Matrix Analysis

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Matrix Operations

- Matrix addition/subtraction
  - Matrices must be of same size.

- Matrix multiplication

m x n   q x p   m x p

$$\begin{bmatrix} a_{11} & a_{12} & . & a_{1n} \\ a_{21} & a_{22} & . & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & . & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & . & b_{1p} \\ b_{21} & b_{22} & . & b_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ b_{q1} & b_{q2} & . & b_{qp} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & . & c_{1p} \\ c_{21} & c_{22} & . & c_{2p} \\ \cdots & \cdots & c_{ij} & \cdots \\ c_{m1} & c_{m2} & . & c_{mp} \end{bmatrix}$$

Condition: n = q

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

$$AB \neq BA$$

Identity Matrix

$$AI = IA = A, \text{ where } I = \begin{bmatrix} 1 & 0 & . & 0 \\ 0 & 1 & . & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & . & 1 \end{bmatrix}$$

Matrix Transpose

$$A = \begin{bmatrix} a_{11} & a_{12} & . & a_{1n} \\ a_{21} & a_{22} & . & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & . & a_{mn} \end{bmatrix}, \; A^T = \begin{bmatrix} a_{11} & a_{21} & . & a_{m1} \\ a_{12} & a_{22} & . & a_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & . & a_{mn} \end{bmatrix}$$

Property: $(AB)^T = B^T A^T$

Symmetric Matrices

$$A = A^T \quad (a_{ij} = a_{ji})$$

Example:

$$\begin{bmatrix} 4 & 5 & -3 \\ 5 & 7 & 2 \\ -3 & 2 & 10 \end{bmatrix}$$

Determinants

2 x 2

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

3 x 3

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

n x n

$$det(A) = \sum_{j=1}^{m} (-1)^{j+k} a_{jk} det(A_{jk}), \text{ for any } k\text{: } 1 \le k \le m$$

Determinants

$$det(AB) = det(A)det(B)$$

$$det(A+B) \neq det(A) + det(B)$$

diagonal matrix:

$$\text{If } A = \begin{bmatrix} a_{11} & 0 & . & 0 \\ 0 & a_{22} & . & 0 \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ 0 & 0 & . & a_{nn} \end{bmatrix}, \text{ then } det(A) = \prod_{i=1}^{n} a_{ii}$$

Matrix Inverse

- The inverse A^{-1} of a matrix A has the property:

$$\boldsymbol{AA^{-1}=A^{-1}A=I}$$

- A^{-1} exists only if $det(A) \neq 0$

Matrix Inverse (cont'd)

- Properties of the inverse:

$$det(A^{-1}) = \frac{1}{det(A)}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(A^T)^{-1} = (A^{-1})^T$$

Pseudo-inverse

- The pseudo-inverse A^+ of a matrix A (could be non-square, e.g., m x n) is given by:

$$A^+ = (A^T A)^{-1} A^T$$

- It can be shown that:

$$A^+ A = I \quad \text{(provided that } (A^T A)^{-1} \text{ exists)}$$

Matrix trace

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

properties:

$$tr(A^T) = tr(A)$$

$$tr(A \pm B) = tr(A) \pm tr(B)$$

$$tr(AB) = tr(BA)$$

$$(\text{in general, } tr(AB) \neq tr(A)tr(B))$$

Rank of matrix

- Equal to the dimension of the largest square sub-matrix of A that has a non-zero determinant.

Example: $\begin{bmatrix} 4 & 5 & 2 & 14 \\ 3 & 9 & 6 & 21 \\ 8 & 10 & 7 & 28 \\ 1 & 2 & 9 & 5 \end{bmatrix}$ has rank 3

$$det(A) = 0, \text{ but } det\left(\begin{bmatrix} 4 & 5 & 2 \\ 3 & 9 & 6 \\ 8 & 10 & 7 \end{bmatrix}\right) = 63 \neq 0$$

Rank of matrix (cont'd)

- **Alternative definition:** the maximum number of linearly independent columns (or rows) of A.

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0 \quad \Longrightarrow \quad c_1 = c_2 = \cdots = c_k = 0$$

Example:

$$1\begin{bmatrix} 4 \\ 3 \\ 8 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 9 \\ 10 \\ 2 \end{bmatrix} + 0\begin{bmatrix} 2 \\ 6 \\ 7 \\ 9 \end{bmatrix} - 1\begin{bmatrix} 14 \\ 21 \\ 28 \\ 5 \end{bmatrix} = 0$$

Therefore, rank is not 4 !

Rank and singular matrices

If A is nxn, $rank(A) = n$ iff A is nonsingular (i.e., invertible).

If A is nxn, $rank(A) = n$ iff $det(A) \neq 0$ (**full rank**).

