# The Behavior of L-graph Automata

1st Elham Raisi Sarbizhan and 2nd Mohammad Mehdi Zahedi

1st *Department of Mathematics*, *Kerman Graduate University of Advanced Technology*, Kerman, Iran, e.raisi@student.kgut.ac.ir

2nd *Department of Mathematics*, *Kerman Graduate University of Advanced Technology*, Kerman, Iran, zahedi_mm@kgut.ac.ir

*Abstract*— **This paper first introduces the L-graph automaton built on an L-graph using a zero-constraint set. Then, we study the behavior of the corresponding automaton when the L-graph is a path L-graph (cycle, complete, fully bipartite). These L-graph automata have some applications in various fields. One of them is the identification of drugs that have the most similar side effects. These concepts and applications have been illustrated with some examples.**

*Index Terms*— **L-graph, L-graph automaton, the behavior of the L-graph automaton**

## I. Introduction

Leonhard Euler first coined the term graph. Since then, graph theory has been used for the big problems of life Operations Research. The working group AIM minimum rank-special graphs introduced the concept of a zero-forcing process for bounding the minimum rank of graph $G$ [1]. After that, many authors have extended this notion [5]. For modeling natural events that are uncertain and vague, it was first modeled by Zadeh using a fuzzy set [19]. Therefore, these applications have been considered by many authors. In 1975, Kaufman used the concept of the fuzzy graph to solve real-world problems [6]. Rosenfeld, on the other hand, has many applications for fuzzy graphs in various fields [13]. Wee [17] and Santos [14] introduced fuzzy automata. Fuzzy finite automata have many applications. The residuated lattice was introduced by Morgan Ward and Robert P. Dilworth in 1939 [16]. Many researchers have considered this concept and used it in many different branches of science. Many authors, such as Ciric and his co-authors [4], Qiu [9] [10], and Tiwari and his co-authors [15], worked on automata theory based on the residuated lattice. Recently, the new definition of graph based on residuated lattice was presented by Zahedi and Raisi Sarbizhan, etc. [11] [12]. Just like fuzzy graphs, there are many applications that Mordeson and his colleagues have recently described in detail in their book [8]. In this paper, the notion of a graph built on a residuated lattice (L-graph) is explained. Moreover, an L-graph automaton is constructed on an L-graph by using zero forcing sets. It also has many applications in various fields. Therefore, we have described a number of these applications in this paper. We have also given some examples and theorems for clarification.

## II. Preliminaries

In this section, the basic notions of graph [18] [3], residuated lattice [16] [2], and L-fuzzy automaton [7] are explained.

*Definition 1:* [1]

- Color-change rule:
  If $G$ is a graph with each vertex colored either white or black, $u$ is a black vertex of $G$, and exactly one neighbor $v$ of $u$ is white, then change the color of $v$ to black.
- Given a coloring of $G$, the derived coloring is the result of applying the color-change rule until no more changes are possible.
- A zero forcing set for a graph $G$ is a subset of vertices $Z$ such that if initially the vertices in $Z$ are colored black and the remaining vertices are colored white, the derived coloring of $G$ is all black.
- $Z(G)$ is the minimum of $|Z|$ over all zero forcing sets $Z \subseteq V(G)$.

*Definition 2:* [16] A residuated lattice is an algebra $L = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ such that

1) $L = (L, \wedge, \vee, 0, 1)$ is a lattice (the corresponding order will be denoted by $\leq$) with the smallest element 0 and the greatest element 1,
2) $L = (L, \otimes, 1)$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $x \otimes 1 = x$ holds),
3) $x \otimes y \leq z$ if and only if $x \leq y \rightarrow z$ holds (adjointness condition).

