Householder Transformation

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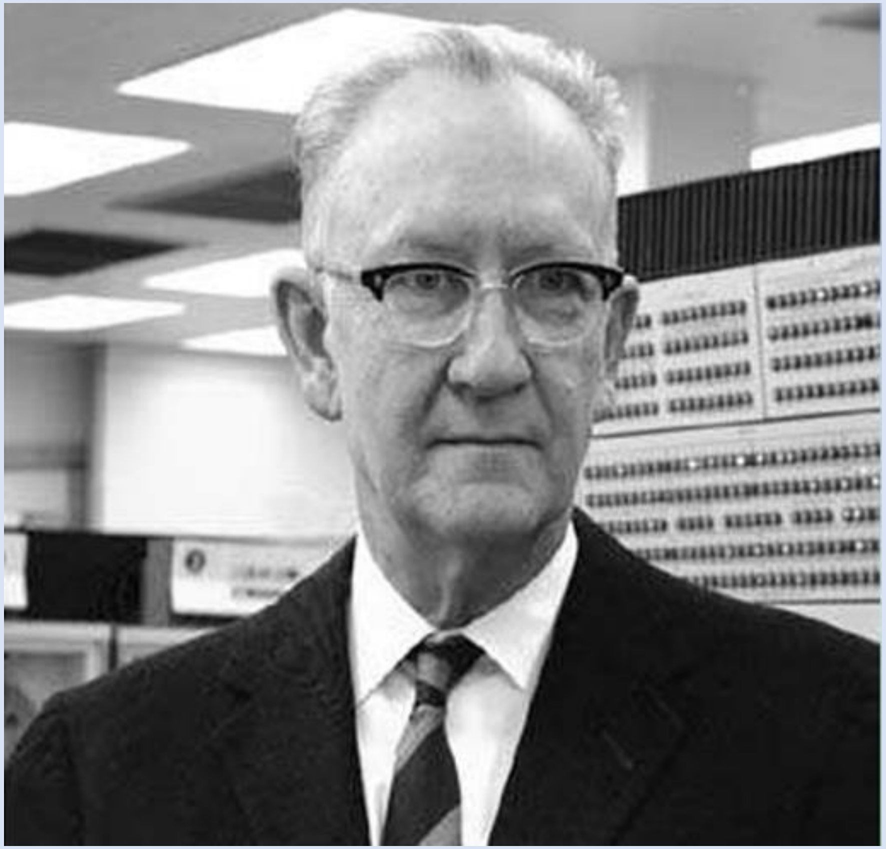
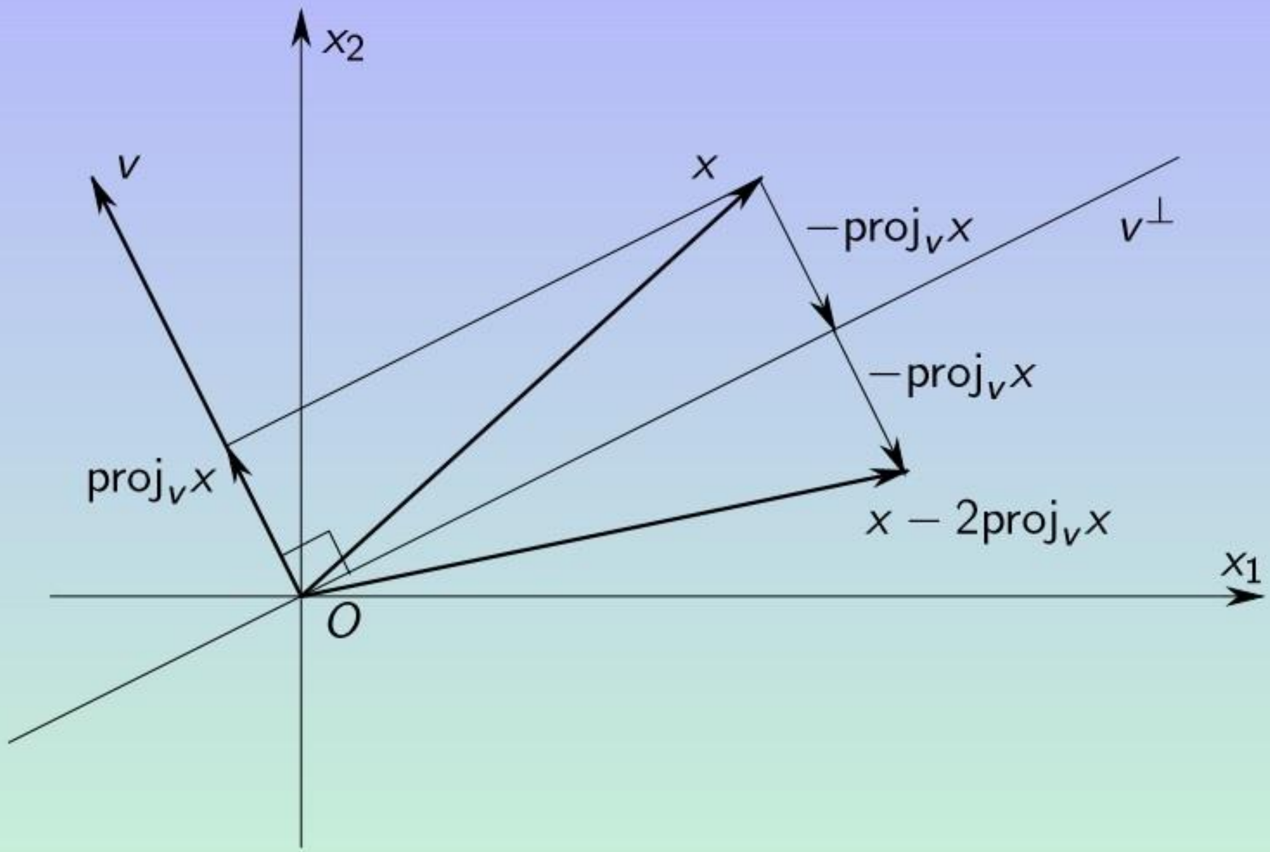
Householder Transformations