If A is nxn, $rank(A) < n$ iff A is singular

Orthogonal matrices

- Notation:

$$A = \begin{bmatrix} a_{11} & a_{12} & . & a_{1n} \\ a_{21} & a_{22} & . & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & . & a_{mn} \end{bmatrix},$$

$$\begin{aligned} u_1^T &= [a_{11}\ a_{12}\ \cdots\ a_{1n}] \\ u_2^T &= [a_{21}\ a_{22}\ \cdots\ a_{2n}] \\ &\dots \\ u_m^T &= [a_{m1}\ a_{m2}\ \cdots\ a_{mn}] \end{aligned}$$

$$A = \begin{bmatrix} u_1^T \\ u_2^T \\ \dots \\ u_m^T \end{bmatrix}$$

- A is orthogonal if:

$$u_j . u_k = 0, \text{ for every } j \neq k \ (u_j \text{ is perpendicular to } u_k)$$

Example:

$$\begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix}$$

Orthonormal matrices

- A is orthonormal if:

(1) $u_k . u_k = 1$ or $\|u_k\| = 1$, for every k

(2) $u_j . u_k = 0$, for every $j \neq k$ (u_j is perpendicular to u_k)

- Note that if A is orthonormal, it easy to find its inverse:

$$AA^T = A^T A = I \quad \text{(i.e., } A^{-1} = A^T\text{)}$$

Property: $\|Av\| = \|v\|$ (does not change the magnitude of v)

Eigenvalues and Eigenvectors

- The vector **v** is an eigenvector of matrix *A* and **λ** is an eigenvalue of *A* if:

$$Av = \lambda v$$ (assume non-zero v)

- **Interpretation:** the linear transformation implied by *A* cannot change the direction of the eigenvectors v, only their magnitude.

Computing λ and v

- To find the eigenvalues λ of a matrix *A*, find the roots of the *characteristic polynomial*:

$$det(A - \lambda I) = 0$$

$$Av = \lambda v$$

Example:

$$A = \begin{bmatrix} 5 & -2 \\ 6 & -2 \end{bmatrix}$$

$$det\left(\begin{bmatrix} 5-\lambda & -2 \\ 6 & -2-\lambda \end{bmatrix}\right) = 0 \text{ or } \lambda^2 - 3\lambda + 2 = 0 \text{ or } \lambda_1 = 1,\ \lambda_2 = 2$$

$$v_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix},\ v_2 = \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

Properties

- Eigenvalues and eigenvectors are only defined for square matrices (i.e., $m = n$)
- Eigenvectors are not unique (e.g., if v is an eigenvector, so is kv)

$$Av = \lambda v$$

- Suppose $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A, then:

(1) $\sum_i \lambda_i = tr(A)$

(2) $\prod_i \lambda_i = det(A)$

(3) if $\lambda = 0$ is an eigenvalue, then the matrix is not invertible

Matrix diagonalization

- Given A, find P such that $P^{-1}AP$ is diagonal (i.e., P diagonalizes A)

- Take $P = [v_1 \; v_2 \; . \; . \; . \; v_n]$, where $v_1, v_2, . \; . \; . \; v_n$ are the eigenvectors of A:

$$Av = \lambda v \;\Rightarrow\; AP = P\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ . & . & . \\ . & . & . \\ . & . & . \\ 0 & 0 & \lambda_n \end{bmatrix} \quad \text{or} \quad P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ . & . & . \\ . & . & . \\ . & . & . \\ 0 & 0 & \lambda_n \end{bmatrix}$$

Matrix diagonalization (cont'd)

Example:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\lambda_1 = 0,\ \lambda_2 = 2,\ v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},\ P^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Are all n x n matrices diagonalizable?

- Only if P^{-1} exists (i.e., *A* must have *n* linearly independent eigenvectors, that is, rank(A)=n)

- If *A* has *n* distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$, then the corresponding eigevectors $v_1, v_2, . . . v_n$ form a basis:

  (1) linearly independent

  (2) span R^n

Diagonalization → Decomposition

- Let us assume that *A* is diagonalizable, then:

$$P^{-1}AP = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ . & . & . \\ . & . & . \\ . & . & . \\ 0 & 0 & \lambda_n \end{bmatrix} \longrightarrow A = P \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ . & . & . \\ . & . & . \\ . & . & . \\ 0 & 0 & \lambda_n \end{bmatrix} P^{-1}$$

Decomposition: symmetric matrices

- The eigenvalues of symmetric matrices are all real.

- The eigenvectors corresponding to distinct eigenvalues are orthogonal.

$$A = P\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 0 & 0 & \lambda_n \end{bmatrix} P^{-1}$$

$P^{-1}=P^T$

→

$$A=PDP^T=\sum_{i=1}^{n} \lambda_i v_i v_i^T$$