*Proposition 1:* [2] Let $(L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ be a residuated lattice. Then the following properties hold:
$(R_1)$ $1 * x = x$, where $* \in \{\wedge, \otimes, \rightarrow\}$,
$(R_2)$ $x \otimes 0 = 0, 1' = 0, 0' = 1$,
$(R_3)$ $x \otimes y \leq x \wedge y \leq x, y$, and $y \leq (x \rightarrow y)$,
$(R_4)$ $(x \rightarrow y) \otimes x \leq y$,
$(R_5)$ $x \leq y$ implies $x * z \leq y * z$, where $* \in \wedge, \vee, \otimes$,
$(R_6)$ $z \otimes (x \wedge y) \leq (z \otimes x) \wedge (z \otimes y)$,
$(R_7)$ $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$.

*Definition 3:* [11] $G = (\alpha, \beta)$ is called an L-graph on $G^*$ if $\alpha : V \rightarrow L$ and $\beta : E \rightarrow L$ are functions, with $\beta(st) \leq \alpha(s) \otimes \alpha(t)$ for every $st \in E$. Besides, if $G^*$ is a path (cycle, bipartite, complete, complete bipartite) graph, then $G$ is called a path (cycle, bipartite, complete, complete bipartite) L-graph on $G^*$.

## III. The L-graph automaton

In this section, we introduce the notion of a related L-graph automaton construct on the L-graph. In this paper, we assume that $L$ is a residuated lattice and $G^*$ is a simple graph $(V, E)$. It proves the behavior of the associated automaton if the L-graph is a path (cycle, complete, fully bipartite) L-graph.

*Definition 4:* Let $G = (\alpha, \beta)$ be an L-graph on $G^*$ and let $Z(G)$ be a zero forcing set of $G^*$. Then an L-graph automaton

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$A(Z(G))$ is defined with respect to $G$ is defined by five-tuple $(Q, X, \mu, F, \sigma)$, where;

(i) $Q = V$ is the finite nonempty set of states,

(ii) $X = \{f, n\}$ is the set of letters of alphabet,

(iii) $\mu : V \times X \times V \longrightarrow L$ is the transition function;
$$\mu(q_i, u, q_j) = \begin{cases} \beta(q_i q_j) & if\ \ u = f, \\ 1 & if\ \ u = n, \end{cases}$$

(iv) $F$ is the set of final states, where $q \in F$ if and only if $q$ does not force any enforce vertices,

(v) $\sigma : V \longrightarrow L$ is the initial distribution;
$$\sigma(q_i) = \begin{cases} 1 & if\ \ q_i \in Z(G), \\ 0 & otherwise, \end{cases} \quad \text{for every } q_i \in Q.$$

Moreover, a new set $Z(A(Z(G))) = \{q_i \in Q | \sigma(q_i) = 1\} = \{q_i \in Q | q_i \in Z(G)\}$ has been defined. Also, a response function of $A(Z(G))$ is defined. A k-behavior of $A(Z(G))$ with threshold $c$ is set

$$B_k(A(Z(G)), c) = \{x \in X^* | \bigvee_{q \in F} r_\mu(x, q) > c \text{ and } |x| \leq k\}.$$

*Definition 5:* Let $A_1 = (Q_1, X, \mu_1, F_1, \sigma_1)$ and $A_2 = (Q_2, X, \mu_2, F_2, \sigma_2)$ be L-graph automata. Then they are equivalent if and only if, $B_k(A_1, c) = B_k(A_2, c)$, for every $k = 0, 1, 2, \ldots$ and for every $c \in L \setminus 1$.

