**  
Coordinate pairing of  
triangular element

| node | $x$   | $y$   | $v$ | $w$ |
|------|-------|-------|-----|-----|
| 1    | $x_1$ | $y_1$ | 0   | 0   |
| 2    | $x_2$ | $y_2$ | 1   | 0   |
| 3    | $x_3$ | $y_3$ | 0   | 1   |

$$N_i(v, w) = N_i(v(x, y), w(x, y)).$$

Specifically, we choose

$$N_1(v, w) = 1 - v - w,$$

$$N_2(v, w) = v,$$

and

$$N_3(v, w) = w.$$

These shape functions also satisfy the Kronecker property stated above and the conditions of polynomial completeness:

$$\sum_{i=1}^3 N_i(v, w) = 1 - v - w + v + w = 1,$$

$$\sum_{i=1}^3 N_i(v, w)x_i = x$$

and

$$\sum_{i=1}^3 N_i(v, w)y_i = y.$$

The last two equations imply that the location of any point inside the element will also be represented by these shape functions as

$$x = N_1(v, w)x_1 + N_2(v, w)x_2 + N_3(v, w)x_3,$$

and

$$y = N_1(v, w)y_1 + N_2(v, w)y_2 + N_3(v, w)y_3.$$

Such elements are called iso-parametric elements since both the geometry and the solution function inside the element are approximated by the same parametric shape functions. Substituting the shape functions we obtain

$$x = (1 - v - w)x_1 + vx_2 + wx_3 = x_1 + (x_2 - x_1)v + (x_3 - x_1)w,$$

and

$$y = (1 - v - w)y_1 + vy_2 + wy_3 = y_1 + (y_2 - y_1)v + (y_3 - y_1)w.$$

This formulation is a crucial component of the standardized element matrix generation as we will see it in the next section.

### 12.5.3 Element matrix generation

In order to compute a particular matrix, for example, the mass matrix  $M$  introduced in Section 12.2, we proceed element by element. We consider all the nodes bounding a particular element and compute the matrix for that particular element. Thus,

$$M_e = \rho \int \int_{x,y \in D_e} N(x,y)^T N(x,y) dx dy,$$

where the  $N(x,y)$  matrix is the local shape function matrix of the particular triangular element and  $D_e$  is its geometric domain.

Using the parametric coordinates, however, the above elemental mass matrix integral may be evaluated as

$$M_e = \rho \int_{v=0}^1 \int_{w=0}^{1-v} N(v,w)^T N(v,w) \det \left( \frac{\partial(x,y)}{\partial(v,w)} \right) dw dv.$$

The matrix of the determinant, called the Jacobian matrix, is computed as follows

$$J = \frac{\partial(x,y)}{\partial(v,w)} = \begin{bmatrix} \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} \end{bmatrix}.$$

For our triangular element this matrix is of the form

$$J = \begin{bmatrix} x_2 - x_1 & y_2 - y_1 \\ x_3 - x_1 & y_3 - y_1 \end{bmatrix}$$

and its determinant is

$$\det(J) = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1).$$

This determinant is a different value for each element, however, since it is a constant, the determinant term could be moved outside of the integral, a fact of further importance. The value of the determinant is indicative of the quality of the finite element. A very small value indicates an ill-shaped finite element that will be detrimental to the solution quality as we will see later.

Since the matrix of the shape functions is organized as

$$N(v,w) = [N_1 \ N_2 \ N_3],$$



the  $N(v, w)^T N(v, w)$  product needed for the mass matrix is of the form

$$N(v, w)^T N(v, w) = \begin{bmatrix} N_1^T N_1 & N_1^T N_2 & N_1^T N_3 \\ N_2^T N_1 & N_2^T N_2 & N_2^T N_3 \\ N_3^T N_1 & N_3^T N_2 & N_3^T N_3 \end{bmatrix}.$$

These terms are only functions of the  $v, w$  parametric variables, hence they may be pre-computed as

$$N(v, w)^T N(v, w) = \begin{bmatrix} (1-v-w)^2 & (1-v-w)v & (1-v-w)w \\ v(1-v-w) & v^2 & vw \\ w(1-v-w) & vw & w^2 \end{bmatrix}.$$