Alston Scott Householder. Born: 5 May 1904 in Rockford, Illinois, USA. Died: 4 July 1993 in Malibu, California, USA.



Householder Transformations



$$\text{proj}_v x = \left(\frac{x \cdot v}{v \cdot v}\right) v.$$

Householder Transformations

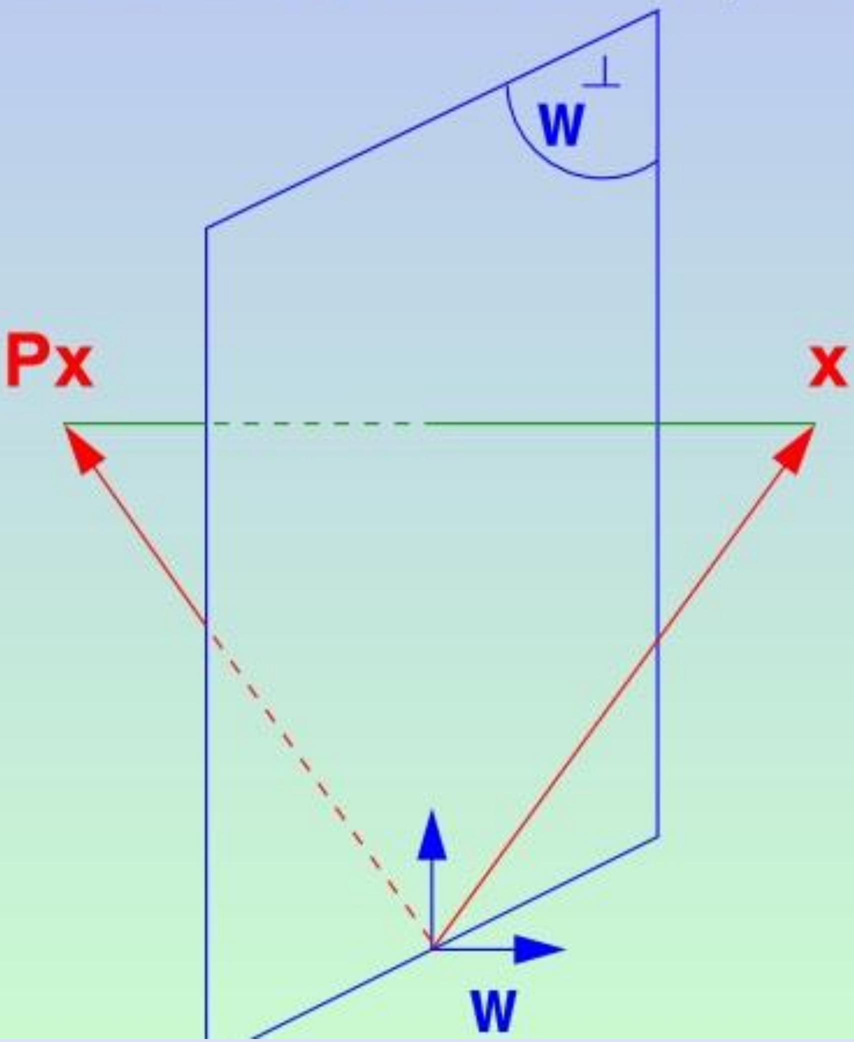
$$
\begin{aligned}
x - 2\mathrm{proj}_v x &= x - 2\left(\frac{x \cdot v}{v \cdot v}\right) v = x - 2\left(\frac{v \cdot x}{v \cdot v}\right) v \\
&= x - \frac{2}{v^T v}(v^T x)v = x - \frac{2}{v^T v} v(v^T x) \\
&= x - \frac{2}{v^T v}(v\,v^T)x = \underbrace{\left(I - \frac{2}{v^T v}(v\,v^T)\right)}_{P} x \\
&= \left(I - \frac{2}{\|v\|_2^2}(v\,v^T)\right) x = \left(I - 2\frac{v}{\|v\|_2}\frac{v^T}{\|v\|_2}\right) x \\
&= \underbrace{\left(I - 2w\,w^T\right)}_{P} x = Px
\end{aligned}
$$