*Example 1:* Suppose $L = ([0, 1], \wedge, \vee, \otimes, \rightarrow, 0, 1)$, where $a \otimes b = ab$, and $a \rightarrow b = \begin{cases} 1 & if\ \ a \leq b, \\ \frac{b}{a} & if\ \ a > b, \end{cases}$ and the L-graph $G = (\alpha, \beta)$ on $G^* = (V, E)$, as in Fig. 1, where $V = \{q_1, q_2, q_3, q_4, q_5, q_6\}$, $E = \{q_1q_2, q_1q_6, q_2q_3, q_2q_6, q_3q_4, q_4q_5, q_4q_6, q_5q_6\}$, $\beta(q_iq_j) = \alpha(q_i) \otimes \alpha(q_j)$, $\alpha(q_1) = 0.1$, $\alpha(q_2) = 0.8$, $\alpha(q_3) = 0.6$, $\alpha(q_4) = 0.9$, $\alpha(q_5) = 0.3$, $\alpha(q_6) = 0.5$, $\beta(q_1q_2) = 0.08$, $\beta(q_1q_6) = 0.05$, $\beta(q_2q_3) = 0.48$, $\beta(q_2q_6) = 0.4$, $\beta(q_3q_4) = 0.54$, $\beta(q_4q_6) = 0.45$, $\beta(q_4q_5) = 0.18$, $\beta(q_5q_6) = 0.15$, and $Z(G) = \{q_1, q_2\}$. Then, corresponding L-graph automaton $A(Z(G)) = (Q, X, \mu, F, \sigma)$ can be determined with respect to $Z(G)$, as in Fig. 2, where $Q = V$, $X = \{n, f\}$, $F = \{q_5\}$, $\mu(q_1, n, q_2) = \mu(q_2, n, q_1) = 1$, $\mu(q_4, n, q_6) = \mu(q_6, n, q_4) = 1$, $\mu(q_2, n, q_6) = \mu(q_6, n, q_2) = 1$, $\mu(q_1, f, q_6) = 0.05$, $\mu(q_2, f, q_3) = 0.48$, $\mu(q_3, f, q_4) = 0.54$, $\mu(q_4, f, q_5) = 0.18$, $\mu(q_6, f, q_5) = 0.15$, $\sigma(q_1) = \sigma(q_2) = 1$, and $\sigma(q_3) = \sigma(q_4) = \sigma(q_5) = \sigma(q_6) = 0$. So,

$$\begin{aligned} r_\mu(f^3, q) &= \sigma(q_2) \otimes \mu(q_2, f, q_3) \otimes \mu(q_3, f, q_4) \otimes \mu(q_4, f, q_5)) \\ &= 0.48 \otimes 0.54 \otimes 0.18 \\ &= 0.046656. \end{aligned}$$

Also,

$$\begin{aligned} r_\mu(f^2, q) &= \sigma(q_1) \otimes \mu(q_1, f, q_6) \otimes \mu(q_6, f, q_5) \\ &= 0.05 \otimes 0.15 \\ &= 0.075. \end{aligned}$$

*Theorem 1:* Let $G = (\alpha, \beta)$ be an L-graph on $G^* = (V, E)$, and let $\beta = \bigvee_{qq' \in E} \beta(qq')$ and $\beta' = \bigwedge_{qq' \in E} \beta(qq')$, Then,

(i) If $G$ is a path L-graph with $n$ vertices, then $A(Z(G)) = (Q, X, \mu, F, \sigma)$ is a related L-graph automaton such that
$$r_\mu(f^{n-1}, q) \leq \otimes_{q \in V} \alpha(q),$$
for every zero forcing set $Z(G)$.

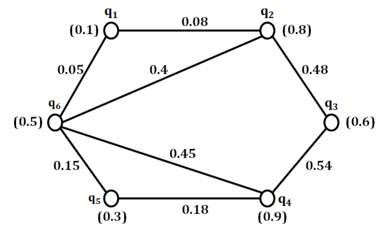


Fig. 1. The L-graph $G$.

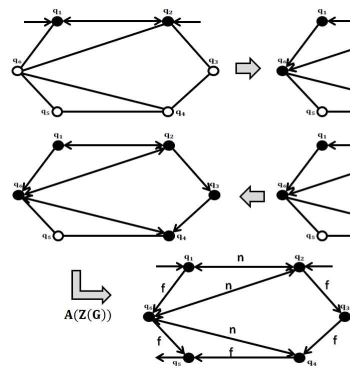


Fig. 2. The L-graph automaton $A(Z(G))$.

(ii) If $G$ is a cycle L-graph with $2k$ vertices, then $A(Z(G)) = (Q, X, \mu, F, \sigma)$ is an associated L-graph automaton such that
$$\otimes_{i=1,\ldots,k-1} \beta' \leq \bigvee_{q \in F} r_\mu(n^* f^{k-1} n^*, q) \leq \otimes_{i=1,\ldots,k-1} \beta,$$
for every zero forcing set $Z(G)$.