The matrix is symmetric and the integral over the parametric domain of the triangular finite element becomes

$$M_e = \rho \det(J) \int_{v=0}^1 \int_{w=0}^{1-v} N(v, w)^T N(v, w) dw dv.$$

In this form the evaluation of the integrals is still cumbersome due to the variable upper limit of the inner integral. They may be further transformed to enable easier integration by the substitution

$$w = \frac{1-v}{2} + \frac{1-v}{2}r,$$

and

$$v = \frac{1}{2} + \frac{1}{2}s.$$

This will of course modify  $N(v, w)$ , a function of  $v, w$ , to  $N(s, r)$ , a function of  $s, r$ , and brings the consequences

$$dv = \frac{1}{2} ds$$

and

$$dw = \frac{1-v}{2} dr.$$

Finally the integrals become

$$M_e = \rho \det(J) \int_{s=-1}^1 \frac{1}{2} \int_{r=-1}^1 N(s, r)^T N(s, r) \left(\frac{1}{4} - \frac{1}{4}s\right) dr ds.$$

These may now be easily integrated for the standard element a priori and only once. During computation of the finite element solution the standard element matrix is multiplied by values in front of the integrals that are characteristic of the shape of the particular element. This is a fundamental aspect of practical finite element analysis.

The generation of the stiffness matrix will require the computation of the  $B$  matrix described in Section 12.2. For our simplified case the matrix is of

the form:

$$B(x, y) = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} \end{bmatrix}.$$

Since the shape functions are defined in terms of the parametric coordinates, the derivatives of the local shape functions are computed by using the chain rule as

$$\frac{\partial N_i}{\partial v} = \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial v}$$

and

$$\frac{\partial N_i}{\partial w} = \frac{\partial N_i}{\partial x} \frac{\partial x}{\partial w} + \frac{\partial N_i}{\partial y} \frac{\partial y}{\partial w}.$$

These relations may be gathered as

$$\begin{bmatrix} \frac{\partial N_i}{\partial v} \\ \frac{\partial N_i}{\partial w} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} \end{bmatrix} \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}.$$

The first term on the right-hand side is

$$\begin{bmatrix} \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} \end{bmatrix} = J,$$

as we found it earlier. Hence

$$\begin{bmatrix} \frac{\partial N_i}{\partial v} \\ \frac{\partial N_i}{\partial w} \end{bmatrix} = J \begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{\partial N_i}{\partial x} \\ \frac{\partial N_i}{\partial y} \end{bmatrix} = J^{-1} \begin{bmatrix} \frac{\partial N_i}{\partial v} \\ \frac{\partial N_i}{\partial w} \end{bmatrix}.$$

The inverse of the Jacobian matrix may be computed by

$$J^{-1} = \frac{\text{adj}(J)}{\det(J)}.$$

This equation clarifies the earlier warning comment about the numerical problems arising from elements with a very small Jacobian determinant that is in the denominator. Hence we now have arrived at the  $B$  matrix with shape function derivatives with respect to the parametric coordinates as

$$B(v, w) = J^{-1} \begin{bmatrix} \frac{\partial N_1}{\partial v} & \frac{\partial N_2}{\partial v} & \frac{\partial N_3}{\partial v} \\ \frac{\partial N_1}{\partial w} & \frac{\partial N_2}{\partial w} & \frac{\partial N_3}{\partial w} \end{bmatrix}.$$

Using the terms of the Jacobian matrix we obtained earlier, the adjoint is

$$\text{adj}(J) = \begin{bmatrix} y_3 - y_1 & x_1 - x_3 \\ y_1 - y_2 & x_2 - x_1 \end{bmatrix},$$

and the determinant becomes

$$\det(J) = (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1).$$

Therefore the inverse matrix contains the element specific coordinates. It is easy to find from the preceding that

$$\begin{aligned} \frac{\partial N_1}{\partial v} &= -1, & \frac{\partial N_1}{\partial w} &= -1, \\ \frac{\partial N_2}{\partial v} &= 1, & \frac{\partial N_2}{\partial w} &= 0, \end{aligned}$$

and

$$\frac{\partial N_3}{\partial v} = 0, \quad \frac{\partial N_3}{\partial w} = 1.$$