where $w = v/\|v\|_2$ is a unit vector in 2-norm.

Householder reflectors are matrices of the form

$$P = I - 2w\,w^T,$$

where w is a unit vector (a vector of 2-norm unity).



Geometrically, Px represents a mirror image of x with respect to the hyperplane $\mathrm{span}\{w\}^\perp$.

Householder Transformations

Example:

$$v = \begin{bmatrix} 9 \\ 1 \\ 5 \\ 1 \end{bmatrix} \qquad P = I - 2\frac{vv^T}{v^Tv} = \begin{bmatrix} -27 & -9 & -45 & -9 \\ -9 & 53 & -5 & -1 \\ -45 & -5 & 29 & -5 \\ -9 & -1 & -5 & 53 \end{bmatrix}$$

$$x = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix} \qquad Px = \begin{bmatrix} -6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

DEF:

$Let\ v \in R^n$ be nonzero $\qquad P = I - \frac{2}{v^Tv} vv^T$

is called **Householder matrix** (Reflection, Transformation)

v is called Householde r vector

Problem 1: Given a vector $x \neq 0$, find w such that the Householder reflection will zero out all but the first entry of x, i.e.

$$Px = (I - 2w\,w^T)x = [\alpha, 0, 0, \cdots, 0] = \alpha e_1,$$

where α is a (free) scalar.
Writing $(I - \beta v\,v^T)x = \alpha e_1$ yields

$$\beta(v^T x)\ v = x - \alpha e_1 \quad \rightarrow \quad v = \frac{1}{\beta(v^T x)}(x - \alpha e_1)$$

Desired v is a multiple of $x - \alpha e_1$, i.e., we can take

$$v = x - \alpha e_1$$

To determine α we just recall that

$$\|Px\| = \|(I - 2w\,w^T)x\|_2 = \|x\|_2 = \|\alpha e_1\|.$$

As a result: $|\alpha| = \|x\|_2$, or

$$\alpha = \pm\|x\|_2$$

Householder Transformations

DEF: Let $v \in R^n$ be nonzero $P = I - \frac{2}{v^T v} v v^T$

is called **Householder matrix** (Reflection, Transformation)

v is called Householder vector

Example

$$v = \begin{bmatrix} 9 \\ 1 \\ 5 \\ 1 \end{bmatrix} \quad x = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix} \quad Px = \begin{bmatrix} -6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Rem:

1) They are rank-1 modifications of the identity
2) They are symmetric and orthogonal
3) They can be used to zero selected components of a vector

$0 \neq x \in R^n$ and want $Px \in span\{e_1\}$

$v = x + \alpha e_1$ where $\alpha = \pm \|x\|_2$

Example

$$x = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix} \quad v = \begin{bmatrix} 9 \\ 1 \\ 5 \\ 1 \end{bmatrix} \quad Px = \begin{bmatrix} -6 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example

$$x = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 3 \\ 1 \end{bmatrix} \quad v = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}$$

QR factorization

$$A = QR$$

We begin with a QR factorization method that utilizes Householder transformations.