(iii) If $G$ is a cyclic L-graph with $2k + 1$ vertices, then $A(Z(G)) = (Q, X, \mu, F, \sigma)$ is an associated L-graph automaton such that
$$\otimes_{i=1,\ldots,k} \beta' \leq r_\mu(n^* f^k, q) \leq \otimes_{i=1,\ldots,k} \beta,$$
for every zero forcing set $Z(G)$.

(iv) If $G$ is a complete L-graph with $n$ vertices, then $A(Z(G)) = (Q, X, \mu, F, \sigma)$ is an associated L-graph automaton such that
$$\beta' \leq r_\mu(f, q) \leq \beta,$$
for any zero forcing set $Z(G)$.

(v) If $G$ is a complete bipartite L-graph with $n$ vertices, then $A(Z(G)) = (Q, X, \mu, F, \sigma)$ is an associated L-graph automaton such that
$$\beta' \leq \bigvee_{q \in F} r_\mu(f, q) \leq \beta,$$
for any zero forcing set $Z(G)$.

*Proof:* (i) $G$ has two distinct zero forcing sets $Z_1(G)$ and $Z_2(G)$, but these related L-graph automata are equivalent and isomorphic. Thus, consider $Z(G) = \{q_1\}$. Thus

$$\begin{aligned} r_\mu(f^{n-1}, q) &= r_\mu(f^{n-1}, q_n) \\ &= \sigma(q_1) \otimes \mu^*(q_1, f^n, q_n) \\ &= \mu(q_1, f, q_2) \otimes \ldots \otimes \mu(q_{n-1}, f, q_n) \\ &\leq \otimes_{q \in V} \alpha(q), \ by \ Proposition \ 1(R_5). \end{aligned}$$

(ii) Since these zero forcing sets are one of the two vertices adjacent to this L-graph, there are $k-1$ vertices in each chain that constraining the other vertex. Suppose $Z(G) = \{q_1, q_2\}$, and $F = \{k+1, k+2\}$. Thus,

$$\begin{aligned} \bigvee_{q \in F} r_\mu(f^{k-1}, q) &= \sigma(q_1) \otimes \mu^*(q_1, f^{k-1}, q_{k+2}) \\ &\vee \ \sigma(q_2) \otimes \mu^*(q_1, f^{k-1}, q_{k+1}) \\ &= \mu^*(q_1, f^{k-1}, q_{k+2}) \vee \mu^*(q_1, f^{k-1}, q_{k+1}) \\ &\leq \otimes_{i=1,\ldots,k-1} \beta, \end{aligned}$$

and

$$\begin{aligned} \bigvee_{q \in F} r_\mu(f^{k-1}, q) &= \mu^*(q_1, f^{k-1}, q_{k+2}) \vee \mu^*(q_1, f^{k-1}, q_{k+1}) \\ &\geq \otimes_{i=1,\ldots,k-1} \beta'. \end{aligned}$$

Moreover, for any $n^* f^{k-1} n^*$, it is similar to the above with some modifications.

(iii) The proof with some terms is similar to the above.

(iv) We know that every zero forcing set has $n-1$ vertices. $Z(G) = \{q_1, q_2, \ldots, q_{n-1}\}$, and $F = \{q_n\}$. Thus,

$$\begin{aligned} r_\mu(f, q) &= \mu(q_1, f, q_n) \vee \ldots \vee \mu(q_{n-1}, f, q_n) \\ &\leq \beta, \end{aligned}$$

and

$$\begin{aligned} r_\mu(f, q) &= \mu(q_1, f, q_n) \vee \ldots \vee \mu(q_{n-1}, f, q_n) \\ &\geq \beta'. \end{aligned}$$