For our specific element we obtain

$$B(v, w) = J^{-1} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

The elemental stiffness matrix, with the inclusion of the material specific elasticity matrix  $D$ , may now be computed as

$$K_e = \det(J) \int_{v=0}^1 \int_{w=0}^{1-v} B(v, w)^T DB(v, w) dw dv.$$

This element stiffness matrix is of order 3 by 3 for our triangular element with a scalar field solution.

The integral transformation shown in connection with the mass matrix is also executed here as

$$K_e = \det(J) \int_{s=-1}^1 \frac{1}{2} \int_{r=-1}^1 B(s, r)^T DB(s, r) \left(\frac{1}{4} - \frac{1}{4}s\right) dr ds.$$

However, due to the content of the  $B$  matrix and the presence of the elasticity matrix, this integral cannot be evaluated a priori; it has to be computed during the solution. For the sake of efficiency the integrals are numerically evaluated, usually by Gaussian quadrature.

The integrals are replaced by weighted sums and the integrand is evaluated at strategically selected points called the Gauss points:

$$K_e = \frac{1}{2} \det(J) \sum_{i=1}^n c_i \sum_{j=1}^n c_j B^T(s_i, r_j) DB(s_i, r_j) \left(\frac{1}{4} - \frac{1}{4}s_i\right).$$

**TABLE 12.2**  
Gauss points and weights

| $n$ | $i$ | $t_i$     | $c_i$    |
|-----|-----|-----------|----------|
| 1   | 1   | 0         | 2        |
| 2   | 1   | -0.577350 | 1        |
| 2   | 2   | 0.577350  | 1        |
| 3   | 1   | -0.774597 | 0.555556 |
| 3   | 2   | 0         | 0.888889 |
| 3   | 3   | 0.774597  | 0.555556 |
| 4   | 1   | -0.861136 | 0.347855 |
| 4   | 2   | -0.339981 | 0.652146 |
| 4   | 3   | 0.339981  | 0.652146 |
| 4   | 4   | 0.861136  | 0.347855 |

Table 12.2 shows the  $s_i = t_i, r_j = t_j$  Gauss point locations and corresponding  $c_i$  weights [2].

For very simple elements first order ( $n = 1$ ) integration suffices. For elements representing more difficult physics the second and third order formulae are used. Fourth order integration is sometimes used for quadratic or higher order elements.

Finally, the elemental load vector is also obtained by integrating with the shape function matrix as

$$F_e = \det(J) \int_{v=0}^1 \int_{w=0}^{1-v} N(v, w)^T \underline{f} dw dv.$$

Here  $\underline{f}$  is the vector of forces acting on the nodes of the element

$$\underline{f} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

We are now in a position to be able assemble the elemental matrices and obtain the solution of the problem on the complete domain.

#### 12.5.4 Element matrix assembly and solution

Since the element matrices have been developed in terms of the local  $v, w$  parametric coordinate system, before assembling they must be transformed

to the global  $x, y$  coordinate system common to all the elements. The coordinates of a point in the two systems are related as

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = T \begin{bmatrix} v \\ w \\ 1 \end{bmatrix}.$$

The transformation matrix is formed as

$$T = \begin{bmatrix} v_x & w_x & x_1 \\ v_y & w_y & y_1 \\ 0 & 0 & 1 \end{bmatrix},$$

where

$$\underline{v} = v_x \underline{i} + v_y \underline{j}$$

and

$$\underline{w} = w_x \underline{i} + w_y \underline{j}$$

are the vectors in the global system defining the local parametric coordinate axes. The point  $(x_1, y_1)$  defines the local element system's origin as was shown in Figure 12.2.