QR factorization

$$A=\begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & x & x & x \end{bmatrix} \qquad H_1A=\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{bmatrix} \qquad H_2=\begin{bmatrix} 1 & \\ & \hat{H}_2 \end{bmatrix}$$

$$H_2H_1A=\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \\ 0 & 0 & x & x \end{bmatrix}$$

$$H_k=\begin{bmatrix} I_{k-1\times k-1} & 0 \\ 0 & \hat{H}_k \end{bmatrix}$$

$$H_nH_{n-1}\cdots H_1A=\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Upper triangular

QR factorization

$$H_n H_{n-1} \cdots H_1 A = \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Upper triangular

$$H_n H_{n-1} \cdots H_1 A = R$$

$$A = \underbrace{H_1^{-1} \cdots H_{n-1}^{-1} H_n^{-1}}_{Q} R$$

$$A = QR$$

$$Q = H_1 \cdots H_n$$

Remark:

H_k orthogonal $\Rightarrow \mathrm{H}_{\mathrm{k}}^{-1}$

Product of orthogonal matrices is an orthogonal

Householder Transformations

DEF: $Let\ v \in R^n$ be nonzero $$P = I - \frac{2}{v^T v} v v^T$$

is called **Householder matrix** (Reflection, Transformation)

v is called Householde r vector

Example (left multiplication)

$$v = \begin{bmatrix} 9 \\ 1 \\ 5 \\ 1 \end{bmatrix} \quad x = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 1 \end{bmatrix} \quad Px = \begin{bmatrix} -6 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad x^T P =$$

Computing Householder

$0 \neq x \in R^n$ and want $Px \in span\{e_1\}$

$v = x + \alpha e_1$ where $\alpha = \pm \|x\|_2$

Choice of sign:

$$v_1 = x_1 \pm \|x\|_2$$

$$v_1 = x_1 - \|x\|_2$$

It is dangerous if x is close to a positive multiple of e_1 because sever cancellation would occur.

Solution:

$$v_1 = x_1 - \| x \|_2 = \frac{x_1^2 - \| x \|_2^2}{x_1 + \| x \|_2} = \frac{-(x_2^2 + \cdots + x_n^2)}{x_1 + \| x \|_2}$$

$case: x_1 > 0$

Householder Transformations

Matlab

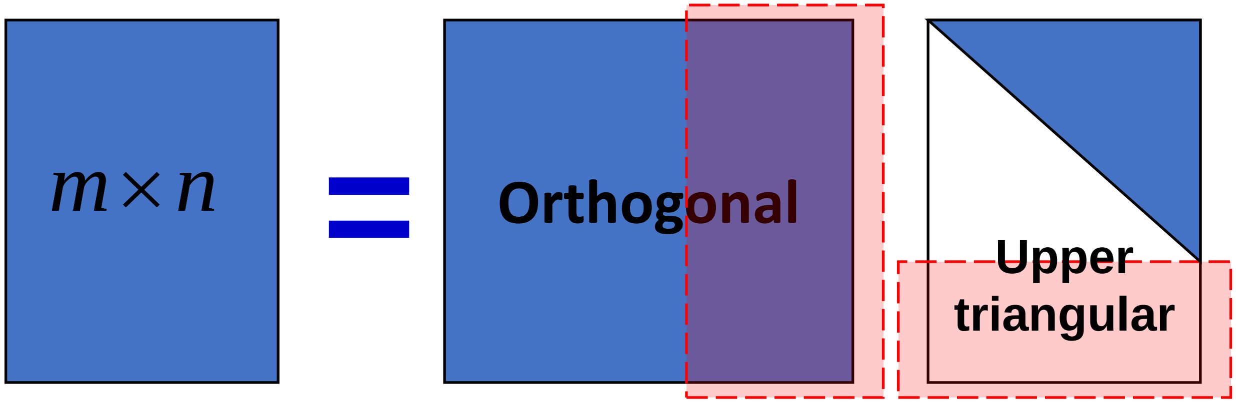
[Q,R] = qr(A), where A is m-by-n, produces an m-by-n upper triangular matrix R and an m-by-m unitary matrix Q so that A = Q*R.