(iv) The proof is similar to the proof above but with some modifications. ■

*Example 2:* Consider $L$ in Example 1 and a complete bipartite L-graph $G = (\alpha, \beta)$ on $G^*$, as in Fig. 3, where $V = \{q_1, q_2, q_3, q_4, q_5\}$, $E = \{q_1q_3, q_1q_4, q_1q_5, q_2q_3, q_2q_4, q_2q_5\}$, $\alpha(q_1) = 0.4$, $\alpha(q_2) = 0.6$, $\alpha(q_3) = 0.3$, $\alpha(q_4) = 0.9$, $\alpha(q_5) = 0.8$, $\beta(q_1q_3) = 0.1$, $\beta(q_1q_4) = 0.3$, $\beta(q_1q_5) = 0.2$, $\beta(q_2q_3) = 0.1$, $\beta(q_2q_4) = 0.5$, $\beta(q_2q_5) = 0.4$, and $Z(G) = \{q_1, q_3, q_4\}$. Therefore, $A(Z(G)) = (Q, X, \mu, F, \sigma)$ is an associated L-graph automaton, shown in Fig. 3, where $Q = V$, $F = \{q_2, q_5\}$, $\mu(q_1, n, q_3) = \mu(q_3, n, q_1) = 1$, $\mu(q_1, n, q_4) = \mu(q_4, n, q_1) = 1$, $\mu(q_1, f, q_5) = 0.2$, $\mu(q_3, f, q_2) = 0.1$, $\mu(q_4, f, q_2) = 0.5$, $\mu(q_2, n, q_5) = \mu(q_5, n, q_2) = 1$, $\sigma(q_1) = \sigma(q_3) = \sigma(q_4) = 1$, and $\sigma(q_2) = \sigma(q_5) = 0$. So,

$$\begin{aligned} r_\mu(f, q) &= (\sigma(q_1) \otimes \mu(q_1, f, q_5)) \vee (\sigma(q_3) \otimes \mu(q_3, f, q_2)) \\ &\vee \ (\sigma(q_4) \otimes \mu(q_4, f, q_2)) \\ &= 0.2 \vee 0.1 \vee 0.5. \end{aligned}$$

Also, for every $x = n^* f n^*$, $r_\mu(x, q) = 0.5$.

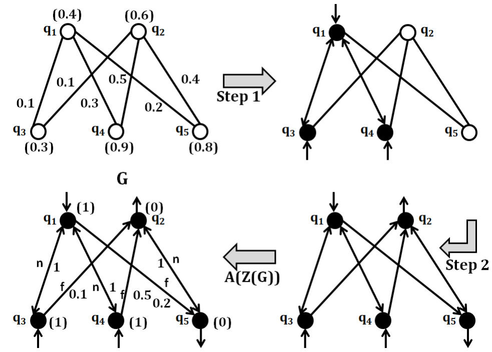


Fig. 3. The L-graph $G$.

## IV. Some applications of L-graph automata

In this section, we discuss the applications of L-graph automata and illustrate them with an example.

*Application 1:* The related L-graph automata have some applications. For example;

a: Assume $n$ drugs with $m$ side effects and $L = (P(X), \cap, \cup, \otimes, \rightarrow, \emptyset, X)$, where $X = \{a_1, a_2, \ldots, a_m\}$ such that $a_i$ per $i$ is labeled for $m$ side effects of these drugs and $A \otimes B = A \cap B$, and $A \rightarrow B = \begin{cases} X & if \ A \subseteq B, \\ B & if \ B \subset A. \end{cases}$ Then, $G = (\alpha, \beta)$ on $G^*$ is the L-graph in which $V = \{q_1, q_2, \ldots, q_n\}$ such that $q_i$ for each $i = 1, 2, \ldots, n$ is labeled with these drugs. Moreover, $q_iq_j \in E$ if and only if these two drugs have at least one similar side effect $\alpha(q_i) = \{a_k | a_k$ is one of side effects of $q_i\}$ and $\beta(q_iq_j) = \alpha(q_i) \otimes \alpha(q_j)$, for each $q_iq_j \in E$. Therefore, $A(Z(G)) = (Q, X, \mu, F, \sigma)$ is the associated L-graph automaton such that $Z(G)$ is a zero-forcing set. Moreover, these behaviors whose words do not have the label $n$ are certain drugs that have the most similar side effects.