The same transformation is applicable to the solution values. The global solution values are related to the local elemental solution values by the same transformation matrix in the form of

$$\begin{bmatrix} u_{e,x} \\ u_{e,y} \\ 1 \end{bmatrix} = T \begin{bmatrix} u_{e,v} \\ u_{e,w} \\ 1 \end{bmatrix}.$$

Hence, the element solutions in the two systems are related as

$$u_e^g = T u_e$$

or

$$u_e = T^{-1} u_e^g.$$

The  $u_e^g$  notation refers to the element solution in the global coordinate system.

Let us now consider an elemental solution with the local element matrix and the local load vector as

$$K_e u_e = F_e.$$

The relationship between the load vector in local terms and its version in the global coordinate system is similar:

$$F_e^g = T F_e,$$

or

$$F_e = T^{-1} F_e^g.$$

Substituting into the elemental solution, we obtain

$$K_e T^{-1} u_e^g = T^{-1} F_e^g,$$

Pre-multiplying by  $T$  and exploiting the emerging identity matrix results in

$$T K_e T^{-1} u_e^g = F_e^g,$$

or

$$K_e^g u_e^g = F_e^g.$$

Here

$$K_e^g = T K_e T^{-1}$$

is the element matrix transformed to global coordinates. This transformation follows the element matrix generation and precedes the assembly process.

Finally, the  $K$  global stiffness matrix is assembled as

$$K = \sum_{e=1}^m L_{ge} K_e^g L_{ge}^T,$$

where  $L_{ge}$  is a Boolean matrix mapping the element local node numbers to the global node numbers. If, for example, the element is described by nodes 1, 2, and 3, then the terms in  $K_e^g$  contribute to the terms of the 1st, 2nd, and 3rd columns and rows of the assembled, global  $K$  matrix. Let us assume that another element is adjacent to the edge between nodes 2 and 3 whose third node is numbered 4. The second element's matrix terms will contribute to the 2nd, 3rd, and 4th columns and rows of the global matrix.

The individual element matrices are mapped to the global matrix that is of size 4 by 4, reflecting the presence of the 4 node points as

$$L_{ge} K_1^g L_{ge}^T = \begin{bmatrix} K_1(1,1) & K_1(1,2) & K_1(1,3) & 0 \\ K_1(2,1) & K_1(2,2) & K_1(2,3) & 0 \\ K_1(3,1) & K_1(3,2) & K_1(3,3) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$L_{ge} K_2^g L_{ge}^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & K_2(1,1) & K_2(1,2) & K_2(1,3) \\ 0 & K_2(2,1) & K_2(2,2) & K_2(2,3) \\ 0 & K_2(3,1) & K_2(3,2) & K_2(3,3) \end{bmatrix}.$$

Here the subscript is the element index  $e = 1, 2$  and the row, column indices in the parenthesis refer to the local element node numbers. The assembled

global finite element matrix is then

$$K = \begin{bmatrix} K_1(1,1) & K_1(1,2) & K_1(1,3) & 0 \\ K_1(2,1) & K_1(2,2) + K_2(1,1) & K_1(2,3) + K_2(1,2) & K_2(1,3) \\ K_1(3,1) & K_1(3,2) + K_2(2,1) & K_1(3,3) + K_2(2,2) & K_2(2,3) \\ 0 & K_2(3,1) & K_2(3,2) & K_2(3,3) \end{bmatrix}.$$

The assembled global load vector is similarly obtained:

$$F = \sum_{i=1}^2 L_{ge} F_{e,i}^g = \begin{bmatrix} F_1(1) \\ F_1(2) + F_2(1) \\ F_1(3) + F_2(2) \\ F_2(3) \end{bmatrix}.$$

The notation convention is the same as in the element matrix assembly.

The global solution is then obtained from the matrix equation

$$K\underline{u} = F,$$

where  $K$  is the global stiffness matrix and  $F$  is the global force vector. The global solution vector is

$$\underline{u} = K^{-1}F = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix},$$

and the solution inside of a particular element is

$$u(x, y) = N_1^i u_1^i + N_2^i u_2^i + N_3^i u_3^i.$$

The superscript  $i$  indicates that the shape functions and node point values are associated with the  $i$ -th element. For the first element in the above hypothetical two element model  $u_j^1 = u_j, j = 1, 2, 3$  and for the second element  $u_j^2 = u_{j+1}, j = 1, 2, 3$ .