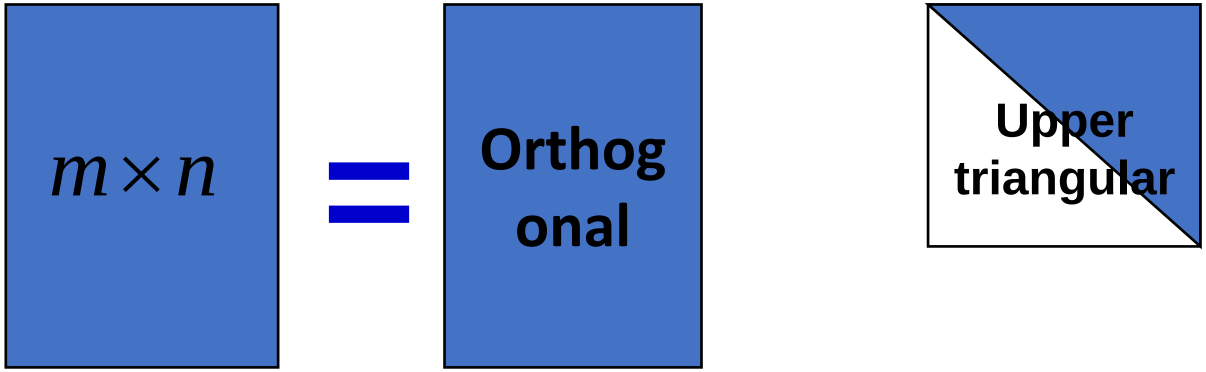
Algorithm 1 Householder Vector. Given $x \in \mathbb{R}^n$, this function computes $v \in \mathbb{R}^n$ with $v(1) = 1$ and $\beta \in \mathbb{R}$ such that $P = I_n - \beta v v^T$ is orthogonal and $Px = \|x\|_2 e_1$.

```
function [v, β] = HouseholderVector(x)
n = length(x)
σ = x(2 : n)^T x(2 : n),  v = [ 1 ; x(2 : n) ]
if σ = 0 then
    β = 0
else
    μ = sqrt(x(1)^2 + σ)
    if x(1) ≤ 0 then
        v(1) = x(1) − μ
    else
        v(1) = −σ/(x(1) + μ)
    end
    β = 2v(1)^2/(σ + v(1)^2),  v = v/v(1)
end
```

Reduced QR factorization

$$A = QR$$





$$A = \hat{Q}\hat{R}$$

Gram-Schmedit

Gram-schmedit Orthogonalization

$\{a_1, a_2, a_3\}$ lin indep Find $\{q_1, q_2, q_3\}$ orthonormal

such that $span\{a_1, a_2, a_3\} = span\{q_1, q_2, q_3\}$

$$span\{a_1, a_2\} = span\{q_1, q_2\}$$

$$span\{a_1\} = span\{q_1\}$$

$$v_1 = a_1$$

$$r_{11} = \|v_1\|$$

$$q_1 = \frac{v_1}{r_{11}}$$

$$r_{12} = q_1^T a_2$$

$$v_2 = a_2 - r_{12} q_1$$

$$r_{22} = \|v_2\|$$

$$q_2 = \frac{v_2}{r_{22}}$$

$$r_{13} = q_1^T a_3 \qquad r_{23} = q_2^T a_3$$

$$v_3 = a_3 - r_{13} q_1 - r_{23} q_2$$

$$r_{33} = \|v_3\|$$

$$q_3 = \frac{v_3}{r_{33}}$$

Gram-Schmedit

$$q_1 = \frac{a_1}{r_{11}},$$

$$q_2 = \frac{a_2 - r_{12}q_1}{r_{22}},$$

$$q_3 = \frac{a_3 - r_{13}q_1 - r_{23}q_2}{r_{33}},$$

$$\vdots$$

$$q_n = \frac{a_n - \sum_{i=1}^{n-1} r_{in}q_i}{r_{nn}}.$$

$$v_j = a_j - (q_1^T a_j)q_1 - (q_2^T a_j)q_2 - \cdots - (q_{j-1}^T a_j)q_{j-1}$$