b: Suppose that $n$ drugs have $m$ health benefits. As in the above method, we can use $A(Z(G)) = (Q, X, \mu, F, \sigma)$, which is the associated L-graph automaton, so $Z(G)$ is a zero forcing set. Moreover, these behaviors whose words do not have the label $n$ are determined to be drugs that have the most similar health benefits.

c: These L-graph automata can also be used to determine the number of articles or books that have the most similar topics.

*Example 3:* Suppose that four drugs $q_1$, $q_2$, $q_3$, and $q_4$ have five side effects, namely, headache, dizziness, nausea, drowsiness, and physical pain, so that drug $q_1$ has two side effects: Headache, dizziness, and nausea, and drug $q_2$ has four side effects: Dizziness, nausea, drowsiness and body pain, the drug $q_3$ two side effects: Nausea and drowsiness, and drug $q_4$ one side effect: physical pain. Suppose $L = (P(X), \cap, \cup, \otimes, \rightarrow, \emptyset, X)$, where $X = \{a_1, a_2, \ldots, a_5\}$, $A \otimes B = A \cap B$, and $A \rightarrow B = \begin{cases} X & if \ A \subseteq B, \\ B & if \ B \subset A. \end{cases}$ Then, $G = (\alpha, \beta)$ on $G^*$ is the L-graph, as in Fig.

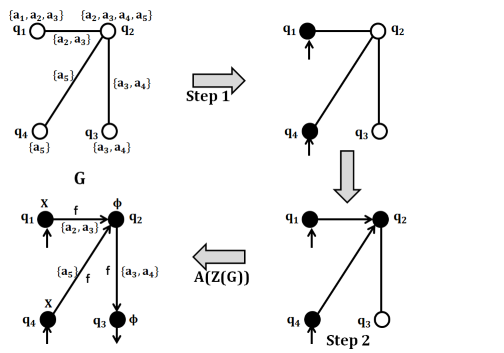


Fig. 4. The L-graph $G$, and the related L-graph automaton $A(Z(G))$.

4, where $V = \{q_1, q_2, q_3, q_4\}$, $E = \{q_1q_2, q_2q_3, q_2q_4\}$, $\alpha(q_1) = \{a_1, a_2, a_3\}$, $\alpha(q_2) = \{a_2, a_3, a_4, a_5\}$, $\alpha(q_3) = \{a_3, a_4\}$, $\alpha(q_4) = \{a_5\}$, $\beta(q_1q_2) = \{a_2, a_3\}$, $\beta(q_2q_3) = \{a_3, a_4\}$, and $\beta(q_2q_4) = \{a_5\}$. For any zero forcing set, these related L-graph automata are equivalent. Consider $Z(G) = \{q_1, q_4\}$. Thus, $A(Z(G)) = (Q, X, \mu, F, \sigma)$ is the related L-graph automaton, as in Fig. 4, where $Q = V$, $\sigma(q_1) = \sigma(q_4) = X$, $\sigma(q_2) = \sigma(q_3) = \emptyset$, $\mu(q_1, f, q_2) = \{a_2, a_3\}$, $\mu(q_4, f, q_2) = \{a_5\}$, and $\mu(q_3, f, q_2) = \{a_3, a_4\}$. Hence,

$$\begin{aligned} r_\mu(f^2, q) &= (\sigma(q_1) \otimes \mu(q_1, f, q_2) \otimes \mu(q_2, f, q_3)) \\ &\vee (\sigma(q_4) \otimes \mu(q_4, f, q_2) \otimes \mu(q_2, f, q_3)) \\ &= ((\{a_2, a_3\}) \otimes (\{a_3, a_4\})) \\ &\vee ((\{a_5\}) \otimes (\{a_3, a_4\})) \\ &= \{a_5\} \vee \emptyset \\ &= \{a_5\}. \end{aligned}$$

drugs $q_1$, $q_2$, and $q_3$ have the most similar side effects.