Naturally the  $M$  matrix is similarly transformed and assembled as

$$M = \sum_{e=1}^m L_{ge} M_e^g L_{ge}^T.$$

This process is the same for any number of elements contained in the finite element discretization of the geometric model.

In conclusion, let us emphasize the fact that in all three mechanical engineering disciplines (structural elasticity, heat conduction, or fluid mechanics) we used the same computational technique to capture the behavior of the physical phenomenon over the geometric domain.

Furthermore, it is important to notice the finite element method's transcendence of the multiple engineering disciplines. For example, as demonstrated in [3], the governing equation in electrostatics is also Poisson's equation, albeit the participating terms have different physical meanings.

The applicability of a certain variational problem to unrelated disciplines is straightforward; one only needs to adhere to the differences in the physics. This fact makes the techniques demonstrated in this book extremely useful in a wide variety of engineering applications.





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## *Closing remarks*

Hopefully the engineers reading this book find its theoretical foundation clear and concise, and the analytic and computational examples enlightening. The focus of the applications on mechanical engineering reflects only the author's personal expertise and is not meant to imply any restriction of applicability to other engineering disciplines.

The book was designed to be a self-contained coverage of the topic specifically addressed to the practicing engineer or engineering student reader. Difficult discussions about spaces of functions and rigorous proofs were avoided to make the topic accessible with a standard engineering mathematics foundation.

With more numerical examples and exercises added, the book may be used as a textbook in the engineering curriculum. It could also be a practical alternative to the abstract approaches frequently used in teaching calculus of variations in advanced mathematics courses.

The reference list reflects the lack of recent attention to the topic of the calculus of variations. The original publications, however, are not only listed here for historical homage. The most readable, despite the archaic style, may be the oldest ones, albeit most of them were not specifically written for an engineering audience. This shortcoming is intended to be corrected by this book.

The reference list is also rather short, containing only those publications that directly influenced the writing of this book. They are all available, some of them in inexpensive reprints, and accessible in English.



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# *Index*

- Archimedes law, 154
- auxiliary function, 4
  
- B-spline, 128
- basis functions, 91
- Beltrami's formula, 10
- Bernoulli, 14
- Bessel equation, 84
- Bessel function, 84
- Bessel functions, 63
- bi-normal vector, 120
- bilinear interpolation, 192
- boundary conditions, 3
- brachistochrone, 10
- buckling problem, 71
- bulk modulus, 187
  
- CAD, 129, 140
- CAM, 43, 140
- catenary curve, 34
- catenoid, 44
- Chebyshev polynomials, 63
- chord, 138
- Christoffel symbols, 115
- compressible fluid, 187
- conservative systems, 145
- constrained splines, 133
- constrained variational problem, 29
- control points, 131
- curvature, 120
- curvature vector, 120
- cylindrical coordinates, 83
  
- d'Alembert operator, 107
- Delaunay triangles, 190
- Dirac function, 104
- direct method, 89
- Dirichlet boundary conditions, 58, 104
  
- eigenfunction, 61
- eigenvalue, 61
- Euler, 20
- Euler equation, 155, 187
- Euler's formula, 70, 75, 82
- Euler's method, 89
- Euler-Lagrange equation, 6
- Euler-Poisson equation, 50
- extremum of functional, 3
  
- Fermat's principle, 18
- finite element matrix generation, 195
- first fundamental quantities, 42, 112
- first variation, 7
- force potential, 145
- frequency, 160
- fundamental lemma, 6
  
- Galerkin's method, 96
- Gamma function, 84
- Gauss points, 198
- Gaussian quadrature, 198
- generalized displacements, 181
- geodesic curvature, 120
- geodesic curve, 111
- great circle, 114
- Green's function, 104
- Green's identity, 40, 45, 46, 60
- Green's theorem, 104, 168
  