$$r_{ij} = q_i^T a_j, \quad i \neq j$$

$$r_{jj} = \|v_j\|$$

$$\left[\begin{array}{c|c|c|c} a_1 & a_2 & \cdots & a_n \end{array}\right] = \left[\begin{array}{c|c|c|c} q_1 & q_2 & \cdots & q_n \end{array}\right] \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & & \vdots \\ & & \ddots & \\ & & & r_{nn} \end{bmatrix}$$

Gram-Schmedit

$$q_1 = \frac{a_1}{r_{11}},$$

$$q_2 = \frac{a_2 - r_{12}q_1}{r_{22}},$$

$$q_3 = \frac{a_3 - r_{13}q_1 - r_{23}q_2}{r_{33}},$$

$$v_j = a_j - (q_1^T a_j)q_1 - (q_2^T a_j)q_2 - \cdots - (q_{j-1}^T a_j)q_{j-1}$$

$$r_{ij} = q_i^T a_j, \quad i \neq j$$

$$r_{jj} = \|v_j\|$$

Example:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$q_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} / \sqrt{2}$$

$$q_2 = (a_2 - \tfrac{1}{\sqrt{2}} q_1)/(\tfrac{1}{\sqrt{2}}) = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

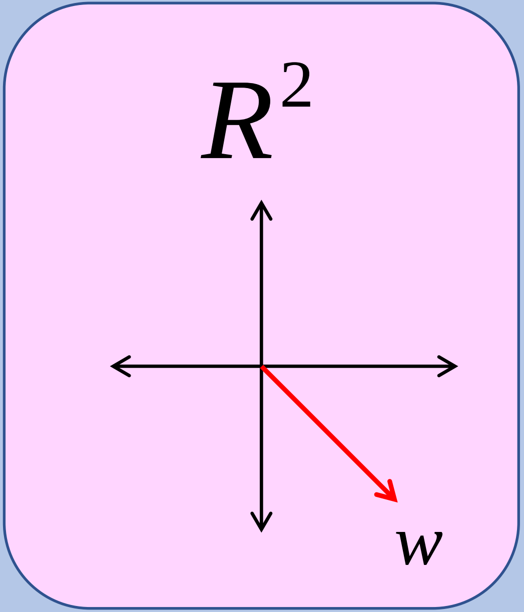
$$q_3 = (a_3 - \sqrt{2}q_1 - \sqrt{2}q_2)/(1) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \sqrt{2} \\ & \frac{1}{\sqrt{2}} & \sqrt{2} \\ & & 1 \end{bmatrix}$$

$$= QR$$

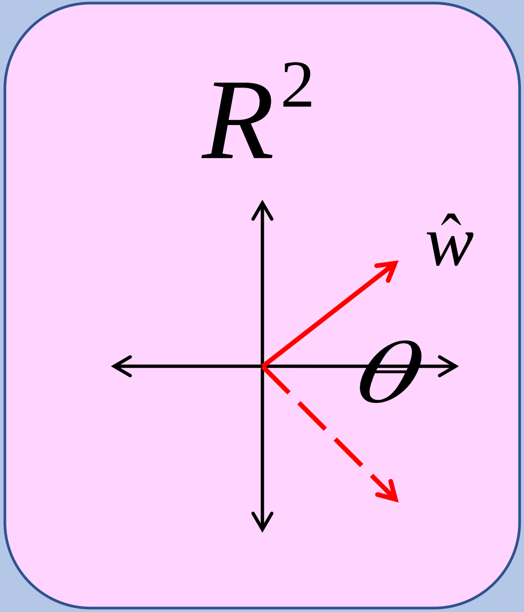
Givens Matrices



$$Q = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$\hat{w} = Q^T w$$

Rotation



Example:

$$Q = \begin{bmatrix} \cos(-60^\circ) & \sin(-60^\circ) \\ -\sin(-60^\circ) & \cos(-60^\circ) \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} \qquad Q^T x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Givens Rotations