## V. Conclusion

In this paper, the associated L-graph automata are introduced, and their behaviors are studied. In addition, some theorems and examples are presented for clarification. We also found some applications for L-graph automata. For example: find drugs that have the most similar treatments. L-Graph automata are used to find the drugs that are most similar in terms of side effects or health benefits. One of the applications of this article is to help medical researchers treat diseases better. In the next article, we will try to establish a connection between the L-graph and the corresponding L-graph automaton to solve more complex problems, such as choosing the best drug for a disease.

## References


[1] AIM Minimum Rank–Special Graphs Work Group, "Zero forcing sets and the minimum rank of graphs," Linear Algebra and its Applications, vol. 428, no. 7, p.p 1628-1648, Apr. 2008.

[2] K. Blount, and C. Tsinakis, "The structure of residuated lattices," International Journal of Algebra and Computation, vol. 13, n. 4, pp. 437– 461, Aug. 2003.

[3] J.A Bondy, and U.S. R. Murty, Graph theory, Landon, UK, Springer London, 2008, pp. 29-86.

[4] M. Ciric, "Lattices of subautomata and direct sum decompositions of automata," Algebra Colloquium, vol. 6, no. 1, pp. 71-88, Dec. 1999.

[5] L. Deaett, and S. A. Meyer, "The minimum rank problem for circulants," Linear Algebra and its Applications, vol. 491, no. 4, pp. 386-418, Feb. 2016.

[6] A. Kauffman, Introduction a La theorie des sous-ensembles flous: A L'Usage Des Ingenieurs (Fuzzy Sets Theory), Tome III, Paris, French, Masson Et Cie, Editeurs, 1975.

[7] J.N Mordeson, and D.S Malik, Fuzzy automata and languages theory and applications, Chapman and Hall, 2002, pp. 32-134.

[8] J. N. Mordeson, and S. Mathew, Mathematics of Uncertainty for Coping with World Challenges: Climate Change, World Hunger, Modern Slavery, Coronavirus, Human Trafficking, Springer International Publishing, 2021, pp. 133-173.

[9] D. Qiu, "Automata theory based on complete residuated lattice-valued logic(I)," Sci China Ser F, vol. 44, no. 6 pp. 419–429, Oct. 2001.

[10] D. Qiu, D. "Automata theory based on complete residuated lattice-valued logic (II)," Sci China Ser F, vol. 45, no. 6, pp. 442–452, Oct. 2002.

[11] E. Raisi Sarbizhan, M.M. Zahedi, and M. Sahmsizade, "L-graph automata and some applications," submitted.

[12] E. Raisi Sarbizhan, and M.M. Zahedi, "Maximal product of two L-graph automata and some applications," submitted.

[13] A. Rosenfeld, "Fuzzy groups," Journal of Mathematical Analysis and Applications, vol. 35, no. 3, p.p 512-517, Sep. 1971.

[14] E.S Santos, "Maxmin automata," Information Control, vol. 13, no. 2, pp.363-377, 1968.

[15] S.p. Tiwari, A.K. Singh, and S. Sharan, "Fuzzy Automata based on Lattice-Ordered Monoid and Associated Topology," Journal of Uncertain Systems, vol. 6, no. 1, pp. 51-55, May. 2012.

[16] M. Ward, and R.P.Dilworth, "Residuated lattices," Transactions of The American Society, vol. 45, no. 3, pp. 335–354, Apr. 1939.

[17] W. G Wee, On generalization of adaptive algorithm and application of the fuzzy sets conceptto pattern classification, Ph.D. Thesis, Purdue University, Lafayette, India, 1967.

[18] D.B. West, Introduction to graph theory, (Vol. 2), Delhi, India, Pearson Education 2001, pp. 19-67.

[19] L.A. Zadeh, "Similarity relations and fuzzy orderings," Information Sciences, vol. 3, no. 2, p.p 177–200, Apr. 1971.