- half-period, 160
- Hamilton's principle, 14, 144
- Hamiltonian, 148
- heat conduction, 185
- helix, 117
- Helmholtz equation, 187
- hemisphere, 114
- Hermite polynomials, 63
- Hooke's law, 169

- inverse problem, 57
- Jacobian matrix, 195
- Jordan curve, 41
- Kantorovich's method, 98
- kinetic energy, 14
- knot points, 129
- knot values, 129
- Kronecker delta, 106
- Kronecker property, 194
- Lagrange multiplier, 28, 54, 59, 61, 65
- Lagrange's equations, 147
- Lagrangian, 146, 147
- Laguerre polynomials, 63
- Lamé constants, 178
- Laplace operator, 57, 58
- Laplace transform, 69
- Laplace's equation, 80, 83
- Legendre equation, 65
- Legendre polynomials, 65
- Legendre test, 8
- Legendre's transformation, 143
- mean curvature, 40
- mean-value theorem, 7
- minimal surfaces, 41
- Minkowski space, 122
- natural frequency, 160
- natural spline, 126
- necessary conditions, 8
- Neumann boundary conditions, 52, 104
- Newton's law, 145
- non-uniform splines, 129
- normal curvature, 120
- normal mode, 160
- normal vector, 120
- polar coordinates, 80
- polynomial completeness, 194
- potential energy, 14
- pressure, 187
- principal curvature, 40
- principle of least action, 20
- principle of minimal potential energy, 31, 170
- Ritz method, 91
- second variation, 8
- shape functions, 191, 192, 195
- Snell's law, 19
- speed of sound, 187
- steady state solution, 184
- strain, 169
- strain energy, 182
- stress, 169
- strong extremum, 8
- Sturm-Liouville equations, 62
- surface of revolution, 43
- tangent constraints, 136
- tangent vector, 120
- tautochrone, 18
- temperature field, 185
- tessellation, 191
- uniform parameterization, 132
- uniform splines, 129
- velocity, 187
- weak extremum, 8
- weighted residuals, 96
- Young's modulus, 169

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## *List of Figures*

|      |  |     |
|------|--|-----|
| 1.1  | Alternative solutions example . . . . .                | 5   |
| 1.2  | Solution of the brachistochrone problem . . . . .      | 17  |
| 1.3  | Trajectory of particle . . . . .                       | 22  |
| 2.1  | Maximum area under curves . . . . .                    | 31  |
| 2.2  | The catenary curve . . . . .                           | 34  |
| 3.1  | Saddle surface . . . . .                               | 42  |
| 3.2  | Catenoid surface . . . . .                             | 44  |
| 5.1  | Eigensolutions of Legendre's equation . . . . .        | 66  |
| 7.1  | Accuracy of the Ritz solution . . . . .                | 94  |
| 7.2  | Domain with overlapping circular regions . . . . .     | 100 |
| 7.3  | Solution of Poisson's equation . . . . .               | 102 |
| 8.1  | Geodesic curve of a cylinder . . . . .                 | 118 |
| 9.1  | Natural spline approximation . . . . .                 | 128 |
| 9.2  | Smooth B-spline approximation . . . . .                | 133 |
| 9.3  | Point constrained B-spline . . . . .                   | 136 |
| 9.4  | Tangent constrained B-spline . . . . .                 | 139 |
| 11.1 | Normal modes of elastic string . . . . .               | 161 |
| 11.2 | Vibration of elastic membrane . . . . .                | 167 |
| 11.3 | Beam cross-section . . . . .                           | 170 |
| 11.4 | Beam profile under its weight . . . . .                | 174 |
| 12.1 | Delaunay triangularization . . . . .                   | 190 |
| 12.2 | Parametric coordinates of triangular element . . . . . | 193 |





---

## *List of Tables*

|      |  |     |
|------|--|-----|
| 5.1  | Eigenpairs of Legendre equation . . . . .          | 66  |
| 7.1  | Accuracy of Euler's method . . . . .               | 91  |
| 12.1 | Coordinate pairing of triangular element . . . . . | 194 |
| 12.2 | Gauss points and weights . . . . .                 | 199 |



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