<u>To zero a specific entry (not all as Householder)</u>

Givens Rotations are of this form:

$$G(i,k,\theta) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & -s & \cdots & c & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix} \begin{matrix} \\ \\ i \\ \\ k \\ \\ \\ \end{matrix}$$

$$\qquad\qquad\qquad\qquad\qquad i \qquad\qquad k$$

where $c = \cos(\theta)$ and $s = \sin(\theta)$

Givens Rotation are orthogonal

$$y = G(i,k,\theta)^T x$$

$$y_j = x_j, \quad j \neq i,k$$

$$y_i = cx_i - sx_k$$

$$y_k = sx_i + cx_k$$

We can force y_k to be zero by setting:

$$c = \frac{x_i}{\sqrt{x_i^2 + x_k^2}}$$

$$s = \frac{-x_k}{\sqrt{x_i^2 + x_k^2}}$$

Householder Transformations

Algorithm 5.1.3 Given scalars a and b, this function computes $c = \cos(\theta)$ and $s = \sin(\theta)$ so

$$\begin{bmatrix} c & s \\ -s & c \end{bmatrix}^T \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix}.$$

```
function: [c, s] = givens(a, b)
    if b = 0
        c = 1; s = 0
    else
        if |b| > |a|
            τ = −a/b; s = 1/√(1 + τ²); c = sτ
        else
            τ = −b/a; c = 1/√(1 + τ²); s = cτ
        end
    end
```

Example:

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \longrightarrow y = Q^T x = \begin{bmatrix} 1 \\ r \\ 3 \\ 0 \end{bmatrix}$$

$$\cos\theta = \frac{1}{\sqrt{5}} \qquad \sin\theta = \frac{-2}{\sqrt{5}}$$

$$G(2,4,\theta)^T x = \begin{bmatrix} 1 \\ \sqrt{20} \\ 3 \\ 0 \end{bmatrix}$$

Applying Givens Rotations

$$G(i, j, \theta)^T A$$

Just effects two rows of A

```
for j = 1:n
    τ1 = A(i, j)
    τ2 = A(k, j)
    A(1, j) = cτ1 − sτ2
    A(2, j) = sτ1 + cτ2
end
```

# of operations = 6n

Householder Transformations

Givens QR

$$\begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \end{bmatrix} \xrightarrow{(3,4)} \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ \times & \times & \times \\ 0 & \times & \times \end{bmatrix} \xrightarrow{(2,3)} \begin{bmatrix} \times & \times & \times \\ \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix} \xrightarrow{(1,2)}$$

$$\begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix} \xrightarrow{(3,4)} \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix} \xrightarrow{(2,3)} \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \\ 0 & 0 & \times \end{bmatrix} \xrightarrow{(3,4)} R$$

Householder Transformations

Theorem: (QR Decomposition)

If A is real m-by-n matrix, then there exist orthogonal matrix Q

such that $$A = QR$$ R upper triangular

Theorem: (QR Decomposition)

If A is real m-by-n matrix matrix of full rank, then A has a unique reduced QR factorization

$$A = \hat{Q}\hat{R}$$ with $r_{ii} > 0$

Householder Transformations

```
function [v]=house(x)
 v=x;
 v(1)=sign(x(1))*norm(x)+x(1);
```

```
function [Q,R]=myqr(A)

 [m,n]=size(A);
 for k=1:n
     x=A(k:m,k)
     [v]=house(x);
     A(k:m,k:n)= A(k:m,k:n) –( 2/v’*v) v*(v’**A(k:m,k:n));
 end
```

Householder Transformations

IMA Journal of Numerical Analysis (2010) **30**, 887–897

Householder triangularization of a quasimatrix

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A standard algorithm for computing the QR factorization of a matrix A is Householder triangularization. Here this idea is generalized to the situation in which A is a quasimatrix, that is, a 'matrix' whose 'columns' are functions defined on an interval $[a, b]$. Applications are mentioned to quasimatrix least squares fitting, singular value decomposition and determination of ranks, norms and condition numbers, and numerical illustrations are presented using the chebfun system.

Questions

Orthogonal Matrices:

1) Two class of orthogonal matrices (small modification from the identity)
   Householder - Givens any others

2) Can we think of Q such that Q(col1) = multiple of e1
   Q(col2) = multiple of e2